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New results on finite-time stability of nonlinear fractional-order multi-time delay systems: Delayed Gronwall inequality approach

Mihailo P. Lazarević[a,*], Stjepko Pišl[a], Darko Radojević[b]

[a] *Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Belgrade, Serbia*
[b] *Dunav Insurance Company, Ustanička 128, 11050 Belgrade, Serbia*

Abstract. In this paper, finite time stability (FTS) analysis of fractional-order nonlinear multi-time delay systems is studied. By use of a fractional Gronwall inequality with time delay, new FTS criteria for proposed systems are established. Two numerical examples are given to illustrate the effectiveness of the obtained theoretical results.

1. Introduction

During past decades, a substantial effort has been made to study the stability and stabilization problem for nonlinear systems with time delays. In practice, there is not only an interest in system stability (e.g. in the sense of Lyapunov), but also in the bounds of system trajectories, (known as non-Lyaponov stability - *finite time stability* (FTS)). For a system, it is said to be FTS once a time interval is fixed, if its state does not exceed some bounds during this time interval, [1]. Control design and stability issues of time-delay systems (TDS) were widely studied due to the effect of delay phenomena on system dynamics, which it often leads to poor performance or even instability, [25],[9]. Recently, the use of fractional calculus methods has been quite prominent in mathematical modeling and control of various processes and phenomena. Also, fractional order dynamical systems have drawn much attention from researchers and engineers over the past few decades, particularly for different kinds of stability, [17]. Stability analysis of the time delay systems of fractional order is more complicated than that of ordinary fractional differential equations, because fractional derivatives are nonlocal and have weakly singular kernels that usually have complicated structures, [18],[10].

Here, we are interested in FTS, where checking the FTS of time-delay fractional-order systems is initially suggested and presented in [11],[12] using (generalized) Gronwall inequality (GGI), and later in [15],[13]. Namely, *Gronwall-type* inequalities, also known as *Gronwall–Bellman* inequalities, are essential tools for analysis of the behavior of solution of differential equations with integer/fractional order and serve to check

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* Corresponding author: Mihailo P. Lazarević
Email addresses: mlazarevic@mas.bg.ac.rs (Mihailo P. Lazarević), stjepko.pisl@gmail.com (Stjepko Pišl), drmasf@yahoo.com (Darko Radojević)

the boundedness property of the considered system, [22],[19],[24],[2],[20],[3],[16]. Recently, authors [4],[5] introduced and applied a *fractional Gronwall inequality with time delay* (FGIT).

To the best of our knowledge, the problem of finding conditions of the FTS for the nonstationary nonlinear fractional-order system with state and control time delays using FGIT has not been considered yet. In this contribution, motivated from the above, we were inspired to study, for the first time, the FTS of a single term $0 < \alpha < 1$ and two term $0 < \beta < \alpha < 1$ *fractional order system with time delay* in state and control, via fractional Gronwall inequality with time delay approach.

The core contributions and novelties of this paper can be stated as follows:

- So far, there are very few research papers on nonlinear nonstationary two term fractional order systems with time delays in the state. Particularly, we consider the case of FOTDS with time delays in state and control.
- By implementing a fractional Gronwall inequality with time delay we derive the FTS conditions, i.e. new criteria are obtained for nonlinear a single term $0 < \alpha < 1$ and two term $0 < \beta < \alpha < 1$ fractional order system with time delays.
- The formulated FTS conditions can be easily validated by two numerical examples.

The rest of this contribution is organized as follows. Some basic definitions and lemmas on fractional calculus as well as on nonlinear time-delay systems of fractional order are given in Section 2. In Section 3, a new criteria of robust FTS of nonstationary nonlinear fractional-order system with state and control time delays is established. Two numerical examples are presented to illustrate the application and verify the effectiveness of theoretical results in Section 4. Finally, this paper ends with a conclusion in Section 5.

2. Preliminaries and problem statement

2.1. Preliminaries

For completeness, this section provides system formulation and some useful properties to derive our required results. The norm $\|(\cdot)\|$ will denote any vector norm, i.e. $\|(\cdot)\|_1$, $\|(\cdot)\|_2$, or $\|(\cdot)\|_\infty$, or the corresponding matrix norm induced by the equivalent vector norm, i.e. 1-, 2-, or ∞- norm, respectively. Also, we introduce some basic notations, definitions of the Caputo and Riemann–Liouville (RL) fractional derivatives, special functions, as well as basic lemmas.

Definition 2.1. *The Riemann–Liouville fractional integral of order α for an integrable function $f(t) : [t_0, \infty) \to \mathbb{R}$ is defined as [18]:*

$$ {}^{\mathrm{RL}}_{t_0}\mathrm{D}_t^{-\alpha} f(t) \equiv {}_{t_0}\mathrm{I}_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s)\mathrm{d}s,\ t \geq t_0,\ \alpha \in \mathbb{C}, t > 0,\ \mathrm{Re}(\alpha) > 0, \tag{1} $$

where $\Gamma(\cdot)$ is the Gamma function, $\Gamma(\xi) = \int_0^\infty s^{\xi-1} e^{-s}\mathrm{d}s$.

Definition 2.2. *[18] The left Caputo fractional derivative of order α, $(n-1 \leq \alpha < n \in \mathbb{Z}^+)$ of the function $f(t)$ is:*

$$ {}^{\mathrm{C}}\mathrm{D}_{t_0,t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\mathrm{d}\tau, \tag{2} $$

where $f^{(n)}(\tau) = \frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}$.

Definition 2.3. *[10] The Mittag-Leffler function with one parameter is given as:*

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)},\ (\alpha = 1,\ E_1(z) = e^z),\ \alpha > 0,\ z \in \mathbb{C}. \tag{3} $$

Definition 2.4. *[10] The Beta function can be defined as follows:*

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\mathrm{d}t = \frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)}, \tag{4}$$

where $p, q > 0$.

Lemma 2.5. *[10]*

$${}_0\mathrm{I}_t^{\alpha}\left({}^c\mathrm{D}_0^{\alpha}x(t)\right) = x(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}x^{(k)}(0),\ n-1<\alpha<n,\ t<0. \tag{5}$$

Lemma 2.6. *Let* $\alpha > \beta > 0$, $n-1<\beta<n$ *and* $x(t) \in AC^n[a,b]$. *Then*

$${}_0\mathrm{I}_t^{\alpha}\left({}^c\mathrm{D}_0^{\alpha}x(t)\right) = {}_0I_t^{\alpha-\beta}x(t) - \sum_{k=0}^{n-1}\frac{t^{k+\alpha-\beta}}{\Gamma(\alpha-\beta+k+1)}x^{(k)}(0). \tag{6}$$

Lemma 2.7. *(Generalized Gronwall Inequality) Suppose* $\alpha > 0$, $z(t)\,\nu(t)$ *are nonnegative and local integrable on* $0 \le t \le T$, $T \le +\infty$ *and* $g(t)$ *is a nonnegative, nondecreasing continuous function defined on* $0 \le t \le T$, $g(t) \le M =$ *const*, $\alpha > 0$ *with*

$$z(t) \le \nu(t) + g(t)\int_0^t (t-s)^{\alpha-1}z(s)\mathrm{d}s \tag{7}$$

on this interval. Then

$$z(t) \le \nu(t) + \int_0^t\left[\sum_{n=1}^{\infty}\frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}a(s)\right]\mathrm{d}s,\ 0\le t<T \tag{8}$$

Corollary 2.8. *Under the hypothesis of Lemma 2.7, let* $\nu(t)$ *be a nondecreasing function on* $[0,T)$. *Then it holds:*

$$z(t) \le \nu(t)E_{\alpha}\left(g(t)\Gamma(\alpha)t^{\alpha}\right) \tag{9}$$

where E_α *is the Mittag-Leffler function.*

Lemma 2.9. *(extended from the GGI), [21] Suppose non-integer orders* $\alpha > 0$, $\beta > 0$, $\nu(t)$ *is a nonnegative function locally integrable on* $[0,T)$, $g_1(t)$ *and* $g_2(t)$ *are nonnegative, nondecreasing, continuous functions defined on* $[0,T)$, $g_1(t) \le N_1$, $g_2(t) \le N_2$, $(N_1, N_2 = const)$. *Suppose* $z(t)$ *is nonnegative and locally integrable on* $[0,T)$ *with*

$$z(t) \le \nu(t) + g_1(t)\int_0^t (t-s)^{\alpha-1}z(s)\mathrm{d}s + g_2(t)\int_0^t (t-s)^{\beta-1}z(s)\mathrm{d}s,\ t\in[0,T). \tag{10}$$

It follows,

$$z(t) \le \nu(t) + \int_0^t\sum_{n=1}^{\infty}\left[g(t)\right]^n\cdot\sum_{k=0}^{n}\frac{C_n^k\left[\Gamma(\alpha)\right]^{n-k}\left[\Gamma(\beta)\right]^k}{\Gamma\left((n-k)\alpha+k\beta\right)}(t-s)^{(n-k)\alpha+k\beta-1}\nu(s)\mathrm{d}s,\ t\in[0,T), \tag{11}$$

where $g(t) = g_1(t) + g_2(t)$ *and* $C_n^k = \frac{n(n-1)(n-1)\cdots(n-k+1)}{k!}$.

Corollary 2.10. *Under the hypothesis of Lemma 2.9, let* $\nu(t)$ *be a nondecreasing function on* $[0,T)$. *Then*

$$z(t) \le \nu(t)E_{\overline{\omega}}\left[g(t)\left(\Gamma(\alpha)t^{\alpha}+\Gamma(\beta)t^{\beta}\right)\right],\ \overline{\omega} = \min(\alpha,\beta). \tag{12}$$

Lemma 2.11. *[5] (Fractional-Order Gronwall Integral Inequality With Time Delay): FOGITD - single case. Assume that $z(t)$, $c(t)$, $d(t)$, $h(t)$ and $k(t)$ are continuous, nonnegative functions on $[t_0, T]$, $\Psi(t)$ is continuous, nonnegative function on $[t_0 - l, T]$ with*

$$\begin{aligned} & z(t) \le c(t) + \frac{d(t)}{\Gamma(\gamma)} \int_{t_0}^{t} (t-s)^{\gamma-1} \left[h(s)z(s) + k(s)z(s-l)\right] \mathrm{d}s, \; t \in [t_0, T], \\ & z(t) \le \varphi(t), \qquad t \in [t_0 - l, t_0]. \end{aligned} \tag{13}$$

Then

$$z(t) \le \left\{ F(t) + G(t) \int_{t_0}^{t} \exp\left(\int_{r}^{t} G(s)H(s)\mathrm{d}s \right) \left[H(r)F(r) + K(r)\Psi(r-l)\right] \mathrm{d}r \right\}^{\frac{1}{q}}, \tag{14}$$

for $[t_0, t_0 + l]$ and

$$z(t) \le \left\{ \begin{array}{l} F(t) + G(t) \int_{t_0}^{t_0+l} \exp\left(\int_{r}^{t_0+l} G(s)H(s)\mathrm{d}s \right) \left[H(r)F(r) + K(r)\Psi(r-l)\right] \mathrm{d}r \\ \exp\left[\int_{t_0+l}^{t_0} \left[K(r)G(r-l) + G(r)H(r)\right] \mathrm{d}r \right] \\ +G(t) \int_{t_0+l}^{t} \exp\left[\int_{r}^{t} \left[K(s)G(s-l) + H(s)G(s)\right] \mathrm{d}s \right] \left[F(r)H(r) + K(r)F(r-l)\right] \mathrm{d}r \end{array} \right\}^{\frac{1}{q}}, \tag{15}$$

for $[t_0 + l, T]$. Furthermore, if $c(t)$, $d(t)$ and $Psi(t)$ are nondecreasing functions, $c(t)$, $d(t)$ and $\Psi(t)$, $c(t_0) = \Psi(t_0)$

$$z(t) \le F^{\frac{1}{q}}(t) \exp\left(\frac{G(t)}{q} \int_{t_0}^{t} \left[K(s) + H(s)\right] \mathrm{d}s \right). \tag{16}$$

For $t \in [t_0, T]$ where $G(t) = 3^{q-1} \frac{d(t)(t-t_0)^{\frac{\gamma-1}{q}}}{\left(\Gamma(\gamma)(p(\gamma-1)+1)^{\frac{1}{p}}\right)^{q}}$, $F(t) = 3^{q-1}c^{q}(t)$, $H(t) = h^{q}(t)$ and $K(t) = k^{q}(t)$, $p, q > 1$ satisfying $\gamma > \frac{1}{q}$ and $\frac{1}{q} + \frac{1}{p} = 1$.

Lemma 2.12. *[6], FOGITD with two different orders. Assume that $c(t)$, $d(t)$, $f(t)$, $k_1(t)$, $k_2(t)$, $h_1(t)$, $h_2(t) \in AB\left([0, T], \mathbb{R}^{+}\right)$, $(0 < T, \infty)$, $\gamma, \lambda > 0$ are continuous, nonnegative functions on $[0, T]$, $\varphi(t)$ is continuous, nonnegative function on $[-l, T]$, $\varphi(t) \in AB\left([-l, 0], \mathbb{R}^{+}\right)$, $c(0) = \varphi(0)$ and $z(t) \in AB\left([-l, T], \mathbb{R}^{+}\right)$*

$$\begin{aligned} & z(t) \le c(t) + \frac{d(t)}{\Gamma(\gamma)} \int_{0}^{t} (t-s)^{\gamma-1} \left[h_1(s)z(s) + k_1(s)z(s-l)\right] \mathrm{d}s \\ & + \frac{f(t)}{\Gamma(\lambda)} \int_{0}^{t} (t-s)^{\lambda-1} \left[h_2(s)z(s) + k_2(s)z(s-l)\right] \mathrm{d}s, \; t \in [0, T] \\ & z(t) \le \varphi(t), \; t \in [-l, 0] \end{aligned} \tag{17}$$

Then

$$z(t) \le F^{\frac{1}{q}}(t) \exp\left(G(t) \cdot \frac{2t}{q} \right), \; t \in [0, T] \tag{18}$$

where

$$Q(t) = 3^{q-1} \left\{ \frac{d(t)}{\Gamma(\gamma)} \frac{t^{\gamma-\frac{1}{q}} \cdot h_1(t)}{(p(\gamma-1)+1)^{\frac{1}{p}}} + \frac{f(t)}{\Gamma(\lambda)} \frac{t^{\lambda-\frac{1}{q}} \cdot h_2(t)}{(p(\lambda-1)+1)^{\frac{1}{p}}} \right\}^{q}, \tag{19}$$

$$R(t) = 3^{q-1} \left\{ \frac{d(t)}{\Gamma(\gamma)} \frac{t^{\gamma-\frac{1}{q}} \cdot k_1(t)}{(p(\gamma-1)+1)^{\frac{1}{p}}} + \frac{f(t)}{\Gamma(\lambda)} \frac{t^{\lambda-\frac{1}{q}} \cdot k_2(t)}{(p(\lambda-1)+1)^{\frac{1}{p}}} \right\}^{q}, \tag{20}$$

$G(t) = \max\{R(t), Q(t)\}$, $p, q > 0$, $F(t) = 3^{q-1} \cdot c^{q}(t)$, $p, q > 0$ *satisfying* $\frac{1}{p} + \frac{1}{q} = 1$, $\gamma > \frac{1}{q}$ *and* $\lambda < \frac{1}{q}$.

Corollary 2.13. *[6], FOGITD with multiple different orders. Assume that* $c(t)$, $d_i(t)$, $k_i(t)$, $h_i(t) \in AB([0,T], \mathbb{R}^+)$, $(0 < T < \infty)$, $\lambda_i > 0$, $i = 1,2,...,n$, *nondecreasing functions,* $\varphi(t) \in AB([-l,0], \mathbb{R}^+)$, $c(0) = \varphi(0)$ *and* $z(t) \in AB([0,T], \mathbb{R}^+)$ *with*

$$\begin{aligned} &z(t) \le c(t) + \sum_{i=1}^{n} \frac{d_i(t)}{\Gamma(\lambda_i)} \int_0^t (t-s)^{\lambda_i - 1} \left[h_i(s)z(s) + k_i(s)z(s-l)\right] \mathrm{d}s, \\ &z(t) \le \varphi(t),\ t \in [-l,0]. \end{aligned} \tag{21}$$

Then,

$$z(t) \le F^{\frac{1}{q}}(t) \exp\left(G(t) \cdot \frac{2t}{q}\right),\ t \in [0,T] \tag{22}$$

where

$$Q(t) = (n+1)^{q-1} \left\{ \sum_{i=1}^{n} \frac{d_i(t)}{\Gamma(\lambda_i)} \frac{t^{\lambda_i - \frac{1}{q}} \cdot h_i(t)}{(p(\lambda_i - 1) + 1)^{\frac{1}{p}}} \right\}^q, R(t) = (n+1)^{q-1} \left\{ \sum_{i=1}^{n} \frac{d_i(t)}{\Gamma(\lambda_i)} \frac{t^{\lambda_i - \frac{1}{q}} \cdot k_i(t)}{(p(\lambda_i - 1) + 1)^{\frac{1}{p}}} \right\}^q \tag{23}$$

$G(t) = \max\{R(t), Q(t)\}$, $p, q > 0$, $F(t) = (n+1)^{q-1} \cdot c^q(t)$, $p, q > 0$ *satisfying* $\frac{1}{p} + \frac{1}{q} = 1$, *and* $\lambda_i = \frac{1}{q}$.

Lemma 2.14. *[7], (Theorem 1.) Let* $\lambda > 0$, $(0 < T < \infty)$, $c(t)$, $d(t)$, $k(t)$, $h(t)$, $z(t)$, *be nonnegative continuous functions defined on* $[t_0, T]$, *as well as* $\varphi(t)$ *be a nonnegative continuous function defined on* $[t_0 - l, t_0]$ *and suppose*

$$\begin{aligned} &z(t) \le c(t) + \frac{d(t)}{\Gamma(\lambda)} \int_{t_0}^t (t-s)^{\lambda - 1} \left[h(s)z(s) + k(s)z(s-l)\right] \mathrm{d}s,\ t \in [t_0, T], \\ &z(t) \le \varphi(t),\ t \in [t_0 - l, t_0]. \end{aligned} \tag{24}$$

Furthermore, if $c(t)$, $d(t)$, $\varphi(t)$ *are nondecreasing,* $c(t_0) = \varphi(t_0)$ *then*

$$z(t) \le c(t) \left[1 + \left(\exp\left(\int_{t_0}^t 2^{q-1} B^q(t) \left(h(s) + k(s)\right)^q \mathrm{d}s\right) - 1\right)^{\frac{1}{q}}\right],\ t \in [t_0, T], \tag{25}$$

where $p, q > 0$ *such that* $\lambda > \frac{1}{q}$, $\frac{1}{p} + \frac{1}{q} = 1$ *and*

$$B(t) = \frac{d(t)(t - t_0)^{\lambda - \frac{1}{q}}}{\Gamma(\lambda)\left(p(\lambda - 1) + 1\right)^{\frac{1}{p}}}. \tag{26}$$

Lemma 2.15. *[8], (Theorem 1.) Assume that* $\lambda > 0$, $(0 < T < \infty)$, $c(t)$, $k(t)$, $h(t)$, $z(t) \in \mathbb{C}([0,T], \mathbb{R}^+)$, *as well as* $\varphi(t) \in \mathbb{C}([-l,0], \mathbb{R}^+)$ *and* $c(t)$, $\varphi(t)$ *be nondecreasing, with* $c(0) = \varphi(0)$. *If* $c(t) \in \mathbb{C}([-l,T], \mathbb{R}^+)$ *and*

$$\begin{aligned} &z(t) \le c(t) + \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda - 1} \left[h(s)z(s) + k(s)z(s-l)\right] \mathrm{d}s, \\ &z(t) \le \varphi(t),\ t \in [-l,0] \end{aligned} \tag{27}$$

then

$$z(t) \le \left[F(t) \exp\left(\int_0^t \left(Q(s) + R(s)\right) \mathrm{d}s\right)\right]^{\zeta},\ t \in [0,T], \tag{28}$$

where

$$R(t) = \frac{4^{\frac{1}{\zeta} - 1}}{\Gamma^{\frac{1}{\zeta}}(\zeta)} \left(B\left(\frac{\xi - \zeta}{1 - \zeta}\right), \frac{1 - \xi}{1 - \zeta}\right)^{\frac{1-\zeta}{\zeta}} t^{\frac{\lambda - \zeta}{\zeta}} \cdot k^{\frac{1}{\zeta}}(t),\ Q(t) = \frac{4^{\frac{1}{\zeta} - 1}}{\Gamma^{\frac{1}{\zeta}}(\zeta)} \left(B\left(\frac{\xi - \zeta}{1 - \zeta}\right), \frac{1 - \xi}{1 - \zeta}\right)^{\frac{1-\zeta}{\zeta}} t^{\frac{\lambda - \zeta}{\zeta}} \cdot h^{\frac{1}{\zeta}}(t), \tag{29}$$

$F(t) = 2^{\frac{1}{\zeta} - 1} \cdot c^{\frac{1}{\zeta}}$, $0 < \zeta < \xi < 1$ *and* $B(\cdot,\cdot)$ *is the Beta function, defined by (4).*

2.2. Problem statement

a) Case: one term $0 < \alpha < 1$

We first consider the following nonstationary nonlinear fractional-order $0 < \alpha < 1$ delay system with state and control time delays given by the following equation:

$$ {}^{c}\mathrm{D}_t^{\alpha}\mathbf{x}(t) = A_0(t)\mathbf{x}(t) + A_1(t)\mathbf{x}(t-l_x) + B_0\mathbf{u}(t) + B_1\mathbf{u}(t-l_u) + D\mathbf{w}(t) + \mathbf{g}(t, x(t), x(t-l_x), \mathbf{w}(t)) \tag{30} $$

with associated continuous functions of initial state and input (control):

$$ \mathbf{x}(t) = \mathbf{\Psi}_x(t),\ t \in [-l_x, 0]\,,\ \mathbf{u}(t) = \mathbf{\Psi}_u(t),\ t \in [-l_u, 0]\,, \tag{31} $$

where l_u is the time input delay, l_x is the time state delay; $x(t) \in \mathbb{R}^n$ is the state vector and $\mathbf{u}(t) \in \mathbb{R}^m$ is the control input; $A_0(t), A_1(t)$, are time-varying matrices as well as B_0, B_1, D denote constant matrices with appropriate dimensions. Behavior of system (30) with given initial function (32) is observed over time interval $J = [t_0, t_0 + T] \in \mathbb{R}$, where T may be either a real positive number or symbol ∞. The $\mathbf{w}(t) \in \mathbb{R}^n$ is the disturbance vector, which has the upper bound as follows: $\|\mathbf{w}(t)\| < \chi_w,\ \chi_w = const > 0, \forall t \in J$. $\mathbf{\Psi}_x(t) \in \mathbb{C}\left([-l_x, 0], \mathbb{R}^n\right)$ is the initial function of $x(t)$ with the norm $\|\Psi_x\|_C = \sup_{-l\leq s\leq 0}\|\Psi_x(s)\|$. Here, the following assumption for the nonlinear perturbation $\mathbf{g}(\cdot)$ is introduced. The nonlinear perturbation $\mathbf{g}\left(t, x(t), x(t-l_x), \mathbf{w}(t)\right)$ satisfies the condition, i.e. there is a continuous function $M(t)$ on $[0, +\infty]$ such that

$$ \|\mathbf{g}(t, x(t), x(t-l_x), \mathbf{w}(t))\| \leq M(t)\left(\|\mathbf{x}(t)\| + \|\mathbf{x}(t-l_x)\| + \|\mathbf{w}(t)\|\right) \tag{32} $$

Also, matrices $A_0(t), A_1(t)$ contain time-varying structural uncertainties $\Delta A_i(t),\ i = 1, 2$ satisfying the following

$$ A_0(t) = A_0 + \Delta A_0(t),\ A_1(t) = A_1 + \Delta A_1(t), \tag{33} $$

where A_0, A_1 are known constant matrices. The norm $\|x(t)\|_\infty$ will be used here as well

$$ \begin{aligned} &\sup_{t\in[0,T]}\|\Delta A_0(t)\| = \Delta a_0,\ \sup_{t\in[0,T]}\|\Delta A_1(t)\| = \Delta a_1, \\ &\sup_{t\in[0,T]}\|A_0(t)\| + \|A_1(t)\| < \infty,\ \sup_{t\in[0,T]}\|G(t)\| = g,\ \sup_{t\in[0,T]}\|M(t)\| = m. \end{aligned} \tag{34} $$

Definition 2.16. *[12],[14]: The nonlinear fractional-order* $0 < \alpha < 1$ *delay system with state and control time delays given by nonhomogenous state equation (30) satisfying initial conditions (31) is finite-time stable w.r.t.* $\left\{\delta, \varepsilon, t_0, \chi_u, \chi_0, J, \|(\cdot)\|\right\}$, $0 < \delta < \varepsilon$, *if and only if:*

$$ \left.\begin{aligned} &\|\mathbf{\Psi}_x\|_C < \delta,\ \|\mathbf{\Psi}_u\|_C < \chi_0, \\ &\|\mathbf{u}(t)\| < \chi_u,\ \forall t \in J, \end{aligned}\right\} \quad \Rightarrow \quad \|\mathbf{x}(t)\| < \varepsilon,\ \forall t \in J. \tag{35} $$

where χ_0, χ_u *are positive constants.*

Definition 2.17. *[12]: The nonlinear fractional-order* $0 < \alpha < 1$ *delay system with state delays given by homogeneous state equation (30)* $\mathbf{u}(t) \equiv 0, \mathbf{u}(t-l_u) \equiv 0$, *satisfying initial conditions (31) is finite-time stable w.r.t.* $\{\delta, \varepsilon, t_0, J, \|(\cdot)\|\}$, $0 < \delta < \varepsilon$, *if and only if:*

$$ \|\mathbf{\Psi}_x\|_C < \delta \quad \Rightarrow \quad \|\mathbf{x}(t)\| < \varepsilon,\ \forall t \in J. \tag{36} $$

Definition 2.18. *[12]: The nonlinear fractional-order* $0 < \alpha < 1$ *delay system with state delays given by homogeneous state equation (30)* $\mathbf{u}(t-l_u) \equiv 0$, *satisfying initial conditions (31) is finite-time stable w.r.t.* $\{\delta, \varepsilon, t_0, \chi_u, J, \|(\cdot)\|\}$, $0 < \delta < \varepsilon$, *if and only if:*

$$ \|\mathbf{\Psi}_x\|_C < \delta, \quad \|\mathbf{u}(t)\| < \chi_u \quad \Rightarrow \quad \|\mathbf{x}(t)\| < \varepsilon,\ \forall t \in J. \tag{37} $$

b) Case: two-term $0 < \beta < \alpha < 1$ *FOTDS*

Further, we will consider the nonstationary nonlinear fractional-order two-term $0 < \beta < \alpha < 1$ delay system with state and control time delays given by the following equation:

$$\begin{aligned}&{}^{c}\mathrm{D}_t^{\alpha}\mathbf{x}(t) = A_0(t)\mathbf{x}(t) + A_1(t)\mathbf{x}(t-l_x) + A_{N2}(t)^{c}\mathrm{D}_t^{\beta}\mathbf{x}(t-l_{xn2}) + B_0\mathbf{u}(t) + B_1\mathbf{u}(t-l_u)\\&+\mathbf{g}(t, x(t), x(t-l_x), \mathbf{w}(t))\end{aligned} \tag{38}$$

with associated continuous functions of initial state and input (control):

$$\mathbf{x}(t) = \mathbf{\Psi}_x(t),\ t \in [-l, 0], \quad \mathbf{u}(t) = \mathbf{\Psi}_u(t),\ t \in [-l_u, 0], \tag{39}$$

where l_u is the time input delay, l_x, l_{n2} are the time state delays, and without losing generality it is assumed that $l_x = l_{n2} = l$.

3. Main results

3.1. Robust FTS of one-term $0 < \alpha < 1$ nonlinear fractional-order multi-time delay system

Theorem 3.1. *The nonstationary nonlinear one-term fractional order time-varying delay system (30) satisfying initial conditions (31) is finite-time stable w.r.t.* $\{\delta, \varepsilon, t_0, \chi_u, \chi_0, J, \|(\cdot)\|\}$, $\delta < \varepsilon$ *if the following condition holds:*

$$3^{\frac{q-1}{q}}\exp\left[\frac{G(t)}{q}\left(a_{om}^q + a_{1m}^q\right)t\right] + \frac{\chi_{0u}^*|t|^\alpha}{\Gamma(\alpha+1)} + \frac{\chi_{10}^* l_u^\alpha}{\Gamma(\alpha+1)} + \frac{\chi_{mw}^*|t|^\alpha}{\Gamma(\alpha+1)} + \frac{\chi_{1u}^*|t-l_u|^\alpha}{\Gamma(\alpha+1)} \le \frac{\varepsilon}{\delta} \tag{40}$$

where $\chi_{0u}^* = \frac{b_0\chi_u}{\delta}$, $\chi_{10}^* = \frac{b_1\chi_0}{\delta}$, $\chi_{1u}^* = \frac{b_1\chi_u}{\delta}$, $\chi_{mw}^* = \frac{d_m\chi_w}{\delta}$, $\|A_0\| = a_0$, $\|A_1\| = a_1$, $\|B_0\| = b_0$, $\|B_1\| = b_1$, $\|D\| = d$, $G(t) = \frac{3^{q-1}t^{\frac{\alpha-1}{q}}}{\left(\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}\right)^q}$

Proof: The fractional order satisfies $0 < \alpha < 1$, and solution can be obtained in the form of the equivalent Volterra integral equation, where is $t_0, l_x = l$

$$\mathbf{x}(t) = \Psi_x(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\begin{bmatrix} A_0(s)\mathbf{x}(s) + A_1(s)\mathbf{x}(s-l) + B_0\mathbf{u}(s) \\ +B_1\mathbf{u}(s-l_u) + D\mathbf{w}(s) + \mathbf{g}(s, x(s), x(t-l), \mathbf{w}(s)) \end{bmatrix}\mathrm{d}s. \tag{41}$$

Applying the norm $\|(\cdot)\|$ and previous assumptions to the previous expression it follows:

$$\|\mathbf{x}(t)\| \le \|\Psi_x(0)\| + \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\begin{bmatrix} \|A_0(s)\|\|\mathbf{x}(s)\| + \|A_1(s)\|\|\mathbf{x}(s-l)\| + \|B_0\|\|\mathbf{u}(s)\| + \|B_1\|\|\mathbf{u}(s-l_u)\| \\ +\|D\|\|\mathbf{w}(s)\| + M(s)\left(\|\mathbf{x}(s)\| + \|\mathbf{x}(s-l)\| + \|\mathbf{w}(s)\|\right) \end{bmatrix}\mathrm{d}s. \tag{42}$$

On the other hand, there are:

$$\begin{aligned}&\left\|A_0(t)\mathbf{x}(t) + A_1(t)\mathbf{x}(t-l) + B_0\mathbf{u}(t) + B_1\mathbf{u}(t-l_u) + D\mathbf{w}(t) + \mathbf{g}(t, x(t), x(t-l), \mathbf{w}(t))\right\|\\&\le \|A_0(t)\|\|\mathbf{x}(t)\| + \|A_1(t)\|\|\mathbf{x}(t-l)\| + \|B_0\|\|\mathbf{u}(t)\|\\&+ \|B_1\|\|\mathbf{u}(t-l_u)\| + \|D\|\|\mathbf{w}(t)\| + M(t)\left(\|x(t)\| + \|x(t-l)\| + \|\mathbf{w}(t)\|\right)\\&\le (a_0 + \Delta a_0 + m)\|\mathbf{x}(t)\| + (a_1 + \Delta a_1 + m)\|\mathbf{x}(t-l)\| + b_1\|\mathbf{u}(t-l_u)\| + (d+m)\|\mathbf{w}(t)\|\\&= a_{om}\|\mathbf{x}(t)\| + a_{1m}\|\mathbf{x}(t-l)\| + b_0\|\mathbf{u}(t)\| + b_1\|\mathbf{u}(t-l_u)\| + d_m\|\mathbf{w}(t)\|.\end{aligned} \tag{43}$$

Combining the previous two expressions, taking into account $\|\mathbf{u}(t)\| < \chi_u$, $\|\mathbf{w}(t)\| < \chi_w$, $\forall t \in J$, it yields:

$$\begin{aligned}&\|\mathbf{x}(t)\| \le \|\Psi_x\|_C + \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\left[a_{0m}\|\mathbf{x}(s)\| + a_{1m}\|\mathbf{x}(s-l)\|\right]\mathrm{d}s\\&+\frac{b_1}{\Gamma(\alpha)}\int_{l_u}^t |t-s|^{\alpha-1}\left[\|\mathbf{u}(s-l_u)\|\right]\mathrm{d}s + \frac{b_0\chi_u|t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1\chi_0 l_u^\alpha}{\Gamma(\alpha+1)} + \frac{d_m\chi_w|t|^\alpha}{\Gamma(\alpha+1)}\end{aligned} \tag{44}$$

or

$$\begin{aligned} \|\mathbf{x}(t)\| \leq \|\Psi_x\|_C + \frac{1}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1} \left[a_{0m}\|\mathbf{x}(s)\| + a_{1m}\|\mathbf{x}(s-l)\|\right] \mathrm{d}s \\ + \frac{b_0 \chi_u |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \chi_0 l_u^\alpha}{\Gamma(\alpha+1)} + \frac{d_m \chi_w |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \chi_u |t-l_u|^\alpha}{\Gamma(\alpha+1)} \end{aligned} \tag{45}$$

Let $t_0 = 0$, $c(t) = \|\Psi_x\|_C$, $d(t) = 1$, $z(t) = \|x(t)\|$, $\varphi(t) = \|\Psi_x\|_C$, $h(t) = a_{0m}$, $k(t) = a_{1m}$. It is obvious that $c(0) = \varphi(0) = \|\Psi_x\|_C$ and $c(t)$, $d(t)$ and $\varphi(t)$ are nondecreasing functions. Applying Lemma 2.11, on the inequality (45) one can gain the following inequality

$$\|\mathbf{x}(t)\| \leq 3^{\frac{q-1}{q}} c(t) \cdot \exp\left(\frac{G(t)}{q} \int_0^t (K(s)+H(s))\, \mathrm{d}s\right) + \frac{b_0 \chi_u |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \chi_0 l_u^\alpha}{\Gamma(\alpha+1)} + \frac{d_m \chi_w |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \chi_u |t-l_u|^\alpha}{\Gamma(\alpha+1)},\ t \in [0, T] \tag{46}$$

where are $G(t) = \frac{3^{q-1} t^{\frac{\alpha-1}{q}}}{\left(\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}\right)^q}$, $H(t) = a_{om}^q$ and $K(t) = a_{1m}^q$, $p, q > 1$ satisfying $\alpha > \frac{1}{q}$ and $\frac{1}{q} + \frac{1}{p} = 1$. Therefore, one has

$$\|\mathbf{x}(t)\| \leq 3^{\frac{q-1}{q}} \|\Psi_x\|_C \exp\left[\frac{G(t)}{q}\left(a_{om}^q + a_{1m}^q\right) t\right] + \frac{b_0 \chi_u |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \chi_0 l_u^\alpha}{\Gamma(\alpha+1)} + \frac{d_m \chi_w |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \chi_u |t-l_u|^\alpha}{\Gamma(\alpha+1)},\ t \in [0, T]. \tag{47}$$

Finally, using the basic condition of Theorem 3.1, and $\|\Psi_x\|_C < \delta$, we can obtain the required FTS condition: $\|\mathbf{x}(t)\| < \varepsilon$, $\forall t \in J$.

Theorem 3.2. *The nonstationary nonlinear one-term fractional order time-varying delay system (30) satisfying initial conditions (31) is finite-time stable w.r.t.* $\{\delta, \varepsilon, t_0, \chi_u, \chi_0, J, \|(\cdot)\|\}$, $\delta < \varepsilon$ *if the following condition holds:*

$$\left[1 + \left(\exp\left(\int_0^t 2^{q-1} B^q(t)\,(h(s)+k(s))^q\, \mathrm{d}s\right) - 1\right)^{\frac{1}{q}}\right] + \frac{\chi_{0u}^* |t|^\alpha}{\Gamma(\alpha+1)} + \frac{\chi_{10}^* l_u^\alpha}{\Gamma(\alpha+1)} + \frac{\chi_{mw}^* |t|^\alpha}{\Gamma(\alpha+1)} + \frac{\chi_{1u}^* |t-l_u|^\alpha}{\Gamma(\alpha+1)} \leq \frac{\varepsilon}{\delta} \tag{48}$$

where $\chi_{0u}^* = \frac{b_0 \chi_u}{\delta}$, $\chi_{10}^* = \frac{b_1 \chi_0}{\delta}$, $\chi_{1u}^* = \frac{b_1 \chi_u}{\delta}$, $\chi_{mw}^* = \frac{d_m \chi_w}{\delta}$, $\|A_0\| = a_0$, $\|A_1\| = a_1$, $\|B_0\| = b_0$, $\|B_1\| = b_1$, $\|D\| = d$, $B(t) = \frac{t^{\frac{\alpha-1}{q}}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}}$, $p, q > 0$, $\alpha > \frac{1}{q}$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof: Applying same procedure from Proof of Theorem 3.1, $t_0 = 0$, $l_x = l$, we can easily get

$$\begin{aligned} \|\mathbf{x}(t)\| \leq \|\Psi_x\|_C + \frac{1}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1} \left[a_{0m}\|\mathbf{x}(s)\| + a_{1m}\|\mathbf{x}(s-l)\|\right] \mathrm{d}s \\ + \frac{b_0 \chi_u |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \chi_0 l_u^\alpha}{\Gamma(\alpha+1)} + \frac{d_m \chi_w |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \chi_u |t-l_u|^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{49}$$

Let $z(t) = \|x(t)\|$, $c(t) = \varphi(t) = \|\Psi_x\|_C$, $d(t) = 1$, $h(t) = a_{0m}$, $k(t) = a_{1m}$, $t_0 = 0$. One may conclude that $c(0) = \varphi(0) = \|\Psi_x\|_C$ and $c(t)$, $d(t)$, $\varphi(t)$ are nondecreasing functions. So, using condition of the Lemma 2.14 we obtain:

$$\begin{aligned} \|\mathbf{x}(t)\| \leq \|\Psi_x\|_C \left[1 + \left(\exp\left(\int_{t_0}^t 2^{q-1} B^q(t)\,(h(s)+k(s))^q\, \mathrm{d}s\right) - 1\right)^{\frac{1}{q}}\right] \\ + \frac{b_0 \chi_u |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \chi_0 l_u^\alpha}{\Gamma(\alpha+1)} + \frac{d_m \chi_w |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \chi_u |t-l_u|^\alpha}{\Gamma(\alpha+1)},\ t \in [0, T]. \end{aligned} \tag{50}$$

Based on Definition 2.17 and the condition of the Theorem 3.2, finally lead us to infer: $\|\mathbf{x}(t)\| < \varepsilon$, $\forall t \in J$.

Remark 3.3. *Let us consider the system* (51), *[12],[15], which is a special case og the system* (30)*, where* $B_1 = 0$, $D = 0$, $\mathbf{g}(\cdot) = 0$, $A_0(t) = A_0$, $A_1(t) = A_1$, $B_0 = B$, $l_x = \tau$, $a_{0m} = a_0 = \|A_0\|$, $a_{1m} = a_1 = \|A_1\|$

$$ {}^{c}\mathrm{D}_t^{\alpha}\mathbf{x}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) + B\mathbf{u}(t). \tag{51}$$

Corollary 3.4. *In this case one can obtain the nonhomogenous system* (51) *is finite-time stable w.r.t.* $\{\delta, \varepsilon, t_0, \chi_u, J, \|(\cdot)\|\}$, $0 < \delta < \varepsilon$, *if*

$$\left[1 + \left(\exp\left(\int_0^t 2^{q-1} B^q(t)\,(a_0+a_1)^q\,\mathrm{d}s\right) - 1\right)^{\frac{1}{q}}\right] + \frac{b_0\chi_u|t|^\alpha}{\delta\cdot\Gamma(\alpha+1)} \le \frac{\varepsilon}{\delta} \tag{52}$$

where satisfying $\alpha > \frac{1}{q}$ *and* $\frac{1}{q} + \frac{1}{p} = 1$. *The previous condition can be written as follows:*

$$\left[1 + \left(\exp\left(2^{q-1}\frac{t^{\alpha q}\cdot(a_0+a_1)^q}{\left[\Gamma(\alpha)\,(1+p(\alpha-1))^{\frac{1}{p}}\right]^q}\right) - 1\right)^{\frac{1}{q}}\right] + \frac{b_0\chi_u|t|^\alpha}{\delta\cdot\Gamma(\alpha+1)} \le \frac{\varepsilon}{\delta} \tag{53}$$

In the homogenous case, we obtain from Theorem 3.2 the following FTS condition

Corollary 3.5. *The homogenous system* (51), $\mathbf{u}(t) \equiv 0$, *is finite-time stable w.r.t.* $\{\delta, \varepsilon, t_0, J, \|(\cdot)\|\}$, $0 < \delta < \varepsilon$, *if*

$$\left[1 + \left(\exp\left(2^{q-1}\frac{t^{\alpha q}\cdot(a_0+a_1)^q}{\left[\Gamma(\alpha)\,(1+p(\alpha-1))^{\frac{1}{p}}\right]^q}\right) - 1\right)^{\frac{1}{q}}\right] \le \frac{\varepsilon}{\delta} \tag{54}$$

satisfying $\alpha > \frac{1}{q}$ *and* $\frac{1}{q} + \frac{1}{p} = 1$.

Theorem 3.6. *The nonstationary nonlinear fractional order time-varying delay system* (30) *satisfying initial conditions* (31) *is finite-time stable w.r.t.* $\{\delta, \varepsilon, t_0, \chi_u, \chi_0, J, \|(\cdot)\|\}$, $\delta < \varepsilon$, *if the following condition holds:*

$$\begin{aligned} &2^{1-\zeta}\exp\left(\frac{\zeta^2}{\lambda}\frac{4^{\frac{1}{\zeta}-1}}{\Gamma^{\frac{1}{\zeta}}(\lambda)}\left(B\left(\frac{\lambda-\zeta}{1-\zeta},\frac{1-\lambda}{1-\zeta}\right)\right)^{\frac{1-\zeta}{\zeta}} t^{\frac{\lambda}{\zeta}}\left(a_{om}^{\frac{1}{\zeta}} + a_{1m}^{\frac{1}{\zeta}}\right)\right) \\ &+ \frac{\chi_{0u}^*|t|^\alpha}{\Gamma(\alpha+1)} + \frac{\chi_{10}^* l_u^\alpha}{\Gamma(\alpha+1)} + \frac{\chi_{mw}^*|t|^\alpha}{\Gamma(\alpha+1)} + \frac{\chi_{1u}^*|t-l_u|^\alpha}{\Gamma(\alpha+1)} \le \frac{\varepsilon}{\delta},\ t\in[0,T] \end{aligned} \tag{55}$$

where are: $\chi_{0u}^* = \frac{b_0\chi_u}{\delta}$, $\chi_{10}^* = \frac{b_1\chi_0}{\delta}$, $\chi_{1u}^* = \frac{b_1\chi_u}{\delta}$, $\chi_{mw}^* = \frac{d_m\chi_w}{\delta}$, $0 < \zeta < \xi < 1$ *and*

$$R(t) = \frac{4^{\frac{1}{\zeta}-1}}{\Gamma^{\frac{1}{\zeta}}(\lambda)}\left(B\left(\frac{\lambda-\zeta}{1-\zeta}\right),\frac{1-\lambda}{1-\zeta}\right)^{\frac{1-\zeta}{\zeta}} t^{\frac{\lambda-\zeta}{\zeta}}\cdot a_{1m}^{\frac{1}{\zeta}},\ Q(t) = \frac{4^{\frac{1}{\zeta}-1}}{\Gamma^{\frac{1}{\zeta}}(\lambda)}\left(B\left(\frac{\lambda-\zeta}{1-\zeta}\right),\frac{1-\lambda}{1-\zeta}\right)^{\frac{1-\zeta}{\zeta}} t^{\frac{\lambda-\zeta}{\zeta}}\cdot a_{om}^{\frac{1}{\zeta}}. \tag{56}$$

Proof: Applying a similar procedure from the previous proof, we have

$$\begin{aligned} \|\mathbf{x}(t)\| &\le \|\Psi_x\|_C + \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\left[a_{0m}\|\mathbf{x}(s)\| + a_{1m}\|\mathbf{x}(s-l)\|\right]\mathrm{d}s \\ &+ \frac{b_0\chi_u|t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1\chi_0 l_u^\alpha}{\Gamma(\alpha+1)} + \frac{d_m\chi_w|t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1\chi_u|t-l_u|^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{57}$$

Using Lemma 2.15 we obtain:

$$\|x(t)\| \le \left[F(t)\exp\left(\int_0^t (Q(s)+R(s))\,\mathrm{d}s\right)\right]^\zeta + \frac{b_0\chi_u|t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1\chi_0 l_u^\alpha}{\Gamma(\alpha+1)} + \frac{d_m\chi_w|t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1\chi_u|t-l_u|^\alpha}{\Gamma(\alpha+1)},\ t\in[0,T]. \tag{58}$$

where

$$F(t)=2^{\frac{1}{\zeta}-1}\cdot\|\Psi_x\|_C^{\frac{1}{\zeta}},\ 0<\zeta<\lambda<1,$$
$$R(t)=\frac{4^{\frac{1}{\zeta}-1}}{\Gamma^{\frac{1}{\zeta}}(\lambda)}\left(B\left(\frac{\lambda-\zeta}{1-\zeta}\right),\frac{1-\lambda}{1-\zeta}\right)^{\frac{1-\zeta}{\zeta}}t^{\frac{\lambda-\zeta}{\zeta}}\cdot a_{1m}^{\frac{1}{\zeta}},\ Q(t)=\frac{4^{\frac{1}{\zeta}-1}}{\Gamma^{\frac{1}{\zeta}}(\lambda)}\left(B\left(\frac{\lambda-\zeta}{1-\zeta}\right),\frac{1-\lambda}{1-\zeta}\right)^{\frac{1-\zeta}{\zeta}}t^{\frac{\lambda-\zeta}{\zeta}}\cdot a_{om}^{\frac{1}{\zeta}} \tag{59}$$

or

$$\|x(t)\|\leq\|\Psi_x\|_C\cdot 2^{1-\zeta}\exp\left(\frac{\zeta^2}{\lambda}\frac{4^{\frac{1}{\zeta}-1}}{\Gamma^{\frac{1}{\zeta}}(\lambda)}\left(B\left(\frac{\lambda-\zeta}{1-\zeta},\frac{1-\lambda}{1-\zeta}\right)\right)^{\frac{1-\zeta}{\zeta}}t^{\frac{\lambda}{\zeta}}\left(a_{om}^{\frac{1}{\zeta}}+a_{1m}^{\frac{1}{\zeta}}\right)\right)$$
$$+\frac{b_0\chi_u|t|^\alpha}{\Gamma(\alpha+1)}+\frac{b_1\chi_0 l_u^\alpha}{\Gamma(\alpha+1)}+\frac{d_m\chi_w|t|^\alpha}{\Gamma(\alpha+1)}+\frac{b_1\chi_u|t-l_u|^\alpha}{\Gamma(\alpha+1)},\ 0<\zeta<\lambda<1,\ t\in[0,T]. \tag{60}$$

Based on Definition 2.4 and the condition of the Theorem 3.2 finally, it follows: $\|\mathbf{x}(t)\|<\varepsilon,\ \forall t\in J$.

Theorem 3.7. *The nonstationary nonlinear two-term fractional order time-varying delay system* (38) *satisfying initial conditions* (39) *is finite-time stable w.r.t.* $\{\delta,\varepsilon,t_0,\chi_u,\chi_0,J,\|(\cdot)\|\}$, $\delta<\varepsilon$, *if the following condition holds:*

$$3^{\frac{q-1}{q}}\cdot\left[1+\frac{a_{n\Delta}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]\cdot\exp\left[\frac{G(t)}{q}2t\right]+\frac{\chi_{0u}^*|t|^\alpha}{\Gamma(\alpha+1)}+\frac{\chi_{10}^* l_u^\alpha}{\Gamma(\alpha+1)}+\frac{\chi_{mw}^*|t|^\alpha}{\Gamma(\alpha+1)}+\frac{\chi_{1u}^*|t-l_u|^\alpha}{\Gamma(\alpha+1)}\leq\frac{\varepsilon}{\delta}, \tag{61}$$

where are: $\chi_{0u}^*=\frac{b_0\chi_u}{\delta},\ \chi_{10}^*=\frac{b_1\chi_0}{\delta},\ \chi_{1u}^*=\frac{b_1\chi_u}{\delta},\ \chi_{mw}^*=\frac{d_m\chi_w}{\delta}$.

Proof: The fractional order satisfies and if an integral of non-order ${}_0\mathrm{I}_t^\alpha$, $t_0=0$ is applied on both sides, taking into account Lemmas 2.5,2.6 and $l_{xn2}=l_x=l$ one can be obtained in the form of the equivalent Volterra integral equation:

$$x(t)=\Psi_x(0)-\Psi_x(-l)\frac{A_{N2}(t)\cdot t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{1}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}A_{N2}(s)x(s-l)\mathrm{d}s+$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left(A_0(s)\mathbf{x}(s)+A_1(s)\mathbf{x}(s-l)+B_0\mathbf{u}(s)+B_1\mathbf{u}(s-l_u)+D\mathbf{w}(s)+\mathbf{g}(s,x(s),x(s-l),\mathbf{w}(s))\right)\mathrm{d}s. \tag{62}$$

By employing the norm $\|(\cdot)\|$ on both sides of the previous expression, one gets

$$\|x(t)\|\leq\|\Psi_x(0)\|_C+\|A_{N2}(t)\|\|\Psi_x(-l)\|\frac{|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{1}{\Gamma(\alpha-\beta)}\int_0^t|t-s|^{\alpha-\beta-1}\|A_{N2}(t)\|\|x(s-l)\|\mathrm{d}s$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t|t-s|^{\alpha-1}\|A_0(s)\mathbf{x}(s)+A_1(s)\mathbf{x}(s-l)+B_0\mathbf{u}(s)+B_1\mathbf{u}(s-l_u)+D\mathbf{w}(s)+\mathbf{g}(s,x(s),x(s-l),\mathbf{w}(s))\|\mathrm{d}s. \tag{63}$$

Consequently, we have

$$\|x(t)\|\leq\|\Psi_x\|_C\left[1+\frac{a_{n\Delta}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]+\frac{a_{n\Delta}}{\Gamma(\alpha-\beta)}\int_0^t|t-s|^{\alpha-\beta-1}\|x(s-l)\|\mathrm{d}s$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t|t-s|^{\alpha-1}\|A_0(s)\mathbf{x}(s)+A_1(s)\mathbf{x}(s-l)+B_0\mathbf{u}(s)+B_1\mathbf{u}(s-l_u)+D\mathbf{w}(s)+\mathbf{g}(s,x(s),x(s-l),\mathbf{w}(s))\|\mathrm{d}s. \tag{64}$$

or, taking into account (43), it yields:

$$\|x(t)\|\leq\|\Psi_x\|_C\left[1+\frac{a_{n\Delta}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]+\frac{a_{n\Delta}}{\Gamma(\alpha-\beta)}\int_0^t|t-s|^{\alpha-\beta-1}\|x(s-l)\|\mathrm{d}s$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t|t-s|^{\alpha-1}\left[a_{0m}\|\mathbf{x}(s)\|+a_{1m}\|\mathbf{x}(s-l)\|\right]\mathrm{d}s+\frac{b_0\chi_u|t|^\alpha}{\Gamma(\alpha+1)}+\frac{b_1\chi_0 l_u^\alpha}{\Gamma(\alpha+1)}+\frac{d_m\chi_w|t|^\alpha}{\Gamma(\alpha+1)}+\frac{b_1\chi_u|t-l_u|^\alpha}{\Gamma(\alpha+1)}. \tag{65}$$

In condensed form, we have

$$\begin{aligned} \|x(t)\| &\le c(t) + \frac{a_{n\Delta}}{\Gamma(\alpha-\beta)} \int_0^t |t-s|^{\alpha-\beta-1} \|x(s-l)\| \mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1} \left[a_{0m} \|\mathbf{x}(s)\| + a_{1m} \|\mathbf{x}(s-l)\| \right] \mathrm{d}s + \frac{b_0 \chi_u |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \chi_0 l_u^\alpha}{\Gamma(\alpha+1)} + \frac{d_m \chi_w |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \chi_u |t-l_u|^\alpha}{\Gamma(\alpha+1)}, \end{aligned} \tag{66}$$

where is $c(t) = \|\Psi_x\|_C \left[1 + \frac{a_{n\Delta}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]$. Let $t_0 = 0$, $c(0) = \|\Psi_x\|_C$, $d(t) = 1$, $f(t) = a_{n\Delta}$, $z(t) = \|x(t)\|$, $\varphi(t) = \|\Psi_x\|_C$, $h_1(t) = a_{0m}$, $k_1(t) = a_{1m}$, $h_2(t) = 0$, $k_2(t) = 1$. It is obvious that $c(0) = \varphi(0) = \|\Psi_x\|_C$ and $c(t)$, $d(t)$, $f(t)$, $h_i(t)$, $k_i(t)$ and $\varphi(t)$ are nondecreasing functions. Using Lemma 2.12 on expression (66) one has

$$\|\mathbf{x}(t)\| \le F^{\frac{1}{q}}(t) \exp\left[\frac{G(t)}{q} 2t\right] + \frac{b_0 \chi_u |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \chi_0 l_u^\alpha}{\Gamma(\alpha+1)} + \frac{d_m \chi_w |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \chi_u |t-l_u|^\alpha}{\Gamma(\alpha+1)}, \; t \in [0,T] \tag{67}$$

where

$$\begin{aligned} F(t) &= 3^{q-1} \cdot c^q(t), \; Q(t) = 3^{q-1} \left\{ \frac{1}{\Gamma(\alpha)} \frac{t^{\frac{\alpha-1}{q}} \cdot a_{om}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \right\}^q, \\ R(t) &= 3^{q-1} \left\{ \frac{1}{\Gamma(\alpha)} \frac{t^{\frac{\alpha-1}{q}} \cdot a_{1m}}{(p(\alpha-1)+1)^{\frac{1}{p}}} + \frac{1}{\Gamma(\alpha-\beta)} \frac{t^{\frac{\alpha-\beta-1}{q}} \cdot a_{n\Delta}}{(p(\alpha-\beta-1)+1)^{\frac{1}{p}}} \right\}^q \end{aligned} \tag{68}$$

$G(t) = \max\{R(t), Q(t)\}$, $p, q > 0$ satisfying $\alpha > \frac{1}{q}$ and $\alpha - \beta > \frac{1}{q}$.

$$\begin{aligned} \|\mathbf{x}(t)\| &\le \|\Psi_x\|_C 3^{\frac{q-1}{q}} \cdot \left[1 + \frac{a_{n\Delta}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right] \cdot \exp\left[\frac{G(t)}{q} 2t\right] \\ &+ \frac{b_0 \chi_u |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \chi_0 l_u^\alpha}{\Gamma(\alpha+1)} + \frac{d_m \chi_w |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \chi_u |t-l_u|^\alpha}{\Gamma(\alpha+1)}, \; t \in [0,T]. \end{aligned} \tag{69}$$

Referring to Definition 2.18 and the condition of Theorem 3.7 we can infer that the system exhibits finite-time stability over the interval $[-l, T]$.

4. Numerical examples

In this section, we will give the following two examples to demonstrate the previous theoretical time delay in state and control:

Example 4.1. *Let us consider the following nonlinear nonhomogenous fractional-order system with a constant the constant time delay:*

$${}^{c}\mathrm{D}_t^\alpha \mathbf{x}(t) = (A_0 + \Delta A_0(t))\,\mathbf{x}(t) + (A_1 + \Delta A_1(t))\,\mathbf{x}(t-l_x) + B_0\mathbf{u}(t) + B_1\mathbf{u}(t-l_u) + \mathbf{g}(t, x(t), \mathbf{x}(t-l_x)), \tag{70}$$

where are:

$$\begin{aligned} A_0 &= \begin{bmatrix} -0.2 & 0 \\ -0.1 & 0.3 \end{bmatrix}, \; \Delta A_0(t) = \begin{bmatrix} -0.02 & 0.01\,(1-\sin t) \\ -0.01 & 0.03\cos t \end{bmatrix}, \; B_0 = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}, \\ A_1 &= \begin{bmatrix} -0.2 & 0.1 \\ 0 & -0.1 \end{bmatrix}, \; \Delta A_1(t) = \begin{bmatrix} -0.05 & 0.01\cos t \\ 0.02\cos t & -0.03 \end{bmatrix}, \; B_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned} \tag{71}$$

and $\mathbf{g}(t, x(t), x(t-l_x)) = (-x_1(t)x_2(t-l_x) - x_2(t)x_1(t-l_x))^T$ *where are* $t_0 = 0$, $l_x = 0.1$, $l_u = 0.04$, $\chi_0 = 0.2$, $\chi_u = 1$, *with associated functions:*

$$\begin{aligned} \mathbf{x}(t) &= \Psi_x(t) = [0.05 \;\; 0.05]^T, \; t \in [t_0 - l_x, t_0] = [-0.1, 0], \\ \mathbf{u}(t) &= \Psi_u(t) = [0.1 \;\; 0]^T, \; t \in [t_0 - l_u, t_0] = [-0.04, 0]. \end{aligned} \tag{72}$$

ε	50	100	150	200
Theorem 3.1	0.275	0.3	0.313	0.323
Theorem 3.2	0.475	0.505	0.518	0.53
Theorem 3.6	1.50	1.67	1.76	1.83

Table 1: The estimated times T_e [s] for $\delta = 0.1$ and ε varies in Example 1

It is easily checked that assuption (32) *is satisfied for* $M = 1$*. From the initial functions and given state equation and using norm* $\|(\cdot)\|_\infty$ *it follows:*

$$\begin{aligned} \|\Psi_x\|_C &= \max_{t\in[-0.1,0]} \|\Psi_x(t)\| = \|\Psi_x\| = 0.05 < \delta = 0.1, \\ \|\Psi_u\|_C &= \max_{t\in[-0.04,0]} \|\Psi_u(t)\| = \|\Psi_u\| = 0.1 < \chi_0 = 0.2. \end{aligned} \tag{73}$$

Moreover, other values are calculated as follows: $a_0 = \|A_0\| = 0.4$, $a_1 = \|A_1\| = 0.3$, $b_0 = \|B_0\| = 3$, $b_1 = \|B_1\| = 2$, $\Delta a_0 = \sup_{t\in[0,T]} \|\Delta A_0(t)\| = 0.04$, $\Delta a_1 = \sup_{t\in[0,T]} \|\Delta A_1(t)\| = 0.06$, $\chi^*_{0u} = \frac{b_0\chi_u}{\delta} = 30$, $\chi^*_{10} = \frac{b_1\chi_0}{\delta} = 6$, $\chi^*_{1u} = \frac{b_1\chi_u}{\delta} = 20$, $\chi^*_{mw} = \frac{d_m\chi_w}{\delta} = 0$*. For* $\alpha = 0.9 < 1$*, one can choose* $q = 2$ *such that* $\alpha > \frac{1}{q}$ *and by calculation from* $\frac{1}{q} + \frac{1}{p} = 1$*, one obtains* $p = 2$ *(Theorem 3.1, Theorem 3.2), as well as parameter* $\zeta = 0.8 < \alpha$ *(Theorem 3.6). The task is to analyze the FTS with respect to* $\{\delta = 0.1, \varepsilon = 50, 100, 150, 200\}$*.*

Table 1 illustrates the effectiveness of obtained results in the system (30) *for estimated times* T_e *for different values of* ε*. From the Table 1 one can conclude that the biggest value of estimated time* T_e *can be obtained using criterion from Theorem 3.6.*

Example 4.2. *Let us consider the following nonlinear fractional-order system* $0 < \beta < \alpha < 1$*, with a constant time delay:*

$$^c\mathrm{D}_t^\alpha \mathbf{x}(t) = A_0(t)\mathbf{x}(t) + A_1(t)\mathbf{x}(t-l) + A_2(t)^c\mathrm{D}_t^\beta \mathbf{x}(t) + B_0\mathbf{u}(t) + \mathbf{g}(t, \mathbf{x}(t), \mathbf{x}(t-l)) + D\mathbf{w}(t) \tag{74}$$

where

$$A_0 = \begin{bmatrix} -0.8 & 0 \\ 0 & -0.5 \end{bmatrix}, A_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, A_2 = \begin{bmatrix} 0.3 & -0.2 \\ 0.4 & 0.1 \end{bmatrix}, B_0 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, D = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix} \tag{75}$$

with the associated continuous function of initial state: $\Psi_x(t) = [0.01, 0.02]^T$, $t \in [-l, 0]$ *and* $\mathbf{g}(t, x(t), x(t-l)) = (-x_1(t)x_2(t-l) \ - x_2(t)x_1(t-l))^T$*, disturbance* $\mathbf{w}(t) = \sin t$*, time-delay* $l = 0.2$ *where upper bound* $\gamma_u = 0.2$ *and* $\alpha = 0.9$, $\beta = 0.1$*. Also, one can take* $q = \frac{3}{2}$ *and by calculation from* $\frac{1}{q} + \frac{1}{p} = 1$*, one obtains* $p = 3$*. It is easily checked that assumption* (32) *is satisfied for* $M = 1$*. Also, one can get* $\|A_0\| = 0.8$, $\|A_1\| = 0.3$, $\|A_2\| = 0.5$, $\|B_0\| = 0.5$, $\delta = 0.1$ *and* $\varepsilon = 100$*. Based on FTS criterion* (61)*, Theorem 3.7, one can obtain the estimated time of FTS of the system* (74) *is* $T_e \approx 0.122$ *s.*

5. Conclusion

In this work, a new and robust FTS for nonstationary nonlinear fractional order time-delay systems with $0 < \beta < \alpha < 1$ is studied. A novel FTS analysis has been derived by applying a fractional Gronwall inequality with time delay. Then, sufficient conditions guaranteeing the FTS of considered systems in finite time are obtained. Finally, two numerical examples have also been provided to illustrate the validity of the proposed procedure and give the estimated time of the FTS.

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