Filomat 38:14 (2024), 4823–4851
https://doi.org/10.2298/FIL2414823W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Stochastic dynamics of a hybrid delay food chain model with harvesting and jumps in an impulsive polluted environment

Sheng Wang[a], **Zeyan Yue**[b,*]

[a] *School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, Henan 454003, PR China*
[b] *School of Mathematical Sciences, Tiangong University, Tianjin 300387, PR China*

Abstract. In this paper, we propose a stochastic hybrid delay food chain model with harvesting (under catch-per-unit-effort hypothesis) and jumps in an impulsive polluted environment. First, we study some stochastic dynamic properties of the system by stochastic analysis techniques, such as stochastic persistence in mean, extinction and global attractivity of the system. Then, we research the optimal harvesting problem of the system by ergodic theory. The accurate expressions for the optimal harvesting effort (OHE, for short) and the maximum of expectation of sustainable yield (MESY, for short) are given. Our results show that the stochastic dynamics and optimal harvesting strategy (OHS, for short) of the system are closely correlated with both time delays and environmental noises. Finally, some numerical simulations are introduced to illustrate the main results.

1. Introduction

In modern natural resource management, one of the most significant problems is to establish ecologically, environmentally and economically reasonable optimal harvesting policy ([1]). Overfishing may lead to the extinction of some species, which will be seriously disruptive to the ecological balance. Many animals are endangered by unrestricted harvesting or hunting. Hence, the study of OHS is undoubtedly significant for the development and utilization of animal resources. Many researchers noted that single-species or two-species population systems cannot fully describe some natural phenomena, but numerous critical behaviors can only be exhibited by models with three or more species ([2], [3]). The classical three-species food chain model with harvesting under catch-per-unit-effort hypothesis can be expressed as follows ([4], [5]):

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1 - a_{11}x_1(t) - a_{12}x_2(t)\right]\mathrm{d}t - h_1x_1(t)\mathrm{d}t, \\ \mathrm{d}x_2(t) = x_2(t)\left[-r_2 + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t)\right]\mathrm{d}t - h_2x_2(t)\mathrm{d}t, \\ \mathrm{d}x_3(t) = x_3(t)\left[-r_3 + a_{32}x_2(t) - a_{33}x_3(t)\right]\mathrm{d}t - h_3x_3(t)\mathrm{d}t, \end{cases} \tag{1}$$

2020 *Mathematics Subject Classification*. 60H10, 60H30, 92D25
Keywords. Stochastic hybrid delay system; Optimal harvesting; Food chain model; Polluted environment; Lévy jumps.
Received: 07 April 2023; Revised: 01 July 2023; Accepted: 13 November 2023
Communicated by Miljana Jovanović
This work is supported by National Natural Science Foundation of China (No. 11901166).
* Corresponding author: Zeyan Yue
Email addresses: wangsheng2017@hpu.edu.cn (Sheng Wang), 2420140208@tiangong.edu.cn (Zeyan Yue)

where $x_1(t)$, $x_2(t)$ and $x_3(t)$ represent, respectively, the population densities of prey, intermediate predator and top predator at time t. r_i and a_{ij} are all positive constants. $h_i \geq 0$ represents the harvesting effort of $x_i(t)$ $(i = 1, 2, 3)$.

In the real world, time delays are common and inevitable, "to ignore time delays is to ignore reality" ([6]). Any species in nature will not always react at once to variation in its own population size or that of an interacting species, but will do so after a time delay preferably ([7]). Thus, incorporating time delays into ecosystems makes them much more realistic than those without time delays ([6], [8], [9]). It is well known that systems with discrete time delays and those with continuously distributed time delays do not contain each other but systems with S-type distributed time delays contain both ([10], [11]). Motivated by the above discussion, introducing S-type distributed time delays into system (1) yields:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1 - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t)\right]\mathrm{d}t - h_1x_1(t)\mathrm{d}t, \\ \mathrm{d}x_2(t) = x_2(t)\left[-r_2 + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t)\right]\mathrm{d}t - h_2x_2(t)\mathrm{d}t, \\ \mathrm{d}x_3(t) = x_3(t)\left[-r_3 + \mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t)\right]\mathrm{d}t - h_3x_3(t)\mathrm{d}t, \end{cases} \tag{2}$$

where $\mathcal{D}_{ji}(x_i)(t) = a_{ji}x_i(t) + \int_{-\tau_{ji}}^{0} x_i(t+\theta)\mathrm{d}\mu_{ji}(\theta)$, $\tau_{ji} > 0$ are time delays, $\tau = \max\left\{\tau_{ji}\right\}$. $\mu_{ji}(\theta)$, $\theta \in [-\tau, 0]$ are nondecreasing bounded variation functions.

On the other hand, the deterministic system has its limitation in mathematical modeling of ecosystems since the parameters involved in the system are unable to capture the influence of environmental noises ([12], [13]). Hence, it is of enormous importance to study the effects of environmental noises on the dynamics of population systems. First, let us take white noises into account. One usually use an average value plus an error term to estimate the growth rate r_1 and death rates r_i $(i = 2, 3)$. In view of the central limit theorem, the error term is normally distributed. Assume that r_i are affected by white noises, i.e., $r_1 \hookrightarrow r_1 + \sigma_1\dot{W}_1(t)$, $-r_2 \hookrightarrow -r_2 + \sigma_2\dot{W}_2(t)$, $-r_3 \hookrightarrow -r_3 + \sigma_3\dot{W}_3(t)$, where $W_i(t)$ are standard Wiener processes defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Then, system (2) becomes

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1 - h_1 - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t)\right]\mathrm{d}t + \sigma_1x_1(t)\mathrm{d}W_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[-r_2 - h_2 + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t)\right]\mathrm{d}t + \sigma_2x_2(t)\mathrm{d}W_2(t), \\ \mathrm{d}x_3(t) = x_3(t)\left[-r_3 - h_3 + \mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t)\right]\mathrm{d}t + \sigma_3x_3(t)\mathrm{d}W_3(t). \end{cases} \tag{3}$$

However, it has been noted that white noises cannot describe some sudden environmental perturbations (for example, typhoon, epidemics, earthquakes and so on) which are often encountered in the growth of species. The majority of scholars claimed that Lévy jumps can be used to describe these sudden environmental perturbations and have introduced Lévy jumps into population systems to analyze the effects of these sudden environmental perturbations ([14], [15], [16], [17], [18]). Incorporating Lévy jumps into system (3) yields:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left\{\left[r_1 - h_1 - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t)\right]\mathrm{d}t + \sigma_1\mathrm{d}W_1(t) + \int_{\mathbb{Z}}\gamma_1(\mu)\widetilde{\mathrm{N}}(\mathrm{d}t, \mathrm{d}\mu)\right\}, \\ \mathrm{d}x_2(t) = x_2(t)\left\{\left[-r_2 - h_2 + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t)\right]\mathrm{d}t + \sigma_2\mathrm{d}W_2(t) + \int_{\mathbb{Z}}\gamma_2(\mu)\widetilde{\mathrm{N}}(\mathrm{d}t, \mathrm{d}\mu)\right\}, \\ \mathrm{d}x_3(t) = x_3(t)\left\{\left[-r_3 - h_3 + \mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t)\right]\mathrm{d}t + \sigma_3\mathrm{d}W_3(t) + \int_{\mathbb{Z}}\gamma_3(\mu)\widetilde{\mathrm{N}}(\mathrm{d}t, \mathrm{d}\mu)\right\}, \end{cases} \tag{4}$$

where $\widetilde{\mathrm{N}}(\mathrm{d}t, \mathrm{d}\mu) = \mathrm{N}(\mathrm{d}t, \mathrm{d}\mu) - \lambda(\mathrm{d}\mu)\mathrm{d}t$, N is a Poisson counting measure with characteristic measure λ on a subset $\mathbb{Z} \subseteq [0, +\infty)$ with $\lambda(\mathbb{Z}) < +\infty$ and $\gamma_i(\mu)$ are bounded functions.

As we all know, there are various types of environmental noises. Now, let us take a further step by considering telephone noises into system (4). It can be described as a random switching between two or more environmental regimes and the switching is memoryless and the waiting time for the next switch has an exponential distribution ([19]). Many academics have argued that parameters in ecosystems often switch because of environmental changes, for example, some species have different growth rates at different temperatures and these changes can be well described by a continuous-time Markov chain $\rho(t)$ with finite-state space ([20], [21], [22], [23], [24], [25], [26]). System (4) under regime switching can be described by the following stochastic hybrid delay system:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[(r_1(\rho(t)) - h_1 - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t))\,\mathrm{d}t + \mathcal{S}_1(t,\rho(t))\right], \\ \mathrm{d}x_2(t) = x_2(t)\left[(-r_2(\rho(t)) - h_2 + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_2(t,\rho(t))\right], \\ \mathrm{d}x_3(t) = x_3(t)\left[(-r_3(\rho(t)) - h_3 + \mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_3(t,\rho(t))\right], \end{cases} \tag{5}$$

where $\mathcal{S}_i(t,\rho(t)) = \sigma_i(\rho(t))\mathrm{d}W_i(t) + \int_{\mathbb{Z}} \gamma_i(\mu,\rho(t))\widetilde{\mathrm{N}}(\mathrm{d}t,\mathrm{d}\mu)$ and $\gamma_i(\mu,\rho(t))$ are bounded functions.

Furthermore, environmental pollution has become an important issue of concern to ecologists all over the world. When considering the impacts of environmental pollution on ecosystems, many models assumed that the emission of pollutants was continuous. However, in reality, the discharge of toxic pollutants was often regular, for example, with the rapid development of industrial and agricultural production, some chemical plants and other industries often periodically discharge sewage or other pollutants into rivers, soil and air ([27]). These pollutants can cause direct damage to ecosystems, such as species extinction. Therefore, it is vital to assess the risk of toxicant on populations ([28]). Motivated by above discussions, we refine system (6) as follows:

$$\begin{cases} \left.\begin{aligned} &\mathrm{d}x_1(t) = x_1(t)\left[(r_1(\rho(t)) - h_1 - r_{11}C_{10}(t) - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t))\,\mathrm{d}t + \mathcal{S}_1(t,\rho(t))\right], \\ &\mathrm{d}x_2(t) = x_2(t)\left[(-r_2(\rho(t)) - h_2 - r_{22}C_{20}(t) + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_2(t,\rho(t))\right], \\ &\mathrm{d}x_3(t) = x_3(t)\left[(-r_3(\rho(t)) - h_3 - r_{33}C_{30}(t) + \mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_3(t,\rho(t))\right], \\ &\mathrm{d}C_{i0}(t) = \left[k_iC_e(t) - (g_i + m_i)\,C_{i0}(t)\right]\mathrm{d}t, \\ &\mathrm{d}C_e(t) = -hC_e(t)\mathrm{d}t, \end{aligned}\right\} t \neq n\gamma, \\ \Delta x_i(t) = 0,\ \Delta C_{i0}(t) = 0,\ \Delta C_e(t) = b,\ t = n\gamma,\ n \in \mathbb{N}_+\ (i = 1,2,3), \end{cases} \tag{6}$$

where $\Delta f(t) = f(t^+) - f(t)$. For other parameters in system (6), see Table 1.

Table 1: Definition of some parameters in system (6)

Parameter	**Definition**
$C_{i0}(t)$	the toxicant concentration in the organism of species i at time t
$C_e(t)$	the toxicant concentration in the environment at time t
r_{ii}	the dose-response rate of species i to the organismal toxicant
k_i	the toxin uptake rate per unit biomass
g_i	the organismal net ingestion rate of toxin
m_i	the organismal deportation rate of toxin
h	the rate of toxin loss in the environment
γ	the period of the impulsive toxicant input
b	the toxicant input amount at every time

The rest of this paper is organized as follows. Section 2 begins with some fundamental assumptions, notations and the theorem of existence and uniqueness of global positive solution to system (6). Section 3 focuses on stochastic persistence in mean and extinction of each species in system (6). Section 4 is devoted to the global attractivity of system (6). Section 5 discusses about the optimal harvesting strategy of system (6). Some numerical simulations are given in Section 6. Finally, we conclude the paper with a perspicuous conclusion and discussion in Section 7.

2. Existence and Uniqueness of Global Positive Solution

We have three fundamental assumptions for system (6).

Assumption 2.1. *$W_1(t)$, $W_2(t)$, $W_3(t)$, $\rho(t)$ and N are mutually independent. $\rho(t)$, taking values in $\mathbb{S} = \{1, 2, ..., S\}$, is irreducible with one unique stationary distribution $\pi = (\pi_1, \pi_2, ..., \pi_S)^{\mathrm{T}}$.*

Assumption 2.2. *$r_j(i) > 0$, $a_{jk} > 0$ and there exist $\gamma_j^*(i) \geq \gamma_{j*}(i) > -1$ such that $\gamma_{j*}(i) \leq \gamma_j(\mu, i) \leq \gamma_j^*(i)$ ($\mu \in \mathbb{Z}$), $\forall i \in \mathbb{S}$, $j, k = 1, 2, 3$.*

Remark 2.3. *Assumption 2.2 implies that the intensities of Lévy jumps are not too big to ensure that the solution will not explode in finite time.*

Assumption 2.4. $0 < k_i \leq g_i + m_i$ $(i = 1, 2, 3)$, $0 < b \leq 1 - \mathrm{e}^{-h\gamma}$.

Remark 2.5. *Assumption 2.4 means $0 \leq C_{i0}(t) < 1$ and $0 \leq C_e(t) < 1$, which must be satisfied to be realistic because $C_{i0}(t)$ and $C_e(t)$ are concentrations of the toxicant ($i = 1, 2, 3$).*

Lemma 2.6. *([29], [30]) $C_{i0}(t)$ involved in system (6) satisfies*

$$\lim_{t\to+\infty} t^{-1} \int_0^t C_{i0}(s)\mathrm{d}s = \frac{k_i b}{h\,(g_i + m_i)\,\gamma} = K_i \quad (i = 1, 2, 3). \tag{7}$$

Denote $\mathbb{R}_+ = (0, +\infty)$, $\overline{\mathbb{R}_+} = [0, +\infty)$ and

$$\begin{cases} A_{ij} = a_{ij} + \displaystyle\int_{-\tau_{ij}}^{0} \mathrm{d}\mu_{ij}(\theta),\ K_i = \frac{k_i b}{h\,(g_i + m_i)\,\gamma}, \\ b_1(\cdot) = r_1(\cdot) - \dfrac{\sigma_1^2(\cdot)}{2} - \displaystyle\int_{\mathbb{Z}} \left[\gamma_j(\mu, \cdot) - \ln(1 + \gamma_j(\mu, \cdot))\right] \lambda(\mathrm{d}\mu), \\ b_j(\cdot) = r_j(\cdot) + \dfrac{\sigma_j^2(\cdot)}{2} + \displaystyle\int_{\mathbb{Z}} \left[\gamma_j(\mu, \cdot) - \ln(1 + \gamma_j(\mu, \cdot))\right] \lambda(\mathrm{d}\mu) \ \ (j = 2, 3)\,, \\ \Sigma_1 = \displaystyle\sum_{i=1}^{S} \pi_i b_1(i) - r_{11}K_1,\ \Sigma_j = -\sum_{i=1}^{S} \pi_i b_j(i) - r_{jj}K_j \ \ (j = 2, 3), \\ B_1 = \Sigma_1 - h_1,\ B_2 = \Sigma_2 - h_2 + \frac{A_{21}}{A_{11}} B_1,\ B_3 = \Sigma_3 - h_3 + \frac{A_{32}}{A_{22}} B_2, \\ |\mathbf{A}| = A_{11}A_{22} + A_{12}A_{21},\ \left|\overline{\mathbf{A}}\right| = A_{22}A_{33} + A_{23}A_{32},\ |\mathbf{A_1}| = A_{22}B_1 - A_{12}\left(B_2 - \frac{A_{21}}{A_{11}}B_1\right),\ |\mathbf{A_2}| = A_{11}B_2, \\ \mathbf{\Delta} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ -A_{21} & A_{22} & A_{23} \\ 0 & -A_{32} & A_{33} \end{pmatrix},\ \mathbf{Q} = \begin{pmatrix} 2\left|\overline{\mathbf{A}}\right| & (A_{21} - A_{12})\,A_{33} & A_{12}A_{23} + A_{21}A_{32} \\ (A_{21} - A_{12})\,A_{33} & 2A_{11}A_{33} & A_{11}\,(A_{32} - A_{23}) \\ A_{12}A_{23} + A_{21}A_{32} & A_{11}\,(A_{32} - A_{23}) & 2\,|\mathbf{A}| \end{pmatrix}. \end{cases}$$

Denote $\mathbf{B} = \left(B_1, B_2 - \frac{A_{21}}{A_{11}}B_1, B_3 - \frac{A_{32}}{A_{22}}B_2\right)^{\mathrm{T}}$, $\mathbf{y} = (y_1, y_2, y_3)^{\mathrm{T}}$. From Cramer's Rule, for system $\mathbf{\Delta y} = \mathbf{B}$, if the determinant of coefficients $|\mathbf{\Delta}| = \det(\mathbf{\Delta})$ is nonzero, then it has a unique solution which is given by $\mathbf{y}^* = |\mathbf{\Delta}|^{-1}\left(|\mathbf{\Delta_1}|, |\mathbf{\Delta_2}|, |\mathbf{\Delta_3}|\right)^{\mathrm{T}}$, where $\mathbf{\Delta_j}$ is $\mathbf{\Delta}$ with column j replaced by $\mathbf{B}$ ($j = 1, 2, 3$).

Denote $\mathbf{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3)^{\mathrm{T}}$, $\mathbf{H} = (h_1, h_2, h_3)^{\mathrm{T}}$, $\overline{\mathbf{\Delta}} = \left(\left|\overline{\mathbf{\Delta_1}}\right|, \left|\overline{\mathbf{\Delta_2}}\right|, \left|\overline{\mathbf{\Delta_3}}\right|\right)^{\mathrm{T}}$, where $\overline{\mathbf{\Delta_j}}$ is $\mathbf{\Delta}$ with column j replaced by $\mathbf{\Sigma}$. By Cramer's Rule, for system $\mathbf{QH} = \overline{\mathbf{\Delta}}$, if $|\mathbf{Q}| \neq 0$, then it has a unique solution which is given by $\left(h_1^*, h_2^*, h_3^*\right)^{\mathrm{T}} = |\mathbf{Q}|^{-1}\left(|\mathbf{Q_1}|, |\mathbf{Q_2}|, |\mathbf{Q_3}|\right)^{\mathrm{T}}$, where $\mathbf{Q_j}$ is $\mathbf{Q}$ with column j replaced by $\overline{\mathbf{\Delta}}$ ($j = 1, 2, 3$).

Theorem 2.7. *For any initial condition $\phi \in C\left([-\tau, 0], \mathbb{R}_+^3\right)$, system (6) has a unique global solution $(x_1(t), x_2(t), x_3(t))^{\mathrm{T}} \in \mathbb{R}_+^3$ on $t \in \mathbb{R}_+$ a.s. Moreover, for any constant $p > 0$, there exist $K_i(p) > 0$ such that $\sup_{t\in\mathbb{R}_+} \mathbb{E}\left[x_i^p(t)\right] \leq K_i(p)$ ($i = 1, 2, 3$).*

Proof. The proof is rather standard and hence is omitted (see e.g. [31]). □

3. Stochastic persistence in mean and Extinction

Lemma 3.1. *([32]) Suppose $Z(t) \in C(\Omega \times [0, +\infty), \mathbb{R}_+)$ and $\lim_{t\to+\infty} \frac{o(t)}{t} = 0$.*

(i) *If there exists constant $\delta_0 > 0$ such that for $t \gg 1$,*

$$\ln Z(t) \leq \delta t - \delta_0 \int_0^t Z(s)\mathrm{d}s + o(t), \tag{8}$$

then

$$\begin{cases} \limsup\limits_{t\to+\infty} t^{-1} \int_0^t Z(s)\mathrm{d}s \leq \dfrac{\delta}{\delta_0} \ a.s. & (\delta \geq 0); \\ \lim\limits_{t\to+\infty} Z(t) = 0 \ a.s. & (\delta < 0). \end{cases} \tag{9}$$

(ii) *If there exist constants $\delta > 0$ and $\delta_0 > 0$ such that for $t \gg 1$,*

$$\ln Z(t) \geq \delta t - \delta_0 \int_0^t Z(s)\mathrm{d}s + o(t), \tag{10}$$

then

$$\liminf_{t\to+\infty} t^{-1} \int_0^t Z(s)\mathrm{d}s \geq \frac{\delta}{\delta_0} \ a.s. \tag{11}$$

Now, let us consider the following auxiliary system:

$$\begin{cases} \left.\begin{aligned} &\mathrm{d}X_1(t) = X_1(t)\left[(r_1(\rho(t)) - h_1 - r_{11}C_{10}(t) - \mathcal{D}_{11}(X_1)(t))\,\mathrm{d}t + \mathcal{S}_1(t,\rho(t))\right], \\ &\mathrm{d}X_2(t) = X_2(t)\left[(-r_2(\rho(t)) - h_2 - r_{22}C_{20}(t) + \mathcal{D}_{21}(X_1)(t) - \mathcal{D}_{22}(X_2)(t))\,\mathrm{d}t + \mathcal{S}_2(t,\rho(t))\right], \\ &\mathrm{d}X_3(t) = X_3(t)\left[(-r_3(\rho(t)) - h_3 - r_{33}C_{30}(t) + \mathcal{D}_{32}(X_2)(t) - \mathcal{D}_{33}(X_3)(t))\,\mathrm{d}t + \mathcal{S}_3(t,\rho(t))\right], \\ &\mathrm{d}C_{i0}(t) = \left[k_iC_e(t) - (g_i + m_i)\,C_{i0}(t)\right]\mathrm{d}t, \\ &\mathrm{d}C_e(t) = -hC_e(t)\mathrm{d}t, \end{aligned}\right\} t \neq n\gamma, \\ \Delta X_i(t) = 0,\ \Delta C_{i0}(t) = 0,\ \Delta C_e(t) = b,\ t = n\gamma,\ n \in \mathbb{N}_+\ (i = 1, 2, 3). \end{cases} \tag{12}$$

Lemma 3.2. *System (12) satisfies Table 2, where*

$$\overline{\mathbf{X}^{\mathrm{T}}(\infty)} = \lim_{t\to+\infty} t^{-1}\left(\int_0^t X_1(s)\mathrm{d}s, \int_0^t X_2(s)\mathrm{d}s, \int_0^t X_3(s)\mathrm{d}s\right).$$

Table 2: Persistent in mean and extinction of system (12)

B_3	B_2	B_1	$\overline{\mathbf{X}^{\mathrm{T}}(\infty)}$
≥ 0	≥ 0	≥ 0	$\left(\frac{B_1}{A_{11}}, \frac{B_2}{A_{22}}, \frac{B_3}{A_{33}}\right)$
< 0	≥ 0	≥ 0	$\left(\frac{B_1}{A_{11}}, \frac{B_2}{A_{22}}, 0\right)$
	< 0	≥ 0	$\left(\frac{B_1}{A_{11}}, 0, 0\right)$
		< 0	$(0, 0, 0)$

Proof. Consider the following stochastic hybrid delay logistic model with Lévy jump in an impulsive polluted environment:

$$\begin{cases} \left.\begin{array}{l} \mathrm{d}X_1(t) = X_1(t)\left[(r_1(\rho(t)) - h_1 - r_{11}C_{10}(t) - \mathcal{D}_{11}(X_1)(t))\,\mathrm{d}t + \mathcal{S}_1(t,\rho(t))\right], \\ \mathrm{d}C_{10}(t) = \left[k_1 C_e(t) - (g_1 + m_1)\,C_{10}(t)\right]\mathrm{d}t, \\ \mathrm{d}C_e(t) = -hC_e(t)\mathrm{d}t, \end{array}\right\} t \neq n\gamma, \\ \Delta X_1(t) = 0,\ \Delta C_{10}(t) = 0,\ \Delta C_e(t) = b,\ t = n\gamma,\ n \in \mathbb{N}_+. \end{cases} \tag{13}$$

Thanks to Lemma 2.6 and Lemma 2.3 in [33], system (13) satisfies

$$\begin{cases} \lim\limits_{t\to+\infty} X_1(t) = 0 \ \ a.s. & (B_1 < 0)\,; \\ \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t X_1(s)\mathrm{d}s = \frac{B_1}{A_{11}} \ \ a.s. & (B_1 \geq 0)\,. \end{cases} \tag{14}$$

By Itô's formula, we compute

$$\ln \mathbf{X}(t) = \mathbf{B}t - \mathbf{A_0}\int_0^t \mathbf{X}(s)\mathrm{d}s + \begin{pmatrix} -\mathcal{T}_{11}(X_1)(t) \\ \mathcal{T}_{21}(X_1)(t) - \mathcal{T}_{22}(X_2)(t) \\ \mathcal{T}_{32}(X_2)(t) - \mathcal{T}_{33}(X_3)(t) \end{pmatrix} + \mathbf{o(t)}, \tag{15}$$

where

$$\ln \mathbf{X}(t) = \begin{pmatrix} \ln X_1(t) \\ \ln X_2(t) \\ \ln X_3(t) \end{pmatrix},\ \int \mathbf{X}(s)\mathrm{d}s = \begin{pmatrix} \int X_1(s)\mathrm{d}s \\ \int X_2(s)\mathrm{d}s \\ \int X_3(s)\mathrm{d}s \end{pmatrix},\ \mathbf{A_0} = \begin{pmatrix} A_{11} & 0 & 0 \\ -A_{21} & A_{22} & 0 \\ 0 & -A_{32} & A_{33} \end{pmatrix},\ \mathbf{o(t)} = o(t)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\mathcal{T}_{ji}(X_i)(t) = \int_{-\tau_{ji}}^0 \int_\theta^0 X_i(s)\mathrm{d}s\mathrm{d}\mu_{ji}(\theta) - \int_{-\tau_{ji}}^0 \int_{t+\theta}^t X_i(s)\mathrm{d}s\mathrm{d}\mu_{ji}(\theta).$$

𝔆𝔞𝔰𝔢 (i) : $B_1 < 0$. Then, $\lim_{t\to+\infty} X_1(t) = 0$ a.s. Hence, for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\ln X_2(t) \leq \left(B_2 - \frac{A_{21}}{A_{11}}B_1 + \epsilon\right)t - a_{22}\int_0^t X_2(s)\mathrm{d}s, \tag{16}$$

which implies $\lim_{t\to+\infty} X_2(t) = 0$ a.s. Similarly, $\lim_{t\to+\infty} X_3(t) = 0$ a.s.

𝔆𝔞𝔰𝔢 (ii) : $B_1 \geq 0$. Consider the following auxiliary function:

$$\mathrm{d}\widetilde{X_2(t)} = \widetilde{X_2(t)}\left[\left(-r_2(\rho(t)) - h_2 - r_{22}C_{20}(t) + \mathcal{D}_{21}(X_1)(t) - a_{22}\widetilde{X_2(t)}\right)\mathrm{d}t + \mathcal{S}_2(t,\rho(t))\right]. \tag{17}$$

Then $X_2(t) \leq \widetilde{X_2(t)}$ a.s. By Itô's formula, we get

$$\ln \widetilde{X_2(t)} = B_2 t - a_{22}\int_0^t \widetilde{X_2(s)}\mathrm{d}s + o(t). \tag{18}$$

In view of Lemma 3.1, we deduce that for arbitrary $\zeta > 0$,

$$\lim_{t\to+\infty} t^{-1}\int_{t-\zeta}^t X_i(s)\mathrm{d}s = 0 \ \ a.s. \ \ (i = 1, 2). \tag{19}$$

Combining (19) with system (15) yields

$$\ln X_2(t) = B_2 t - A_{22}\int_0^t X_2(s)\mathrm{d}s + o(t). \tag{20}$$

Thanks to Lemma 3.1, we deduce

$$\begin{cases} \lim\limits_{t\to+\infty} X_2(t) = 0 \ \ a.s. & (B_2 < 0)\,; \\ \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t X_2(s)\mathrm{d}s = \frac{B_2}{A_{22}} \ \ a.s. & (B_2 \geq 0)\,. \end{cases} \tag{21}$$

If $B_2 < 0$, then for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\ln X_3(t) \leq \left(B_3 - \frac{A_{32}}{A_{22}}B_2 + \epsilon\right)t - a_{33}\int_0^t X_3(s)\mathrm{d}s, \tag{22}$$

which implies $\lim_{t\to+\infty} X_3(t) = 0$ a.s.

$\mathfrak{Case}$ (iii) : $B_1 \geq 0$, $B_2 \geq 0$. Consider the following SDDE:

$$\mathrm{d}\widetilde{X_3(t)} = \widetilde{X_3(t)}\left[\left(-r_3(\rho(t)) - h_3 - r_{33}C_{30}(t) + \mathcal{D}_{32}(X_2)(t) - a_{33}\widetilde{X_3(t)}\right)\mathrm{d}t + \mathcal{S}_3(t,\rho(t))\right]. \tag{23}$$

Then, $X_3(t) \leq \widetilde{X_3(t)}$ a.s. By Itô's formula,

$$\ln\widetilde{X_3(t)} = B_3 t - a_{33}\int_0^t \widetilde{X_3(s)}\mathrm{d}s + o(t). \tag{24}$$

Thanks to Lemma 3.1, we derive that for arbitrary $\zeta > 0$,

$$\lim_{t\to+\infty} t^{-1}\int_{t-\zeta}^t X_i(s)\mathrm{d}s = 0 \ \ a.s. \ \ (i = 1,2,3). \tag{25}$$

Hence, for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\ln X_3(t) = B_3 t - A_{33}\int_0^t X_3(s)\mathrm{d}s + o(t). \tag{26}$$

In view of Lemma 3.1, we obtain

$$\begin{cases} \lim\limits_{t\to+\infty} X_3(t) = 0 \ \ a.s. & (B_3 < 0)\,; \\ \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t X_3(s)\mathrm{d}s = \frac{B_3}{A_{33}} \ \ a.s. & (B_3 \geq 0)\,. \end{cases} \tag{27}$$

The proof is complete. □

Lemma 3.3. *For system (6):*
(i) $\limsup_{t\to+\infty} t^{-1}\ln x_i(t) \leq 0$ *a.s.* ($i = 1,2,3$).
(ii) $\lim_{t\to+\infty} x_i(t) = 0 \Rightarrow \lim_{t\to+\infty} x_j(t) = 0$ *a.s.* ($1 \leq i < j \leq 3$).

Proof. From Lemma 3.2, system (12) satisfies $\lim_{t\to+\infty} t^{-1}\ln X_i(t) = 0$ a.s. ($i = 1,2,3$). By the stochastic comparison theorem, we obtain the desired assertion (i). The proof of (ii) is similar to that of Lemma 3.2 and here is omitted. □

Assumption 3.4. $A_{22}A_{33}\,|\mathbf{A}| > A_{12}A_{21}A_{23}A_{32}$.

Theorem 3.5. *Under Assumption 3.4. System (6) satisfies Table 3, where*

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \lim_{t\to+\infty} t^{-1}\left(\int_0^t x_1(s)\mathrm{d}s, \int_0^t x_2(s)\mathrm{d}s, \int_0^t x_3(s)\mathrm{d}s\right).$$

Table 3: Persistent in mean and extinction of system (6)

$\lvert\mathbf{\Delta_3}\rvert$	$\lvert\mathbf{A_2}\rvert$	B_1	$\overline{\mathbf{x}^{\mathrm{T}}(\infty)}$
+			$\left(\frac{\lvert\mathbf{\Delta_1}\rvert}{\lvert\mathbf{\Delta}\rvert}, \frac{\lvert\mathbf{\Delta_2}\rvert}{\lvert\mathbf{\Delta}\rvert}, \frac{\lvert\mathbf{\Delta_3}\rvert}{\lvert\mathbf{\Delta}\rvert}\right)$
−	+		$\left(\frac{\lvert\mathbf{A_1}\rvert}{\lvert\mathbf{A}\rvert}, \frac{\lvert\mathbf{A_2}\rvert}{\lvert\mathbf{A}\rvert}, 0\right)$
	−	+	$\left(\frac{B_1}{A_{11}}, 0, 0\right)$
		−	$(0, 0, 0)$

Proof. Compute $|\mathbf{\Delta_3}| < A_{32}\,|\mathbf{A_2}| < A_{21}A_{32}B_1$. Thanks to Lemma 3.2, for any $\zeta > 0$,

$$\lim_{t\to+\infty} t^{-1}\int_{t-\zeta}^{t} x_i(s)\mathrm{d}s = 0 \ a.s.\ (i = 1, 2, 3). \tag{28}$$

By Itô's formula, we compute

$$\ln \mathbf{x}(t) = (\mathbf{\Sigma} - \mathbf{h})\,t - \mathbf{\Delta}\int_0^t \mathbf{x}(s)\mathrm{d}s + \mathbf{o}(\mathbf{t}). \tag{29}$$

$\mathfrak{Case}$ (i) : $|\mathbf{\Delta_3}| > 0$. Thanks to system (29), we compute

$$\lim_{t\to+\infty} t^{-1}\left(A_{21}A_{32}\ln x_1(t) + A_{11}A_{32}\ln x_2(t) + |\mathbf{A}|\ln x_3(t) + |\mathbf{\Delta}|\int_0^t x_3(s)\mathrm{d}s\right) = |\mathbf{\Delta_3}|. \tag{30}$$

By Lemma 3.3 (i) and Lemma 3.1, we deduce

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \ge \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} \ a.s. \tag{31}$$

Based on system (29), we derive

$$\lim_{t\to+\infty} t^{-1}\left(A_{22}\ln x_1(t) - A_{12}\ln x_2(t) + |\mathbf{A}|\int_0^t x_1(s)\mathrm{d}s - A_{12}A_{23}\int_0^t x_3(s)\mathrm{d}s\right) = |\mathbf{A_1}|. \tag{32}$$

By Lemma 3.3 (i), for $\forall\epsilon \in (0, 1)$ and $t \gg 1$,

$$A_{22}\ln x_1(t) \le \left(|\mathbf{A_1}| + A_{12}A_{23}\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s + \epsilon\right)t - |\mathbf{A}|\int_0^t x_1(s)\mathrm{d}s. \tag{33}$$

In view of (31), we deduce

$$|\mathbf{A_1}| + A_{12}A_{23}\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \ge |\mathbf{A_1}| + A_{12}A_{23}\frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} = |\mathbf{A}|\,\frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|} > 0. \tag{34}$$

Thanks to Lemma 3.1, we obtain

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \le |\mathbf{A}|^{-1}\left(|\mathbf{A_1}| + A_{12}A_{23}\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s\right) \triangleq \Gamma_{x_1}^{sup}. \tag{35}$$

According to (31), (35) and system (29), for $\forall\epsilon \in (0, 1)$ and $t \gg 1$,

$$\ln x_2(t) \le \left(B_2 - \frac{A_{21}}{A_{11}}B_1 + A_{21}\Gamma_{x_1}^{sup} - A_{23}\frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} + \epsilon\right)t - A_{22}\int_0^t x_2(s)\mathrm{d}s. \tag{36}$$

Combining (34) with (35) yields

$$\begin{aligned} & B_2 - \frac{A_{21}}{A_{11}} B_1 + A_{21} \Gamma_{x_1}^{sup} - A_{23} \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} \\ \geq\ & B_2 - \frac{A_{21}}{A_{11}} B_1 + A_{21} |\mathbf{A}|^{-1} \left(|\mathbf{A_1}| + A_{12} A_{23} \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} \right) - A_{23} \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} = A_{22} \frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|} > 0. \end{aligned} \tag{37}$$

Thanks to Lemma 3.1, we obtain

$$\limsup_{t\to+\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s \leq A_{22}^{-1} \left(B_2 - \frac{A_{21}}{A_{11}} B_1 + A_{21} \Gamma_{x_1}^{sup} - A_{23} \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} \right). \tag{38}$$

Therefore, for $\forall \epsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_3(t) \leq \left[B_3 - \frac{A_{32}}{A_{22}} B_2 + \frac{A_{32}}{A_{22}} \left(B_2 - \frac{A_{21}}{A_{11}} B_1 + A_{21} \Gamma_{x_1}^{sup} - A_{23} \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} \right) + \epsilon \right] t - A_{33} \int_0^t x_3(s)\mathrm{d}s. \tag{39}$$

Thanks to (37), we obtain

$$B_3 - \frac{A_{32}}{A_{22}} B_2 + \frac{A_{32}}{A_{22}} \left(B_2 - \frac{A_{21}}{A_{11}} B_1 + A_{21} \Gamma_{x_1}^{sup} - A_{23} \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} \right) \geq B_3 - \frac{A_{32}}{A_{22}} B_2 + A_{32} \frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|} = A_{33} \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} > 0. \tag{40}$$

According to Lemma 3.1, we obtain

$$\limsup_{t\to+\infty} t^{-1} \int_0^t x_3(s)\mathrm{d}s \leq A_{33}^{-1} \left[B_3 - \frac{A_{21}A_{32}}{A_{11}A_{22}} B_1 + \frac{A_{32}}{A_{22}} \left(A_{21} \Gamma_{x_1}^{sup} - A_{23} \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} \right) \right] \ a.s. \tag{41}$$

Under Assumption 3.4, (41) is equal to

$$\limsup_{t\to+\infty} t^{-1} \int_0^t x_3(s)\mathrm{d}s \leq \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} \ a.s. \tag{42}$$

Combining (31) with (42) yields

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_3(s)\mathrm{d}s = \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} \ a.s. \tag{43}$$

Therefore, $\Gamma_{x_1}^{sup} = \frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|}$ and

$$\limsup_{t\to+\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s \leq \frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|} \ a.s. \tag{44}$$

In view of system (29), we compute

$$\lim_{t\to+\infty} t^{-1} \left(A_{21} \ln x_1(t) + A_{11} \ln x_2(t) + |\mathbf{A}| \int_0^t x_2(s)\mathrm{d}s + A_{11}A_{23} \int_0^t x_3(s)\mathrm{d}s \right) = |\mathbf{A_2}|. \tag{45}$$

Thanks to Lemma 3.3 (i), for $\forall \epsilon \in (0,1)$ and $t \gg 1$,

$$A_{11} \ln x_2(t) \geq \left(|\mathbf{A_2}| - A_{11}A_{23} \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} - \epsilon \right) t - |\mathbf{A}| \int_0^t x_2(s)\mathrm{d}s, \tag{46}$$

where $|\mathbf{A_2}| - A_{11}A_{23} \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} = |\mathbf{A}| \frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|} > 0$. According to Lemma 3.1,

$$\liminf_{t\to+\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s \geq \frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|} \ a.s. \tag{47}$$

Combining (44) with (47) yields

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|}\ a.s. \tag{48}$$

Substituting (48) into system (29) yields

$$\lim_{t\to+\infty} t^{-1}\left(\ln x_1(t) + A_{11}\int_0^t x_1(s)\mathrm{d}s\right) = A_{11}\frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|}\ a.s. \tag{49}$$

Therefore, by Lemma 3.1, we obtain

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|}\ a.s. \tag{50}$$

Case (ii) : $|\mathbf{A_2}| > 0 > |\mathbf{\Delta_3}|$. From (30) and Lemma 3.3 (ii), $\lim_{t\to+\infty} x_3(t) = 0$ a.s. Hence, (32) and (45) are transformed into

$$\begin{aligned}
&\lim_{t\to+\infty} t^{-1}\left(A_{22}\ln x_1(t) - A_{12}\ln x_2(t) + |\mathbf{A}|\int_0^t x_1(s)\mathrm{d}s\right) = |\mathbf{A_1}|,\\
&\lim_{t\to+\infty} t^{-1}\left(A_{21}\ln x_1(t) + A_{11}\ln x_2(t) + |\mathbf{A}|\int_0^t x_2(s)\mathrm{d}s\right) = |\mathbf{A_2}|.
\end{aligned} \tag{51}$$

Based on Lemma 3.3 (i), for $\forall\epsilon\in(0,1)$ and $t \gg 1$,

$$\begin{aligned}
&A_{22}\ln x_1(t) \le (|\mathbf{A_1}| + \epsilon)\,t - |\mathbf{A}|\int_0^t x_1(s)\mathrm{d}s,\\
&A_{11}\ln x_2(t) \ge (|\mathbf{A_2}| - \epsilon)\,t - |\mathbf{A}|\int_0^t x_2(s)\mathrm{d}s.
\end{aligned} \tag{52}$$

Thanks to (52) and Lemma 3.1, we have

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \le \frac{|\mathbf{A_1}|}{|\mathbf{A}|}\ a.s. \tag{53-1}$$

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \ge \frac{|\mathbf{A_2}|}{|\mathbf{A}|}\ a.s. \tag{53-2}$$

Substituting (53-1) into system (29) yields that for $\forall\epsilon\in(0,1)$ and $t \gg 1$,

$$\ln x_2(t) \le \left(B_2 - \frac{A_{21}}{A_{11}}B_1 + A_{21}\frac{|\mathbf{A_1}|}{|\mathbf{A}|} + \epsilon\right)t - A_{22}\int_0^t x_2(s)\mathrm{d}s. \tag{54}$$

On the basis of Lemma 3.1, we have

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \le A_{22}^{-1}\left(B_2 - \frac{A_{21}}{A_{11}}B_1 + A_{21}\frac{|\mathbf{A_1}|}{|\mathbf{A}|}\right) = \frac{|\mathbf{A_2}|}{|\mathbf{A}|}\ a.s. \tag{55}$$

Therefore, combining (53-2) and (55), we get

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{A_2}|}{|\mathbf{A}|}\ a.s. \tag{56}$$

By (56) and system (29), we compute

$$\lim_{t\to+\infty} t^{-1}\left(\ln x_1(t) + A_{11}\int_0^t x_1(s)\mathrm{d}s\right) = B_1 - A_{12}\frac{|\mathbf{A_2}|}{|\mathbf{A}|} = A_{11}\frac{|\mathbf{A_1}|}{|\mathbf{A}|}\ a.s. \tag{57}$$

Hence, in view of Lemma 3.1, we obtain

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{A_1}|}{|\mathbf{A}|}\ a.s. \tag{58}$$

$\mathfrak{Case}$ (iii) : $B_1 > 0 > |\mathbf{A_2}|$. Then, $\lim_{t\to+\infty} x_3(t) = 0$ a.s. Thanks to (51) and Lemma 3.3 (ii), we deduce that $\lim_{t\to+\infty} x_2(t) = 0$ a.s. Hence,

$$\ln x_1(t) = B_1 t - A_{11}\int_0^t x_1(s)\mathrm{d}s + o(t). \tag{59}$$

From Lemma 3.1, we obtain

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{B_1}{A_{11}}\ a.s. \tag{60}$$

$\mathfrak{Case}$ (iv) : $B_1 < 0$. By Lemma 3.2, $\lim_{t\to+\infty} x_i(t) = 0$ a.s. $(i = 1, 2, 3)$. □

Remark 3.6. *If $\mathbb{S} = \{1\}$, $h_i = r_{ii} = 0$ and $\mu_{ji}(\theta) = C_{ji}$, then system (6) becomes model (3) discussed in [34]. Hence, Theorem 3.5 contains Lemma 4 in [34] as a special case.*

Remark 3.7. *If $\mathbb{S} = \{1\}$, $h_i = r_{ii} = \gamma_i(\mu, 1) = 0$, $\mu_{ii}(\theta) = C_{ii}$, $a_{ij} = 0$ $(i \neq j)$, and $\mu_{ij}(\theta)$ are defined as follows:*

$$\mu_{12}(\theta) = \begin{cases} a_{12}^*, & -\tau_{12}^* \le \theta \le 0, \\ 0, & -\tau_{12} \le \theta < -\tau_{12}^*, \end{cases} \quad \mu_{21}(\theta) = \begin{cases} a_{21}^*, & -\tau_{21}^* \le \theta \le 0, \\ 0, & -\tau_{21} \le \theta < -\tau_{21}^*, \end{cases}$$

$$\mu_{23}(\theta) = \begin{cases} a_{23}^*, & -\tau_{23}^* \le \theta \le 0, \\ 0, & -\tau_{23} \le \theta < -\tau_{23}^*, \end{cases} \quad \mu_{32}(\theta) = \begin{cases} a_{32}^*, & -\tau_{32}^* \le \theta \le 0, \\ 0, & -\tau_{32} \le \theta < -\tau_{32}^*, \end{cases}$$

then system (6) becomes the following system discussed in [35]:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1 - a_{11}x_1(t) - a_{12}^*x_2\left(t - \tau_{12}^*\right)\right]\mathrm{d}t + \sigma_1 x_1(t)\mathrm{d}W_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[-r_2 + a_{21}^*x_1\left(t - \tau_{21}^*\right) - a_{22}x_2(t) - a_{23}^*x_3\left(t - \tau_{23}^*\right)\right]\mathrm{d}t + \sigma_2 x_2(t)\mathrm{d}W_2(t), \\ \mathrm{d}x_3(t) = x_3(t)\left[-r_3 + a_{32}^*x_2\left(t - \tau_{32}^*\right) - a_{33}x_3(t)\right]\mathrm{d}t + \sigma_3 x_3(t)\mathrm{d}W_3(t). \end{cases}$$

Hence, Theorem 3.5 contains Lemma 2.5 in [35] as a special case.

Remark 3.8. *If $\mathbb{S} = \{1\}$, $h_i = r_{ii} = 0$, $\mu_{ii}(\theta) = C_{ii}$, $a_{ij} = 0$ $(i \neq j)$ and $\mu_{ij}(\theta)$ are defined as follows:*

$$\mu_{12}(\theta) = \begin{cases} \widetilde{a_{12}}, & -\tau_1 \le \theta \le 0, \\ 0, & -\tau_{12} \le \theta < -\tau_1, \end{cases} \quad \mu_{21}(\theta) = \begin{cases} \widetilde{a_{21}}, & -\tau_2 \le \theta \le 0, \\ 0, & -\tau_{21} \le \theta < -\tau_2, \end{cases}$$

$$\mu_{23}(\theta) = \begin{cases} \widetilde{a_{23}}, & -\tau_3 \le \theta \le 0, \\ 0, & -\tau_{23} \le \theta < -\tau_3, \end{cases} \quad \mu_{32}(\theta) = \begin{cases} \widetilde{a_{32}}, & -\tau_4 \le \theta \le 0, \\ 0, & -\tau_{32} \le \theta < -\tau_4, \end{cases}$$

then system (6) becomes the following system discussed in [36]:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[\left(r_1 - a_{11}x_1(t) - \widetilde{a_{12}}x_2\left(t - \tau_1\right)\right)\mathrm{d}t + \mathcal{S}_1(t, 1)\right], \\ \mathrm{d}x_2(t) = x_2(t)\left[\left(-r_2 + \widetilde{a_{21}}x_1\left(t - \tau_2\right) - a_{22}x_2(t) - \widetilde{a_{23}}x_3\left(t - \tau_3\right)\right)\mathrm{d}t + \mathcal{S}_2(t, 1)\right], \\ \mathrm{d}x_3(t) = x_3(t)\left[\left(-r_3 + \widetilde{a_{32}}x_2\left(t - \tau_4\right) - a_{33}x_3(t)\right)\mathrm{d}t + \mathcal{S}_3(t, 1)\right]. \end{cases}$$

Hence, Theorem 3.5 contains Theorem 2 in [36] as a special case.

4. Global attractivity

Assumption 4.1. $2a_{jj} > \sum_{i=1}^{3} A_{ij}$ $(j = 1, 2, 3)$.

Theorem 4.2. *Under Assumption 4.1. System (6) is globally attractive, namely, for any ϕ and $\phi^* \in C\left([-\tau, 0], \mathbb{R}_+^3\right)$, $\lim_{t\to+\infty} \mathbb{E}\left[\left\|\mathbf{x}(t;\phi) - \mathbf{x}(t;\phi^*)\right\|\right] = 0$, where $\mathbf{x}(t;\phi) = \left(x_1(t;\phi), x_2(t;\phi), x_3(t;\phi)\right)^{\mathrm{T}}$ is the solution to system (6) with $\phi \in C\left([-\tau, 0], \mathbb{R}_+^3\right)$.*

Proof. We prove that

$$\lim_{t\to+\infty} \mathbb{E}\left|x_i(t;\phi) - x_i(t;\phi^*)\right| = 0 \ (i = 1, 2, 3). \tag{61}$$

For $(i, j) \neq (1, 3)$ and $(i, j) \neq (3, 1)$, we define function as follows

$$W(t;\phi,\phi^*) = \sum_{i=1}^{3} \left|\ln\left(\frac{x_i(t;\phi^*)}{x_i(t;\phi)}\right)\right| + \sum_{i,j=1}^{3} \int_{-\tau_{ji}}^{0} \int_{t+\theta}^{t} \left|x_i(s;\phi^*) - x_i(s;\phi)\right| \mathrm{d}s\mathrm{d}\mu_{ji}(\theta).$$

Applying Itô's formula, we obtain

$$\mathcal{L}\left[W(t;\phi,\phi^*)\right] \leq -\sum_{j=1}^{3} \left(2a_{jj} - \sum_{i=1}^{3} A_{ij}\right) \left|x_j(t;\phi^*) - x_j(t;\phi)\right|. \tag{62}$$

According to (62), we deduce

$$\int_0^{+\infty} \mathbb{E}\left[\left|x_i(t;\phi^*) - x_i(t;\phi)\right|\right] \mathrm{d}t < +\infty \quad (i = 1, 2, 3). \tag{63}$$

Define $H_i(t) = \mathbb{E}\left[\left|x_i(t;\phi^*) - x_i(t;\phi)\right|\right]$ $(i = 1, 2, 3)$. Then,

$$|H_i(t_2) - H_i(t_1)| \leq \mathbb{E}\left[\left|x_i(t_2;\phi^*) - x_i(t_1;\phi^*)\right|\right] + \mathbb{E}\left[\left|x_i(t_2;\phi) - x_i(t_1;\phi)\right|\right]. \tag{64}$$

Denote $\max_{i\in\mathbb{S}} r_j(i) = r_j^*$, $\max_{i\in\mathbb{S}} \left|\sigma_j(i)\right| = \sigma_j^*$, $\sup_{s\geq 0} C_j(s) = C_j^*$, $\sup_{(\mu,i)\in\mathbb{Z}\times\mathbb{S}} \left|\gamma_j(\mu, i)\right| = \gamma_j^*$, $L_j = \max\{r_j^*, \sigma_j^*, C_j^*, \gamma_j^*\}$. In view of system (6) and Hölder's inequality, for $t_2 > t_1$ and $p > 1$,

$$\left(\mathbb{E}\left[|x_j(t_2) - x_j(t_1)|\right]\right)^p \leq \mathbb{E}\left[|x_j(t_2) - x_j(t_1)|^p\right] \leq 3^{p-1} \sum_{i=1}^{3} \Upsilon_i, \tag{65}$$

where Υ_i will be listed later. By Theorem 7.1 in [37], for $p \geq 2$, we obtain

$$\Upsilon_2 = \mathbb{E}\left[\left|\int_{t_1}^{t_2} \sigma_j(\rho(s))x_j(s)\mathrm{d}W_j(s)\right|^p\right] \leq L_j^p \left(\tfrac{p(p-1)}{2}\right)^{\frac{p}{2}} (t_2 - t_1)^{\frac{p-2}{2}} \int_{t_1}^{t_2} \mathbb{E}\left[x_j^p(s)\right] \mathrm{d}s. \tag{66}$$

From Hölder's inequality, we derive

$$\begin{aligned}
\Upsilon_1 &= \mathbb{E}\left[\left(\int_{t_1}^{t_2} x_j(s)\left(L_j + h_j + r_{jj}L_j + \sum_{i=1}^{3} \mathcal{D}_{ji}(x_i)(s)\right)\mathrm{d}s\right)^p\right] \\
&\leq 9^{p-1}\left[L_j^p + h_j^p + \left(r_{jj}L_j\right)^p\right](t_2 - t_1)^{p-1} \int_{t_1}^{t_2} \mathbb{E}\left[x_j^p(s)\right]\mathrm{d}s + 9^{p-1} \sum_{i=1}^{3} a_{ji}^p (t_2 - t_1)^{p-1} \int_{t_1}^{t_2} \mathbb{E}\left[x_i^p(s)x_j^p(s)\right]\mathrm{d}s \\
&\quad + 9^{p-1} \sum_{i=1}^{3} (t_2 - t_1)^{p-1} \mathbb{E}\left[\int_{t_1}^{t_2} \left(\int_{-\tau_{ji}}^{0} x_i(s+\theta)x_j(s)\mathrm{d}\mu_{ji}(\theta)\right)^p \mathrm{d}s\right].
\end{aligned} \tag{67}$$

On the basis of Hölder's inequality, we get

$$\begin{aligned}&\mathbb{E}\left[\int_{t_1}^{t_2}\left(\int_{-\tau_{ji}}^{0} x_j(s)x_i(s+\theta)\mathrm{d}\mu_{ji}(\theta)\right)^p \mathrm{d}s\right]\\&\leq \frac{1}{2}\left(\int_{-\tau_{ji}}^{0}\mathrm{d}\mu_{ji}(\theta)\right)^p\int_{t_1}^{t_2}\mathbb{E}\left[x_j^{2p}(s)\right]\mathrm{d}s+\frac{1}{2}\left(\int_{-\tau_{ji}}^{0}\mathrm{d}\mu_{ji}(\theta)\right)^{p-1}\int_{t_1}^{t_2}\int_{-\tau_{ji}}^{0}\mathbb{E}\left[x_i^{2p}(s+\theta)\right]\mathrm{d}\mu_{ji}(\theta)\mathrm{d}s.\end{aligned}\tag{68}$$

According to the Kunita's first inequality in [38], for $p > 2$, we get

$$\begin{aligned}\Upsilon_3 &= \mathbb{E}\left[\left|\int_{t_1}^{t_2}\int_{\mathbb{Z}} x_j(s)\gamma_j(\mu,\rho(s))\widetilde{N}(\mathrm{d}s,\mathrm{d}\mu)\right|^p\right]\\&\leq D(p)\left\{\mathbb{E}\left[\left(\int_{t_1}^{t_2}\int_{\mathbb{Z}}\left|x_j(s)\gamma_j(\mu,\rho(s))\right|^2\lambda(\mathrm{d}\mu)\mathrm{d}s\right)^{\frac{p}{2}}\right]+\mathbb{E}\left[\int_{t_1}^{t_2}\int_{\mathbb{Z}}\left|x_j(s)\gamma_j(\mu,\rho(s))\right|^p\lambda(\mathrm{d}\mu)\mathrm{d}s\right]\right\}\\&\leq D(p)\left\{L_j^p\left(\int_{\mathbb{Z}}\lambda(\mathrm{d}\mu)\right)^{\frac{p}{2}}|t_2-t_1|^{\frac{p-2}{2}}\int_{t_1}^{t_2}\mathbb{E}\left[x_j^p(s)\right]\mathrm{d}s+L_j^p\int_{\mathbb{Z}}\lambda(\mathrm{d}\mu)\int_{t_1}^{t_2}\mathbb{E}\left[x_j^p(s)\right]\mathrm{d}s\right\}.\end{aligned}\tag{69}$$

Thanks to (65)-(69), for $p > 2$ and $|t_2 - t_1| \leq \frac{1}{2}$, there is a constant $M > 0$ such that

$$\left(\mathbb{E}\left[|x_j(t_2)-x_j(t_1)|\right]\right)^p \leq M\,|t_2-t_1|\,. \tag{70}$$

Combining (64) with (70) yields

$$\left|H_j(t_2)-H_j(t_1)\right| \leq 2(M|t_2-t_1|)^{\frac{1}{p}}. \tag{71}$$

Consequently, (61) follows from (63), (71) and Barbalat's conclusion in [39]. □

5. Optimal harvesting strategy

Now, let us consider the optimal harvesting problem of system (6). Our goal is to find the optimal harvesting effort $\mathbf{H} = (h_1, h_2, h_3)^{\mathrm{T}}$ such that (**i**) All of $x_1(t)$, $x_2(t)$ and $x_3(t)$ are not extinct; (**ii**) The expectation of sustained yield $Y(\mathbf{H}) = \lim_{t\to+\infty}\mathbb{E}\left[\mathbf{H}^{\mathrm{T}}\mathbf{x}(t)\right]$ is maximum.

Denote $\mathbf{H} = (h_1, h_2, h_3)^{\mathrm{T}} \in \overline{\mathbb{R}_+^3} \Leftrightarrow h_i \in \overline{\mathbb{R}_+}$ $(i = 1, 2, 3)$.

Theorem 5.1. *Under Assumption 3.4. Let*

$$Y^*(\mathbf{H}) = -\frac{\mathbf{H}^{\mathrm{T}}\mathbf{Q}\mathbf{H}}{2} + \mathbf{H}^{\mathrm{T}}\overline{\mathbf{\Delta}}. \tag{72}$$

(i) *If*

$$|\mathbf{\Delta_3}|\,(\mathbf{H}^*)\,|_{\mathbf{H}^*\in\overline{\mathbb{R}_+^3}} > 0,\quad 4A_{11}\left|\overline{\mathbf{A}}\right| > A_{33}\,|A_{12}-A_{21}|^2\,,\quad |\mathbf{Q}| > 0, \tag{73}$$

then the OHE is $\mathbf{H}^* = \left(h_1^*, h_2^*, h_3^*\right)^{\mathrm{T}}$ *and MESY* $= \frac{Y^*(\mathbf{H}^*)}{|\mathbf{\Delta}|}$.

(ii) *If one of the following conditions holds, then the OHS does not exist:*

(a) $0 > |\mathbf{\Delta_3}|\,(\mathbf{H}^*)$;

(𝔟) $\mathbf{H}^* \notin \overline{\mathbb{R}^3_+}$;

(𝔠) $4A_{11}\left|\overline{\mathbf{A}}\right| < A_{33}\left|A_{12} - A_{21}\right|^2$;

(𝔡) $|\mathbf{Q}| < 0$.

Proof. Based on Theorem 2.7, there exists a constant $C(p) > 0$ such that

$$t^{-1}\int_0^t \mathbb{E}\left[\sum_{i=1}^{3} x_i^p(s)\right]\mathrm{d}s \le C(p). \tag{74}$$

Thanks to Theorem 3.1.1 in [40], Theorem 3.1 in [41] and Theorem 3.2.6 in [42], we deduce that $(x_1(t), x_2(t), x_3(t), \rho(t))^{\mathrm{T}}$ has a unique ergodic invariant measure $\nu(\cdot\times\cdot)$ in $\mathbb{R}^3_+ \times \mathbb{S}$. Hence,

$$\sum_{k=1}^{S}\int_{\mathbb{R}^3_+}\theta_i \nu\left(\mathrm{d}\theta_1, \mathrm{d}\theta_2, \mathrm{d}\theta_3, k\right) = \lim_{t\to+\infty} t^{-1}\int_0^t x_i(s)\mathrm{d}s \quad a.s. \quad (i = 1, 2, 3). \tag{75}$$

Define $\mathcal{U} = \left\{\mathbf{H} \in \overline{\mathbb{R}^3_+} \mid |\mathbf{\Delta_3}|(\mathbf{H}) > 0\right\}$. By Theorem 3.5, for any $\mathbf{H} \in \mathcal{U}$, we have

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_i(s)\mathrm{d}s = \frac{|\mathbf{\Delta_i}|}{|\mathbf{\Delta}|} \quad a.s. \quad (i = 1, 2, 3). \tag{76}$$

On the other hand, if the OHE $\mathbf{H}^*$ exists, then $\mathbf{H}^* \in \mathcal{U}$.

Proof of (i). Under condition (73), for $\mathbf{H} \in \mathcal{U}$, we have

$$\lim_{t\to+\infty} t^{-1}\int_0^t \mathbf{H}^{\mathrm{T}}\mathbf{x}(s)\mathrm{d}s = \frac{Y^*(\mathbf{H})}{|\mathbf{\Delta}|}. \tag{77}$$

Let $\varrho(\cdot\times\cdot)$ be the stationary probability density of system (6), then we get

$$Y(\mathbf{H}) = \lim_{t\to+\infty}\mathbb{E}\left[\mathbf{H}^{\mathrm{T}}\mathbf{x}(t)\right] = \sum_{k=1}^{S}\int_{\mathbb{R}^3_+}\mathbf{H}^{\mathrm{T}}\theta\varrho(\theta, k)\mathrm{d}\theta. \tag{78}$$

Noting that system (6) has a unique ergodic invariant measure $\nu(\cdot\times\cdot)$ and that there exists a one-to-one correspondence between $\varrho(\cdot\times\cdot)$ and $\nu(\cdot\times\cdot)$, we deduce

$$\sum_{k=1}^{S}\int_{\mathbb{R}^3_+}\mathbf{H}^{\mathrm{T}}\theta\varrho(\theta, k)\mathrm{d}\theta = \sum_{k=1}^{S}\int_{\mathbb{R}^3_+}\mathbf{H}^{\mathrm{T}}\theta\nu(\mathrm{d}\theta, k). \tag{79}$$

In view of (75), (77), (78) and (79), we deduce $Y(\mathbf{H}) = \frac{Y^*(\mathbf{H})}{|\mathbf{\Delta}|}$. Solving $\frac{\mathrm{d}Y^*(\mathbf{H})}{\mathrm{d}\mathbf{H}} = \mathbf{0}$ yields $\mathbf{H}^* = |\mathbf{Q}|^{-1}\left(|\mathbf{Q_1}|, |\mathbf{Q_2}|, |\mathbf{Q_3}|\right)^{\mathrm{T}}$. Compute the Hessian matrix $\mathbf{\Lambda}$ of $Y^*(\mathbf{H})$ is $-\mathbf{Q}$. $-2\left|\overline{\mathbf{A}}\right| < 0$ and $4A_{11}\left|\overline{\mathbf{A}}\right| > A_{33}\left|A_{12} - A_{21}\right|^2$ implies that $\mathbf{\Lambda}$ is negative definite. Hence, $Y^*(\mathbf{H})$ has a unique maximum, and the unique maximum value point of $Y^*(\mathbf{H})$ is $\mathbf{H}^*$.

Proof of (ii). Thanks to Theorem 3.5, under condition (𝔞), the OHS does not exist. Finally, let us show that if the following condition holds, then the OHS does not exist (i.e. prove (𝔠) and (𝔡)):

$$\begin{cases} |\mathbf{\Delta_3}|(\mathbf{H}^*)\big|_{\mathbf{H}^*\in\overline{\mathbb{R}^3_+}} > 0, \\ 4A_{11}\left|\overline{\mathbf{A}}\right| < A_{33}\left|A_{12} - A_{21}\right|^2 \ \ or \ \ |\mathbf{Q}| < 0. \end{cases} \tag{80}$$

Clearly, $\mathbf{\Lambda}$ is not positive semidefinite. (80) implies that $\mathbf{\Lambda}$ is not negative semidefinite. Hence, $\mathbf{\Lambda}$ is indefinite. Thus, $Y^*(\mathbf{H})$ does not exist extreme point. So, the OHS does not exist. □

Remark 5.2. *If* $\mathbb{S} = \{1\}$, $r_{ii} = \gamma_i(\mu, 1) = 0$, $a_{ij} = 0$ $(i \neq j)$ *and* $\mu_{ii}(\theta) = C_{ii}$*, then system (6) becomes the following system discussed in [7]:*

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1 - h_1 - a_{11}x_1(t) - \int_{-\tau_{12}}^{0} x_2(t+\theta)\mathrm{d}\mu_{12}(\theta)\right]\mathrm{d}t + \sigma_1 x_1(t)\mathrm{d}W_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[-r_2 - h_2 + \int_{-\tau_{21}}^{0} x_1(t+\theta)\mathrm{d}\mu_{21}(\theta) - a_{22}x_2(t) - \int_{-\tau_{23}}^{0} x_3(t+\theta)\mathrm{d}\mu_{23}(\theta)\right]\mathrm{d}t + \sigma_2 x_2(t)\mathrm{d}W_2(t), \\ \mathrm{d}x_3(t) = x_3(t)\left[-r_3 - h_3 + \int_{-\tau_{32}}^{0} x_2(t+\theta)\mathrm{d}\mu_{32}(\theta) - a_{33}x_3(t)\right]\mathrm{d}t + \sigma_3 x_3(t)\mathrm{d}W_3(t). \end{cases}$$

Hence, Theorem 5.1 contains Theorem 4 in [7] as a special case.

Remark 5.3. *If* $\mathbb{S} = \{1\}$, $r_{ii} = \gamma_i(\mu, 1) = 0$ *and* $\mu_{ij}(\theta) = C_{ij}$*, then system (6) becomes*

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1 - h_1 - a_{11}x_1(t) - a_{12}x_2(t)\right]\mathrm{d}t + \sigma_1 x_1(t)\mathrm{d}W_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[-r_2 - h_2 + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t)\right]\mathrm{d}t + \sigma_2 x_2(t)\mathrm{d}W_2(t), \\ \mathrm{d}x_3(t) = x_3(t)\left[-r_3 - h_3 + a_{32}x_2(t) - a_{33}x_3(t)\right]\mathrm{d}t + \sigma_3 x_3(t)\mathrm{d}W_3(t). \end{cases}$$

Therefore, Theorem 5.1 contains Theorem 2 in [3] as a special case.

6. Numerical simulation

In this section we introduce some examples and figures to illustrate our main results. For simplicity, we suppose that $\mathbb{S} = \{1, 2\}$. Then system (6) is a hybrid system of the following two subsystems:

$$\begin{cases} \left.\begin{array}{l} \mathrm{d}x_1(t) = x_1(t)\left[(r_1(1) - h_1 - r_{11}C_{10}(t) - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t))\,\mathrm{d}t + \mathcal{S}_1(t,1)\right], \\ \mathrm{d}x_2(t) = x_2(t)\left[(-r_2(1) - h_2 - r_{22}C_{20}(t) + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_2(t,1)\right], \\ \mathrm{d}x_3(t) = x_3(t)\left[(-r_3(1) - h_3 - r_{33}C_{30}(t) + \mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_3(t,1)\right], \\ \mathrm{d}C_{i0}(t) = \left[0.1C_e(t) - (0.1 + 0.1)\,C_{i0}(t)\right]\mathrm{d}t, \\ \mathrm{d}C_e(t) = -0.5C_e(t)\mathrm{d}t, \end{array}\right\} t \neq 12n, \\ \Delta x_i(t) = 0,\ \Delta C_{i0}(t) = 0,\ \Delta C_e(t) = 0.6,\ t = 12n,\ n \in \mathbb{N}_+\ (i = 1, 2, 3), \end{cases} \tag{81}$$

and

$$\begin{cases} \left.\begin{array}{l} \mathrm{d}x_1(t) = x_1(t)\left[(r_1(2) - h_1 - r_{11}C_{10}(t) - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t))\,\mathrm{d}t + \mathcal{S}_1(t,2)\right], \\ \mathrm{d}x_2(t) = x_2(t)\left[(-r_2(2) - h_2 - r_{22}C_{20}(t) + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_2(t,2)\right], \\ \mathrm{d}x_3(t) = x_3(t)\left[(-r_3(2) - h_3 - r_{33}C_{30}(t) + \mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_3(t,2)\right], \\ \mathrm{d}C_{i0}(t) = \left[0.1C_e(t) - (0.1 + 0.1)\,C_{i0}(t)\right]\mathrm{d}t, \\ \mathrm{d}C_e(t) = -0.5C_e(t)\mathrm{d}t, \end{array}\right\} t \neq 12n, \\ \Delta x_i(t) = 0,\ \Delta C_{i0}(t) = 0,\ \Delta C_e(t) = 0.6,\ t = 12n,\ n \in \mathbb{N}_+\ (i = 1, 2, 3). \end{cases} \tag{82}$$

Let $\tau_{ji} = \ln 2$, $\mu_{ji}(\theta) = \mu_{ji}\mathrm{e}^{\theta}$, $\gamma_j(\mu, i) = \gamma_j(i)$ and $\lambda(\mathbb{Z}) = 1$. In the following examples, we choose the initial conditions $x_1(\theta) = 2\mathrm{e}^{\theta}$, $x_2(\theta) = 1.2\mathrm{e}^{\theta}$, $x_3(\theta) = 0.5\mathrm{e}^{\theta}$, $\theta \in [-\ln 2, 0]$.

Denote

$$\mathrm{Param(i)} = \begin{pmatrix} r_{11} & a_{11} & a_{12} & 0 & \mu_{11} & \mu_{12} & 0 & \sigma_1(i) & \gamma_1(i) \\ r_{22} & a_{21} & a_{22} & a_{23} & \mu_{21} & \mu_{22} & \mu_{23} & \sigma_2(i) & \gamma_2(i) \\ r_{33} & 0 & a_{32} & a_{33} & 0 & \mu_{32} & \mu_{33} & \sigma_3(i) & \gamma_3(i) \end{pmatrix}.$$

Then system (6) may be regarded as the result of regime switching between subsystems (81) and (82) with the following parameters, respectively,

$$\text{Param}(1) = \begin{pmatrix} 0.4 & 0.3 & 0.1 & 0 & 0.1 & 0.1 & 0 & 0.1 & 0.1 \\ 0.4 & 0.5 & 0.4 & 0.1 & 0.2 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.4 & 0 & 0.4 & 0.5 & 0 & 0.1 & 0.2 & 0.1 & 0.1 \end{pmatrix},$$

$$\text{Param}(2) = \begin{pmatrix} 0.4 & 0.3 & 0.1 & 0 & 0.1 & 0.1 & 0 & 1.2 & 0.2 \\ 0.4 & 0.5 & 0.4 & 0.1 & 0.2 & 0.1 & 0.1 & 0.2 & 0.2 \\ 0.4 & 0 & 0.4 & 0.5 & 0 & 0.1 & 0.2 & 0.2 & 0.2 \end{pmatrix}.$$

Compute $|\mathbf{\Delta}| = 0.172125$ and $|\mathbf{A}| = 0.2475$.

6.1. example 1

Consider the effects of telephone noise on the persistence in mean and extinction of system (6). Let $h_1 = h_2 = h_3 = 0$. In regime 1, we choose $r_1(1) = 0.9$, $r_2(1) = 0.5$, $r_3(1) = 0.3$. Compute

$$|\mathbf{\Delta_1}| = 0.30825 + 0.27\ln 1.1,\ \ |\mathbf{\Delta_2}| = 0.1700625 + 0.5175\ln 1.1,\ \ |\mathbf{\Delta_3}| = 0.005625 + 0.675\ln 1.1 > 0.$$

Based on Theorem 3.5, all species in subsystem (81) are persistent in mean and

$$\begin{cases} \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|} = 1.9404 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|} = 1.2746 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_3(s)\mathrm{d}s = \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} = 0.4064 \quad a.s. \end{cases} \tag{83}$$

In regime 2, we choose $r_1(2) = 0.5$, $r_2(2) = 0.3$, $r_3(2) = 0.2$. Calculate $B_1 = -0.44 + \ln 1.2 < 0$. Thanks to Theorem 3.5, all species in subsystem (82) are extinctive.

Case 1. $(\pi_1, \pi_2) = (0.9, 0.1)$. We gain

$$|\mathbf{\Delta_1}| = 0.266445 + 0.243\ln 1.1 + 0.027\ln 1.2,\ \ |\mathbf{\Delta_2}| = 0.12818625 + 0.46575\ln 1.1 + 0.05175\ln 1.2,$$
$$|\mathbf{\Delta_3}| = -0.0262125 + 0.6075\ln 1.1 + 0.0675\ln 1.2 > 0.$$

On the basis of Theorem 3.5, all species in system (6) are persistent in mean and

$$\begin{cases} \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|} = 1.7111 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|} = 1.0574 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_3(s)\mathrm{d}s = \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} = 0.2556 \quad a.s. \end{cases} \tag{84}$$

Case 2. $(\pi_1, \pi_2) = (0.5, 0.5)$. We have

$$|\mathbf{\Delta_3}| = -0.1535625 + 0.3375\ln 1.1 + 0.3375\ln 1.2 < 0,$$
$$|\mathbf{A_1}| = 0.16275 + 0.15\ln 1.1 + 0.15\ln 1.2,\ \ |\mathbf{A_2}| = -0.103375 + 0.475\ln 1.1 + 0.475\ln 1.2 > 0.$$

In view of Theorem 3.5, $x_1(t)$ and $x_2(t)$ are persistent in mean, while $x_3(t)$ is extinctive and

$$\begin{cases} \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{A_1}|}{|\mathbf{A}|} = 0.8258 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{A_2}|}{|\mathbf{A}|} = 0.1152 \quad a.s. \end{cases} \tag{85}$$

Case 3. $(\pi_1, \pi_2) = (0.4, 0.6)$. We deduce

$$|\mathbf{A_2}| = -0.1733 + 0.38 \ln 1.1 + 0.57 \ln 1.2 < 0, \; B_1 = 0.046 + 0.4 \ln 1.1 + 0.6 \ln 1.2 > 0.$$

From Theorem 3.5, $x_1(t)$ is persistent in mean, while $x_2(t)$ and $x_3(t)$ are extinctive and

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{B_1}{A_{11}} = 0.5529 \quad a.s. \tag{86}$$

Case 4. $(\pi_1, \pi_2) = (0.2, 0.8)$. We get $B_1 = -0.197 + 0.2 \ln 1.1 + 0.8 \ln 1.2 < 0$. By Theorem 3.5, all species in system (6) are extinctive.

6.2. example 2

Consider the influences of telephone noise on the existence of the OHS. Compute

$$A_{22}A_{33}\,|\mathbf{A}| - A_{12}A_{21}A_{23}A_{32} = 0.0608 > 0, \; |\mathbf{Q}| = 0.07746 > 0, \; 4A_{11}\left|\overline{\mathbf{A}}\right| - A_{33}\,|A_{12} - A_{21}|^2 = 0.3510 > 0.$$

In regime 1, we choose $r_1(1) = 1$, $r_2(1) = 0.2$, $r_3(1) = 0.1$. We gain

$$h_1^* = 0.2320 > 0, \; h_2^* = 0.5495 > 0, \; h_3^* = 0.1377 > 0, \; |\mathbf{\Delta_3}|\,(\mathbf{H}^*)\,|_{\mathbf{H}^*\in\overline{\mathbb{R}^3_+}} = 0.0104 > 0.$$

According to Theorem 5.1 (i), the OHE in subsystem (81) is

$$\mathbf{H}^* = (0.2320, 0.5495, 0.1377)^{\mathrm{T}} \tag{87}$$

and

$$MESY = \frac{Y^*(\mathbf{H}^*)}{|\mathbf{\Delta}|} = 0.8016. \tag{88}$$

In regime 2, we choose $r_1(2) = 1.1$, $r_2(2) = 0.2$, $r_3(2) = 0.2$, $\sigma_1(2) = 0.4$ and the values of other parameters are the same with those in subsystem (82). We have

$$h_1^* = 0.2746 > 0, \; h_2^* = 0.5577 > 0, \; h_3^* = 0.0444 > 0, \; |\mathbf{\Delta_3}|\,(\mathbf{H}^*)\,|_{\mathbf{H}^*\in\overline{\mathbb{R}^3_+}} = -0.0121 < 0.$$

Based on Theorem 5.1 (ii) , the OHS in subsystem (82) does not exist.

Case 1. $(\pi_1, \pi_2) = (0.9, 0.1)$. Then

$$h_1^* = 0.2363 > 0, \; h_2^* = 0.5503 > 0, \; h_3^* = 0.1284 > 0, \; |\mathbf{\Delta_3}|\,(\mathbf{H}^*)\,|_{\mathbf{H}^*\in\overline{\mathbb{R}^3_+}} = 0.0082 > 0.$$

In view of Theorem 5.1 (i) , the OHE in system (6) is

$$\mathbf{H}^* = (0.2363, 0.5503, 0.1284)^{\mathrm{T}} \tag{89}$$

and

$$MESY = \frac{Y^*(\mathbf{H}^*)}{|\mathbf{\Delta}|} = 0.8012. \tag{90}$$

Case 2. $(\pi_1, \pi_2) = (0.5, 0.5)$. Then

$$h_1^* = 0.2533 > 0, \; h_2^* = 0.5536 > 0, \; h_3^* = 0.0911 > 0, \; |\mathbf{\Delta_3}|\,(\mathbf{H}^*)\,|_{\mathbf{H}^*\in\overline{\mathbb{R}^3_+}} = -0.00083 < 0.$$

By Theorem 5.1 (ii), the OHS in system (6) does not exist.

6.3. example 3

Consider the effects of Lévy jumps on the stochastic dynamics of population systems. Let $r_1(1) = 0.8$, $r_2(1) = 0.5$, $r_3(1) = 0.3$. We study the effects of Lévy jumps on the persistence in mean and extinction of the species by changing the values of $\gamma_j(1)$ and the remaining parameters of the examples are the same with those in system (81).

6.3.1. The effects of $\gamma_j(1)$ on the persistence in mean and extinction of system (81)

Case 1. Let $\gamma_1(1) = \gamma_2(1) = 0.1$, $\gamma_{\mathbf{3}}(\mathbf{1}) = \mathbf{-0.9}$. We derive

$$|\mathbf{\Delta_3}| = 0.226165 + 0.2475 \ln 0.1 + 0.4275 \ln 1.1 < 0,$$
$$|\mathbf{A_1}| = 0.3975 + 0.3 \ln 1.1, \;\; |\mathbf{A_2}| = 0.18625 + 0.95 \ln 1.1 > 0.$$

By Theorem 3.5, $x_1(t)$ and $x_2(t)$ are persistent in mean, while $x_3(t)$ is extinctive (see Figure 1(a)) and

$$\begin{cases} \lim\limits_{t\to+\infty} t^{-1} \displaystyle\int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{A_1}|}{|\mathbf{A}|} = 1.7216 \;\; a.s. \\ \lim\limits_{t\to+\infty} t^{-1} \displaystyle\int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{A_2}|}{|\mathbf{A}|} = 1.1184 \;\; a.s. \end{cases} \tag{91}$$

Let $\gamma_1(1) = \gamma_2(1) = 0.1$, $\gamma_{\mathbf{3}}(\mathbf{1}) = \mathbf{0.2}$. We calculate

$$|\mathbf{\Delta_1}| = 0.27225 + 0.2475 \ln 1.1 + 0.0225 \ln 1.2, \;\; |\mathbf{\Delta_2}| = 0.1393125 + 0.57 \ln 1.1 - 0.0525 \ln 1.2,$$
$$|\mathbf{\Delta_3}| = -0.046125 + 0.4275 \ln 1.1 + 0.2475 \ln 1.2 > 0.$$

According to Theorem 3.5, all species in system (81) are persistent in mean (see Figure 1(b)) and

$$\begin{cases} \lim\limits_{t\to+\infty} t^{-1} \displaystyle\int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|} = 1.7426 \;\; a.s. \\ \lim\limits_{t\to+\infty} t^{-1} \displaystyle\int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|} = 1.0694 \;\; a.s. \\ \lim\limits_{t\to+\infty} t^{-1} \displaystyle\int_0^t x_3(s)\mathrm{d}s = \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} = 0.2309 \;\; a.s. \end{cases} \tag{92}$$

Let $\gamma_1(1) = \gamma_2(1) = 0.1$, $\gamma_{\mathbf{3}}(\mathbf{1}) = \mathbf{1.2}$. We deduce

$$|\mathbf{\Delta_3}| = -0.293625 + 0.4275 \ln 1.1 + 0.2475 \ln 2.2 < 0, \;\; |\mathbf{A_2}| = 0.18625 + 0.95 \ln 1.1 > 0.$$

On the basis of Theorem 3.5, $x_1(t)$ and $x_2(t)$ are persistent in mean, while $x_3(t)$ is extinctive and (91) holds (see Figure 1(c)).

Table 4: Changes of $\gamma_3(1)$ when $\gamma_1(1) = \gamma_2(1) = 0.1$ in system (81)

$\gamma_1(1)$	$\gamma_2(1)$	$\gamma_3(1)$	$\overline{\mathbf{x}^{\mathrm{T}}(\infty)}$	Figure
0.1	0.1	**−0.9**	(1.7216, 1.1184, 0)	1(*a*)
0.1	0.1	**0.2**	(1.7426, 1.0694, 0.2309)	1(*b*)
0.1	0.1	**1.2**	(1.7216, 1.1184, 0)	1(*c*)

By comparing, as $\gamma_3(1)$ increases (see Table 4), $x_3(t)$ goes from extinction to persistence in mean and then extinction again, while $x_1(t)$ and $x_2(t)$ remain persistence.

Case 2. Let $\gamma_1(1) = 0.1$, $\gamma_{\mathbf{2}}(\mathbf{1}) = \mathbf{-0.8}$, $\gamma_3(1) = -0.9$. Then

$$|\mathbf{A_2}| = 0.50125 + 0.6\ln 1.1 + 0.35\ln 0.2 < 0, \ B_1 = 0.675 + \ln 1.1 > 0.$$

From Theorem 3.5, $x_1(t)$ is persistent in mean, while $x_2(t)$ and $x_3(t)$ are extinctive (see Figure 1(d)) and

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{B_1}{A_{11}} = 2.2009 \quad a.s. \tag{93}$$

Let $\gamma_1(1) = 0.1$, $\gamma_{\mathbf{2}}(\mathbf{1}) = \mathbf{1.9}$, $\gamma_3(1) = -0.9$. Then

$$|\mathbf{A_2}| = -0.44375 + 0.6\ln 1.1 + 0.35\ln 2.9 < 0, \ B_1 = 0.675 + \ln 1.1 > 0.$$

According to Theorem 3.5, $x_1(t)$ is persistent in mean, while $x_2(t)$ and $x_3(t)$ are extinctive and (93) holds (see Figure 1(e)) .

Table 5: Changes of $\gamma_2(1)$ when $\gamma_1(1) = 0.1$ and $\gamma_3(1) = -0.9$ in system (81)

$\gamma_1(1)$	$\gamma_2(1)$	$\gamma_3(1)$	$\overline{\mathbf{x}^{\mathrm{T}}(\infty)}$	Figure
0.1	**−0.8**	−0.9	(2.2009, 0, 0)	1(*d*)
0.1	**0.1**	−0.9	(1.7216, 1.1184, 0)	1(*a*)
0.1	**1.9**	−0.9	(2.2009, 0, 0)	1(*e*)

Let $\gamma_1(1) = 0.1$, $\gamma_{\mathbf{2}}(\mathbf{1}) = \mathbf{-0.8}$, $\gamma_3(1) = 1.2$. Compute

$$|\mathbf{A_2}| = 0.50125 + 0.6\ln 1.1 + 0.35\ln 0.2 < 0, \ B_1 = 0.675 + \ln 1.1 > 0.$$

By Theorem 3.5, $x_1(t)$ is persistent in mean, while $x_2(t)$ and $x_3(t)$ are extinctive and (93) holds (see Figure 1(f)).

Let $\gamma_1(1) = 0.1$, $\gamma_{\mathbf{2}}(\mathbf{1}) = \mathbf{1.9}$, $\gamma_3(1) = 1.2$. Compute

$$|\mathbf{A_2}| = -0.44375 + 0.6\ln 1.1 + 0.35\ln 2.9 < 0, \ B_1 = 0.675 + \ln 1.1 > 0.$$

On the basis of Theorem 3.5, $x_1(t)$ is persistent in mean, while $x_2(t)$ and $x_3(t)$ are extinctive and (93) holds (see Figure 1(g)).

Table 6: Changes of $\gamma_2(1)$ when $\gamma_1(1) = 0.1$ and $\gamma_3(1) = 1.2$ in system (81)

$\gamma_1(1)$	$\gamma_2(1)$	$\gamma_3(1)$	$\overline{\mathbf{x}^{\mathrm{T}}(\infty)}$	Figure
0.1	**−0.8**	1.2	(2.2009, 0, 0)	1(*f*)
0.1	**0.1**	1.2	(1.7216, 1.1184, 0)	1(*c*)
0.1	**1.9**	1.2	(2.2009, 0, 0)	1(*g*)

By contrast, for fixed $\gamma_3(1) = -0.9$ or $\gamma_3(1) = 1.2$, $x_3(t)$ is extinctive. With the increasing of $\gamma_2(1)$ (see Tables 5 and 6), $x_2(t)$ goes from extinction to persistence in mean and then extinction again, while $x_1(t)$ remains persistence.

Case 3. Let $\gamma_{\mathbf{1}}(\mathbf{1}) = \mathbf{-0.9}$, $\gamma_2(1) = -0.8$, $\gamma_3(1) = -0.9$. Calculate $B_1 = 1.675 + \ln 0.1 < 0$. Based on Theorem 3.5, all species are extinctive (see Figure 2(a)).

Let $\gamma_{\mathbf{1}}(\mathbf{1}) = \mathbf{1.9}$, $\gamma_2(1) = -0.8$, $\gamma_3(1) = -0.9$. Calculate $B_1 = -1.125 + \ln 2.9 < 0$. On the basis of Theorem 3.5, all species are extinctive (see Figure 2(b)).

Table 7: Changes of $\gamma_1(1)$ when $\gamma_2(1) = -0.8$ and $\gamma_3(1) = -0.9$ in system (81)

$\gamma_1(1)$	$\gamma_2(1)$	$\gamma_3(1)$	$\overline{\mathbf{x}^T(\infty)}$	Figure
−0.9	−0.8	−0.9	(0, 0, 0)	2(a)
0.1	−0.8	−0.9	(2.2009, 0, 0)	1(d)
1.9	−0.8	−0.9	(0, 0, 0)	2(b)

Table 8: Changes of $\gamma_1(1)$ when $\gamma_2(1) = 1.9$ and $\gamma_3(1) = -0.9$ in system (81)

$\gamma_1(1)$	$\gamma_2(1)$	$\gamma_3(1)$	$\overline{\mathbf{x}^T(\infty)}$	Figure
−0.9	1.9	−0.9	(0, 0, 0)	2(c)
0.1	1.9	−0.9	(2.2009, 0, 0)	1(e)
1.9	1.9	−0.9	(0, 0, 0)	2(d)

Let $\boldsymbol{\gamma_1(1) = -0.9}$, $\gamma_2(1) = 1.9$, $\gamma_3(1) = -0.9$. We derive $B_1 = 1.675 + \ln 0.1 < 0$. In view of Theorem 3.5, all species are extinctive (see Figure 2(c)).

Let $\boldsymbol{\gamma_1(1) = 1.9}$, $\gamma_2(1) = 1.9$, $\gamma_3(1) = -0.9$. We derive $B_1 = -1.125 + \ln 2.9 < 0$. By Theorem 3.5, all species are extinctive (see Figure 2(d)).

Let $\boldsymbol{\gamma_1(1) = -0.9}$, $\gamma_2(1) = -0.8$, $\gamma_3(1) = 1.2$. We deduce $B_1 = 1.675 + \ln 0.1 < 0$. Based on Theorem 3.5, all species are extinctive (see Figure 2(e)).

Let $\boldsymbol{\gamma_1(1) = 1.9}$, $\gamma_2(1) = -0.8$, $\gamma_3(1) = 1.2$. We deduce $B_1 = -1.125 + \ln 2.9 < 0$. In view of Theorem 3.5, all species are extinctive (see Figure 2(f)).

Table 9: Changes of $\gamma_1(1)$ when $\gamma_2(1) = -0.8$ and $\gamma_3(1) = 1.2$ in system (81)

$\gamma_1(1)$	$\gamma_2(1)$	$\gamma_3(1)$	$\overline{\mathbf{x}^T(\infty)}$	Figure
−0.9	−0.8	1.2	(0, 0, 0)	2(e)
0.1	−0.8	1.2	(2.2009, 0, 0)	1(f)
1.9	−0.8	1.2	(0, 0, 0)	2(f)

Let $\boldsymbol{\gamma_1(1) = -0.9}$, $\gamma_2(1) = 1.9$, $\gamma_3(1) = 1.2$. We have $B_1 = 1.675 + \ln 0.1 < 0$. From Theorem 3.5, all species are extinctive (see Figure 2(g)).

Let $\boldsymbol{\gamma_1(1) = 1.9}$, $\gamma_2(1) = 1.9$, $\gamma_3(1) = 1.2$. We have $B_1 = -1.125 + \ln 2.9 < 0$. On the basis of Theorem 3.5, all species are extinctive (see Figure 2(h)).

Table 10: Changes of $\gamma_1(1)$ when $\gamma_2(1) = 1.9$ and $\gamma_3(1) = 1.2$ in system (81)

$\gamma_1(1)$	$\gamma_2(1)$	$\gamma_3(1)$	$\overline{\mathbf{x}^T(\infty)}$	Figure
−0.9	1.9	1.2	(0, 0, 0)	2(g)
0.1	1.9	1.2	(2.2009, 0, 0)	1(g)
1.9	1.9	1.2	(0, 0, 0)	2(h)

Through comparison, when $x_2(t)$ and $x_3(t)$ are extinctive, as $\gamma_1(1)$ increases, $x_1(t)$ goes from extinction to persistence in mean and then extinction again (see Tables 7, 8, 9 and 10).

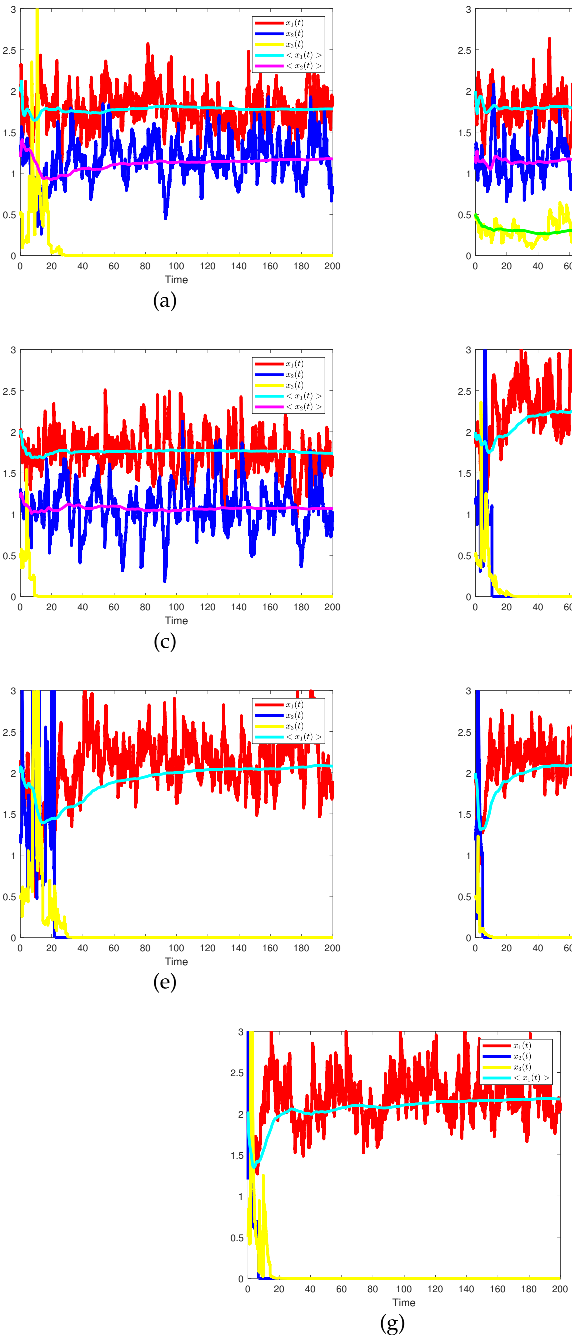

Figure 1: The sample paths of system (81) for **Case 1** and **Case 2** in 6.3.1. For the parameters of each subfigure, see Tables 4, 5 and 6. (a) and (c) represent that both $x_1(t)$ and $x_2(t)$ are persistent in mean, while $x_3(t)$ is extinctive; (b) represents that all species are persistent in mean; (d), (e), (f) and (g) represent that in **Case 2**, $x_1(t)$ is persistent in mean, while $x_2(t)$ and $x_3(t)$ are extinctive.

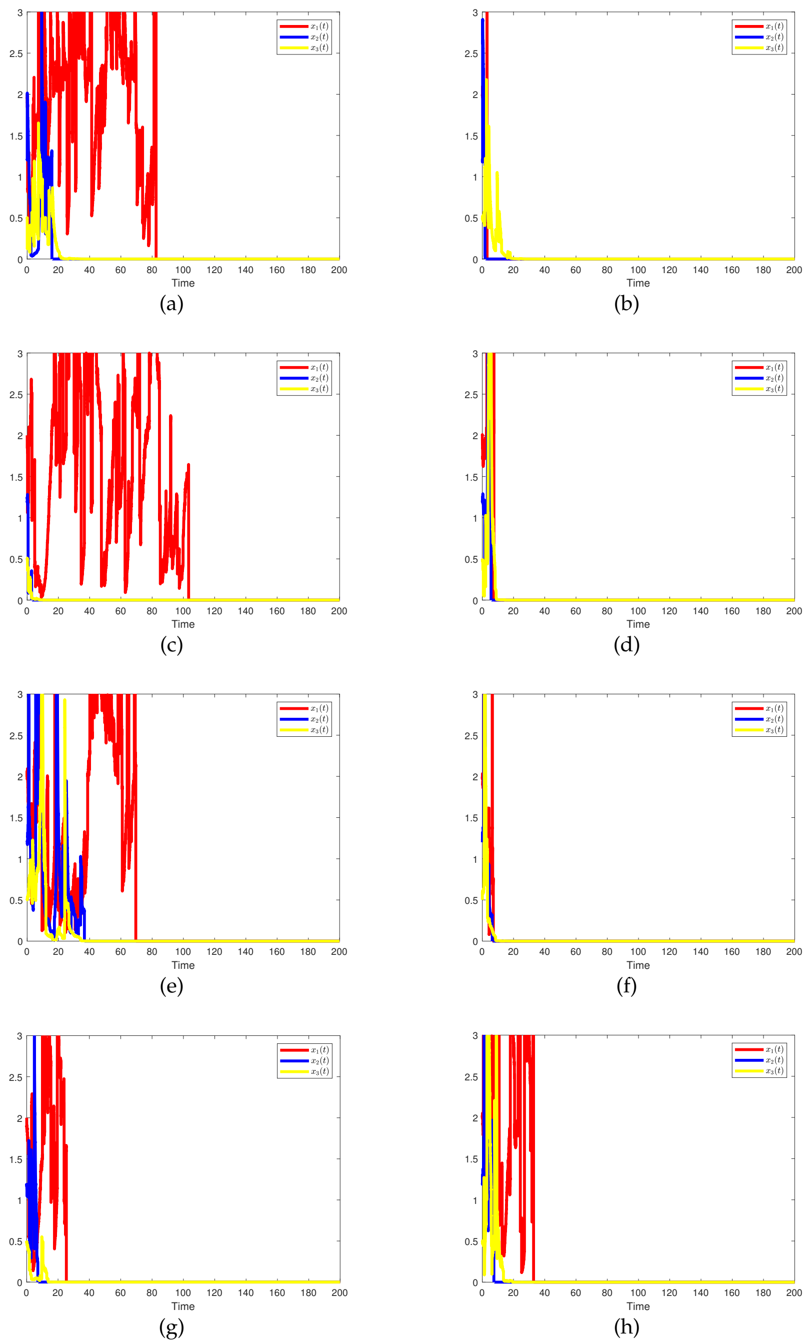

Figure 2: The sample paths of system (81) for **Case 3** in 6.3.1. For the parameters of each subfigure, see Tables 7, 8, 9 and 10. These subfigures represent that all species in **Case 3** are extinctive.

6.3.2. The effects of $\gamma_1(1)$ on the persistence in mean and extinction of system (81)

Case 1. Let $\gamma_1(1) = -0.8$. Compute $B_1 = 1.575 + \ln 0.2 < 0$. Based on Theorem 3.5, all species in system (81) are extinctive.

Let $\gamma_1(1) = -0.7$. We derive

$$|\mathbf{A_2}| = 0.66625 + 0.35 \ln 1.1 + 0.6 \ln 0.3 < 0, \ B_1 = 1.475 + \ln 0.3 > 0.$$

By Theorem 3.5, $x_1(t)$ is persistent in mean, while $x_2(t)$ and $x_3(t)$ are extinctive and

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{B_1}{A_{11}} = 0.7744 \quad a.s. \tag{94}$$

Let $\gamma_1(1) = -0.5$. We calculate

$$|\mathbf{\Delta_3}| = 0.140625 + 0.405 \ln 1.1 + 0.27 \ln 0.5 < 0,$$
$$|\mathbf{A_1}| = 0.6675 - 0.15 \ln 1.1 + 0.45 \ln 0.5, \ |\mathbf{A_2}| = 0.54625 + 0.35 \ln 1.1 + 0.6 \ln 0.5 > 0.$$

From Theorem 3.5, $x_1(t)$ and $x_2(t)$ are persistent in mean, while $x_3(t)$ is extinctive and

$$\begin{cases} \lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{A_1}|}{|\mathbf{A}|} = 1.3789 \quad a.s. \\ \lim_{t\to+\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{A_2}|}{|\mathbf{A}|} = 0.6615 \quad a.s. \end{cases} \tag{95}$$

Let $\gamma_1(1) = -0.2$. We deduce

$$|\mathbf{\Delta_1}| = 0.37575 - 0.0675 \ln 1.1 + 0.3375 \ln 0.8, \ |\mathbf{\Delta_2}| = 0.2420625 + 0.1575 \ln 1.1 + 0.36 \ln 0.8,$$
$$|\mathbf{\Delta_3}| = 0.059625 + 0.405 \ln 1.1 + 0.27 \ln 0.8 > 0.$$

On the basis of Theorem 3.5, all species in system (81) are persistent in mean and

$$\begin{cases} \lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|} = 1.7081 \quad a.s. \\ \lim_{t\to+\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|} = 1.0268 \quad a.s. \\ \lim_{t\to+\infty} t^{-1} \int_0^t x_3(s)\mathrm{d}s = \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} = 0.2206 \quad a.s. \end{cases} \tag{96}$$

Case 2. Let $\gamma_1(1) = 0.3$. We have

$$|\mathbf{\Delta_1}| = 0.207 - 0.0675 \ln 1.1 + 0.3375 \ln 1.3, \ |\mathbf{\Delta_2}| = 0.0620625 + 0.1575 \ln 1.1 + 0.36 \ln 1.3,$$
$$|\mathbf{\Delta_3}| = -0.075375 + 0.405 \ln 1.1 + 0.27 \ln 1.3 > 0.$$

In view of Theorem 3.5, all species in system (81) are persistent in mean and

$$\begin{cases} \lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|} = 1.6797 \quad a.s. \\ \lim_{t\to+\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|} = 0.9965 \quad a.s. \\ \lim_{t\to+\infty} t^{-1} \int_0^t x_3(s)\mathrm{d}s = \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} = 0.1979 \quad a.s. \end{cases} \tag{97}$$

Let $\gamma_1(1) = 0.7$. We compute

$$|\mathbf{\Delta_3}| = -0.183375 + 0.405 \ln 1.1 + 0.27 \ln 1.7 < 0,$$

$$|\mathbf{A_1}| = 0.1275 - 0.15 \ln 1.1 + 0.45 \ln 1.7, \ |\mathbf{A_2}| = -0.17375 + 0.35 \ln 1.1 + 0.6 \ln 1.7 > 0.$$

Based on Theorem 3.5, $x_1(t)$ and $x_2(t)$ are persistent in mean, while $x_3(t)$ is extinctive and

$$\begin{cases} \lim\limits_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \dfrac{|\mathbf{A_1}|}{|\mathbf{A}|} = 1.4222 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s = \dfrac{|\mathbf{A_2}|}{|\mathbf{A}|} = 0.7191 \quad a.s. \end{cases} \tag{98}$$

Let $\gamma_1(1) = 1.3$. We gain

$$|\mathbf{A_2}| = -0.53375 + 0.35 \ln 1.1 + 0.6 \ln 2.3 < 0, \ B_1 = -0.525 + \ln 2.3 > 0.$$

According to Theorem 3.5, $x_1(t)$ is persistent in mean, while $x_2(t)$ and $x_3(t)$ are extinctive and

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{B_1}{A_{11}} = 0.8797 \quad a.s. \tag{99}$$

Let $\gamma_1(1) = 1.9$. We get $B_1 = -1.125 + \ln 2.9 < 0$. By Theorem 3.5, all species in system (81) are extinctive.

6.4. example 4

Consider the influences of Lévy jumps on the existence of the OHS. Let $r_1(1) = 0.9$, $r_2(1) = 0.1$ and $r_3(1) = 0.1$. By changing the values of $\gamma_j(1)$ to see the effects of Lévy jumps on the existence of the OHS and the rest parameters of the examples are the same with those in system (81).

6.4.1. The influence of $\gamma_3(1)$ on the existence of the OHS

Case 1. Let $\gamma_3(1) = -0.4$. Then

$$h_1^* = 0.1756 > 0, \ h_2^* = 0.5753 > 0, \ h_3^* = 0.0898 > 0, \ |\mathbf{\Delta_3}|(\mathbf{H}^*)\big|_{\mathbf{H}^*\in\overline{\mathbb{R}^3_+}} = -0.0040 < 0.$$

According to Theorem 5.1 (ii), the OHS in system (81) does not exist.

Let $\gamma_3(1) = -0.2$. Then

$$h_1^* = 0.1602 > 0, \ h_2^* = 0.5602 > 0, \ h_3^* = 0.1459 > 0, \ |\mathbf{\Delta_3}|(\mathbf{H}^*)\big|_{\mathbf{H}^*\in\overline{\mathbb{R}^3_+}} = 0.0103 > 0.$$

Based on Theorem 5.1 (i), the OHE in system (81) is

$$\mathbf{H}^* = (0.1602, 0.5602, 0.1459)^{\mathrm{T}} \tag{100}$$

and

$$MESY = \frac{Y^*(\mathbf{H}^*)}{|\mathbf{\Delta}|} = 0.6941. \tag{101}$$

Case 2. Let $\gamma_3(1) = 0.3$. Compute

$$h_1^* = 0.1628 > 0, \ h_2^* = 0.5627 > 0, \ h_3^* = 0.1366 > 0, \ |\mathbf{\Delta_3}|(\mathbf{H}^*)\big|_{\mathbf{H}^*\in\overline{\mathbb{R}^3_+}} = 0.0079 > 0.$$

In view of Theorem 5.1 (i), the OHE in system (81) is

$$\mathbf{H}^* = (0.1628, 0.5627, 0.1366)^{\mathrm{T}} \tag{102}$$

and

$$MESY = \frac{Y^*(\mathbf{H}^*)}{|\mathbf{\Delta}|} = 0.6934. \tag{103}$$

Let $\gamma_3(1) = 0.6$. Compute

$$h_1^* = 0.1789 > 0, \ h_2^* = 0.5786 > 0, \ h_3^* = 0.0775 > 0, \ |\mathbf{\Delta_3}|(\mathbf{H}^*)\big|_{\mathbf{H}^*\in\overline{\mathbb{R}^3_+}} = -0.0072 < 0.$$

By Theorem 5.1 (ii), the OHS in system (81) does not exist.

6.4.2. The influence of $\gamma_1(1)$ on the existence of the OHS

Case 1. Let $\gamma_1(1) = -0.6$. Calculate

$$h_1^* = 0.1414 > 0,\ h_2^* = 0.3280 > 0,\ h_3^* = 0.0455 > 0,\ |\mathbf{\Delta_3}|\,(\mathbf{H}^*)\,|_{\mathbf{H}^*\in\overline{\mathbb{R}_+^3}} = -0.0028 < 0.$$

According to Theorem 5.1 (ii), the OHS in system (81) does not exist.

Let $\gamma_1(1) = -0.3$. Calculate

$$h_1^* = 0.1544 > 0,\ h_2^* = 0.5188 > 0,\ h_3^* = 0.1390 > 0,\ |\mathbf{\Delta_3}|\,(\mathbf{H}^*)\,|_{\mathbf{H}^*\in\overline{\mathbb{R}_+^3}} = 0.0106 > 0.$$

Based on Theorem 5.1 (i), the OHE in system (81) is

$$\mathbf{H}^* = (0.1544, 0.5188, 0.1390)^{\mathrm{T}} \tag{104}$$

and

$$MESY = \frac{\Upsilon^*(\mathbf{H}^*)}{|\mathbf{\Delta}|} = 0.6090. \tag{105}$$

Case 2. Let $\gamma_1(1) = 0.4$. Then

$$h_1^* = 0.1541 > 0,\ h_2^* = 0.5138 > 0,\ h_3^* = 0.1365 > 0,\ |\mathbf{\Delta_3}|\,(\mathbf{H}^*)\,|_{\mathbf{H}^*\in\overline{\mathbb{R}_+^3}} = 0.0103 > 0.$$

In view of Theorem 5.1 (i), the OHE in system (81) is

$$\mathbf{H}^* = (0.1541, 0.5138, 0.1365)^{\mathrm{T}} \tag{106}$$

and

$$MESY = \frac{\Upsilon^*(\mathbf{H}^*)}{|\mathbf{\Delta}|} = 0.5980. \tag{107}$$

Let $\gamma_1(1) = 0.9$. Then

$$h_1^* = 0.1443 > 0,\ h_2^* = 0.3707 > 0,\ h_3^* = 0.0664 > 0,\ |\mathbf{\Delta_3}|\,(\mathbf{H}^*)\,|_{\mathbf{H}^*\in\overline{\mathbb{R}_+^3}} = -0.0002 < 0.$$

From Theorem 5.1 (ii), the OHS in system (81) does not exist.

7. Conclusion and Discussion

The interdisciplinary stochastic population dynamics which is of great theoretical and practical significance has been a hot topic of international research, but there are still many urgently problems to be solved, for example, what is the critical effects of environmental stochasticity on coexistence, extinction and optimal harvesting of populations? From the viewpoint of biology, the sudden catastrophic shocks may seriously affect the asymptotical behaviors of ecosystems. As is well known, white noises cannot describe sudden environmental perturbations. Therefore, in this paper, we use Lévy jumps to describe these sudden environmental perturbations. Besides, the fertility and mortality of the species are inevitably affected by temperature, humidity, nutrients, acidity and other factors. We utilize a continuous-time Markov chain $\rho(t)$ with finite-state space to model the telephone noises in the environment and propose a stochastic hybrid delay three-species food chain model with harvesting and jumps in an impulsive polluted environment. Our main results are Theorem 3.5 and Theorem 5.1. Theorem 3.5 establishes sufficient and necessary criteria for stochastic persistence in mean and extinction of each species in system (6). Theorem 5.1 provides sufficient and necessary conditions for the existence of optimal harvesting policy and gives the accurate expressions for the optimal harvesting effort and the maximum of expectation of sustainable yield. Our results reveal that both environmental noises and time delays have close relationship with the persistence, extinction and existence of optimal harvesting policy for system (6).

First of all, let us consider the effects of telephone noise on the stochastic dynamics of system (6). Many interesting and surprising results about influences of regime switching on the dynamical properties of system (6) are obtained. According to Subsection 6.1, we observe that all species in subsystem (81) are persistent in mean while all species in subsystem (82) are extinctive. For the stochastic hybrid system (6), as the result of regime switching, in **Case 1**, all species are persistent in mean; in **Case 2**, both $x_1(t)$ and $x_2(t)$ are persistent in mean, while $x_3(t)$ is extinctive; in **Case 3**, $x_1(t)$ is persistent in mean, while both $x_2(t)$ and $x_3(t)$ are extinctive; in **Case 4**, all species are extinctive. By comparing the above four cases, one can find that regime switching can change the properties of ecosystems significantly. More precisely, Theorem 3.5 gives a fascinating result that if some subsystems are persistent in mean while some are extinctive, as the result of regime switching, every species in the hybrid system may be persistent in mean or extinctive.

Moreover, regime switching can also affect the existence of OHS. There are two methods to study the optimal harvesting problem of stochastic population systems. One is to use the explicit solution of the system or to use the explicit solution of the corresponding Fokker-Planck equation ([43], [44]). Due to the fact that for most stochastic population systems, the corresponding Fokker-Planck equation can not be solved explicitly, in this paper, we use another approach based on the ergodic theory ([45]) to study the optimal harvesting problem of system (6). This method does not need to solve the Fokker-Planck equation. Based on Subsection 6.2, the OHS of subsystem (81) exists while the OHS of subsystem (82) does not exist. Thanks to regime switching, for the stochastic hybrid system (6), in **Case 1**, the OHS exists; in **Case 2**, the OHS does not exist. Through the above discussion, one can draw a conclusion that if the OHS of some subsystems are existent while some are nonexistent, thanks to regime switching, the OHS of the stochastic hybrid system may be existent or nonexistent.

Next, let us consider the effects of Lévy jumps on the stochastic dynamics of system (6). For simplicity, we only consider the influence of Lévy jumps on the persistence in mean and extinction of system (81). Theorem 3.5 reveals that the stochastic persistence in mean and extinction of top predator in system (81) depends only on the sign of $|\mathbf{\Delta_3}|$. Compute

$$\begin{aligned}|\mathbf{\Delta_3}| =&A_{21}A_{32}\left(r_1(1)-\frac{\sigma_1^2(1)}{2}-\gamma_1(1)+\ln\left(1+\gamma_1(1)\right)-h_1-r_{11}K_1\right)\\&-A_{11}A_{32}\left(r_2(1)+\frac{\sigma_2^2(1)}{2}+\gamma_2(1)-\ln\left(1+\gamma_2(1)\right)+h_2+r_{22}K_2\right)\\&-|\mathbf{A}|\left(r_3(1)+\frac{\sigma_3^2(1)}{2}+\gamma_3(1)-\ln\left(1+\gamma_3(1)\right)+h_3+r_{33}K_3\right).\end{aligned} \tag{108}$$

Then, $\frac{\partial|\mathbf{\Delta_3}|}{\partial\gamma_3(1)} = -|\mathbf{A}|\frac{\gamma_3(1)}{1+\gamma_3(1)}$ and $\lim_{\gamma_3(1)\to(-1)^+}|\mathbf{\Delta_3}| = \lim_{\gamma_3(1)\to+\infty}|\mathbf{\Delta_3}| = -\infty$. Assume that $|\mathbf{\Delta_3}||_{\gamma_3(1)=0} > 0$, if $-1 < \gamma_3(1) < 0$, with the increasing of $\gamma_3(1)$, the top predator goes from extinction to persistence in mean; if $\gamma_3(1) > 0$, with the increasing of $\gamma_3(1)$, the top predator goes from persistence in mean to extinction. Regardless of either case, $\gamma_3(1)$ has no influence on the persistence of prey and intermediate predator. More distinctly, let us see Table 4 in Subsection 6.3.1, by comparing Figure 1(a), 1(b) and 1(c) we find that with the increasing of $\gamma_3(1)$, the top predator goes from extinction to persistence in mean and then extinction again, while both prey and intermediate predator remain persistence in mean. In a real ecosystem where the top predator dies out, the intermediate predator can be persistent better. Theorem 3.5 shows that persistence in mean and extinction of intermediate predator in system (81) depends only on the symbol of $|\mathbf{A_2}|$ when top predator is extinctive. Compute

$$\begin{aligned}|\mathbf{A_2}| =&A_{21}\left(r_1(1)-\frac{\sigma_1^2(1)}{2}-\gamma_1(1)+\ln\left(1+\gamma_1(1)\right)-h_1-r_{11}K_1\right)\\&-A_{11}\left(r_2(1)+\frac{\sigma_2^2(1)}{2}+\gamma_2(1)-\ln\left(1+\gamma_2(1)\right)+h_2+r_{22}K_2\right).\end{aligned} \tag{109}$$

Then, $\frac{\partial|\mathbf{A_2}|}{\partial\gamma_2(1)} = -A_{11}\frac{\gamma_2(1)}{1+\gamma_2(1)}$ and $\lim_{\gamma_2(1)\to(-1)^+}|\mathbf{A_2}| = \lim_{\gamma_2(1)\to+\infty}|\mathbf{A_2}| = -\infty$. Assume that $|\mathbf{A_2}|\,|_{\gamma_2(1)=0} > 0$, if $-1 < \gamma_2(1) < 0$, with the increasing of $\gamma_2(1)$, the intermediate predator goes from extinction to persistence in mean; if $\gamma_2(1) > 0$, with the increasing of $\gamma_2(1)$, the intermediate predator goes from persistence in mean to extinction. In either case, the prey is still persistent in mean. In view of Table 5 and Table 6 in Subsection 6.3.1, by comparing Figure 1(d), 1(a), 1(e) (or Figure 1(f), 1(c) and 1(g)), one can observe that with the increasing of $\gamma_2(1)$, the intermediate predator goes from extinction to persistence in mean and then extinction again, while the prey remains persistence in mean. Furthermore, if both $x_2(t)$ and $x_3(t)$ are extinctive, then the stochastic persistence in mean and extinction of the prey in system (81) depends only on the symbol of B_1. Compute

$$B_1 = \left(r_1(1) - \frac{\sigma_1^2(1)}{2} - \gamma_1(1) + \ln(1+\gamma_1(1)) - h_1 - r_{11}K_1\right). \tag{110}$$

Then, $\frac{\partial B_1}{\partial\gamma_1(1)} = -\frac{\gamma_1(1)}{1+\gamma_1(1)}$ and $\lim_{\gamma_1(1)\to(-1)^+} B_1 = \lim_{\gamma_1(1)\to+\infty} B_1 = -\infty$. Assume that $B_1|_{\gamma_1(1)=0} > 0$, if $-1 < \gamma_1(1) < 0$, with the increasing of $\gamma_1(1)$, the prey goes from extinction to persistence in mean; if $\gamma_1(1) > 0$, with the increasing of $\gamma_1(1)$, the prey goes from persistence in mean to extinction. By comparing the final column of Table 7, 8, 9 and 10 in Subsection 6.3.1, when both the intermediate predator and top predator are extinctive, as $\gamma_1(1)$ increases, the prey goes from extinction to persistence in mean and then extinction again.

On the other hand, according to (108), (109) and (110), we obtain that the stochastic persistence in mean and extinction of every species are closely related to $\gamma_1(1)$. Therefore, it is natural to consider the effect of $\gamma_1(1)$ on the stochastic dynamics of system (81). Compute $\frac{\partial|\mathbf{\Delta_3}|}{\partial\gamma_1(1)} = -A_{21}A_{32}\frac{\gamma_1(1)}{1+\gamma_1(1)}$, $\frac{\partial|\mathbf{A_2}|}{\partial\gamma_1(1)} = -A_{21}\frac{\gamma_1(1)}{1+\gamma_1(1)}$ and $\frac{\partial B_1}{\partial\gamma_1(1)} = -\frac{\gamma_1(1)}{1+\gamma_1(1)}$. Thus, if $-1 < \gamma_1(1) < 0$, with the increasing of $\gamma_1(1)$, the persistent levels of all species increase (see **Case 1** in Subsection 6.3.2); if $\gamma_1(1) > 0$, with the increasing of $\gamma_1(1)$, the persistent levels of all species decrease (see **Case 2** in Subsection 6.3.2).

Finally, let us take $\gamma_1(1)$ and $\gamma_3(1)$ as examples to consider the effects of Lévy jumps on the existence of OHS. Apparently,

$$\begin{aligned}|\mathbf{\Delta_3}|(\mathbf{H}^*) =& A_{21}A_{32}\left(r_1(1) - \frac{\sigma_1^2(1)}{2} - \gamma_1(1) + \ln(1+\gamma_1(1)) - h_1^* - r_{11}K_1\right)\\ &- A_{11}A_{32}\left(r_2(1) + \frac{\sigma_2^2(1)}{2} + \gamma_2(1) - \ln(1+\gamma_2(1)) + h_2^* + r_{22}K_2\right)\\ &- |\mathbf{A}|\left(r_3(1) + \frac{\sigma_3^2(1)}{2} + \gamma_3(1) - \ln(1+\gamma_3(1)) + h_3^* + r_{33}K_3\right).\end{aligned} \tag{111}$$

Then, $\frac{\partial|\mathbf{\Delta_3}|(\mathbf{H}^*)}{\partial\gamma_3(1)} = -|\mathbf{A}|\frac{\gamma_3(1)}{1+\gamma_3(1)}$ and $\lim_{\gamma_3(1)\to(-1)^+}|\mathbf{\Delta_3}|(\mathbf{H}^*) = \lim_{\gamma_3(1)\to+\infty}|\mathbf{\Delta_3}|(\mathbf{H}^*) = -\infty$. Assume $|\mathbf{\Delta_3}|(\mathbf{H}^*)\,|_{\gamma_3(1)=0} > 0$, if $-1 < \gamma_3(1) < 0$, with the increasing of $\gamma_3(1)$, the OHS will appear (see **Case 1** in Subsection 6.4.1); if $\gamma_3(1) > 0$, with the increasing of $\gamma_3(1)$, the OHS may be disappear (see **Case 2** in Subsection 6.4.1). The similar result can be obtained for $\gamma_1(1)$ (see Subsection 6.4.2).

Biodiversity is the basis for the survival and development of human society. Many aspects of our clothing and food are closely related to the maintenance of biodiversity. Nowadays, human activities lead to the destruction of the living environment of wild animals and even disappear. In order to maintain biodiversity, help people rationally develop resources and make natural resources sustainable, we can take the following measures:

• Controlling Lévy jumps in a reasonable range: protecting natural habitats and minimizing deforestation, land development and ecological destruction;

• Increasing the period of the impulsive toxicant input and decreasing the toxicant input amount: reducing environmental pollution from industry, agriculture and urbanization, such as improving the treatment of waste water and waste gas.

Some interesting topics deserve further investigation. It is interesting to consider other parameters are also affected by telephone noises, for instance, the following model:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[(r_1(\rho(t)) - h_1 - a_{11}(\rho(t))x_1(t) - a_{12}(\rho(t))x_2(t))\,\mathrm{d}t + \mathcal{S}_1(t,\rho(t))\right], \\ \mathrm{d}x_2(t) = x_2(t)\left[(-r_2(\rho(t)) - h_2 + a_{21}(\rho(t))x_1(t) - a_{22}(\rho(t))x_2(t) - a_{23}(\rho(t))x_3(t))\,\mathrm{d}t + \mathcal{S}_2(t,\rho(t))\right], \\ \mathrm{d}x_3(t) = x_3(t)\left[(-r_3(\rho(t)) - h_3 + a_{32}(\rho(t))x_2(t) - a_{33}(\rho(t))x_3(t))\,\mathrm{d}t + \mathcal{S}_3(t,\rho(t))\right]. \end{cases}$$

It is also meaningful to study the optimal harvesting problem of the stochastic multi-population food chain model and other stochastic population systems, for instance, competitive systems and cooperative systems. One may also propose some more realistic systems, such as considering the generalized functional responses and the influences of impulsive perturbations. We will investigate these problems in the future.

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