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A wave and beam equations with structural damping of negative order

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Abstract. We introduce a mathematical model in $\mathbb{R}^n$ for linear classical wave and beam equations with structural damping of negative order, namely, the Riesz potential of frictional damping of the form $(-\Delta)^{-\delta}u_t(t,x)$, $t \in [0,\infty)$, $x \in \mathbb{R}^n$ and $0 < \delta < n/2$.

Our approach is based on the Fourier analysis and the main purpose is to derive the L^2-energy estimates for the solution. In comparison to the well-known results for classical waves with frictional damping $u_t(t,x)$, the proposed damping produces the regularity-loss type and leads to a new fast decay rate. As a result, we will introduce a new classification of the damping mechanism $(-\Delta)^{\gamma}u_t(t,x)$ according to the fractional power $\gamma \in \mathbb{R}$.

1. Introduction

We are interested to know how, mathematically, structural damping of negative order (the Riesz potential of frictional damping)

$$(-\Delta)^{-\delta}u_t(t,x) = c_n \int_{\mathbb{R}^n} \frac{u_t(t,y)}{|x-y|^{n-2\delta}}dy, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+, \ 0 < \delta < n/2,$$

affects the qualitative properties of solutions to linear classical damped wave and beam equations. The two models we have in mind are

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^{\sigma}u(t,x) + au_t(t,x) + b(-\Delta)^{-\delta}u_t(t,x) = 0 & x \in \mathbb{R}^n, \ t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \ t = 0, \end{cases} \tag{1}$$

where $(a = 0, b = 1)$ or $(a = 1, b = 1)$ with $\sigma \geq 1$, $0 < \delta < n/2$. The operators $(-\Delta)^{\sigma}$ and $(-\Delta)^{-\delta}$ denote the fractional Laplacian and the Riesz potential with symbols $|\xi|^{2\sigma}$ and $|\xi|^{-2\delta}$ respectively, more precisely, in the Fourier analysis they can be defined as follows

$$(-\Delta)^{\theta}f(x) = \mathcal{F}^{-1}\left(|\xi|^{2\theta}\mathcal{F}(f)(\xi)\right)(x), \quad x, \xi \in \mathbb{R}^n, \ |\xi|^2 = \xi_1^2 + \cdots + \xi_n^2,$$

where $\theta = \sigma, -\delta$, that is the Riesz potential is the inverse operation of the fractional Laplace operator.

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The above term is a nonlocal (in space) damping mechanism. The nonlocal (in space) concept means: if $f(x,t)$ is thought of as a density at the point x and time t and $G(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^n} G(y-x)f(y,t)dy = (G * f)(x,t)$ is the rate at which individuals are arriving at position x from all other places and $-f(x,t) = -\int_{\mathbb{R}^n} G(y-x)f(x,t)dy$ is the rate at which they are leaving location x to travel to all other sites (see [10]). On the other hand, in the theory of wave equations, the first order time derivative $u_t(t,x)$ represents the frictional damping, it is a local force that is approximately proportional in size to the velocity at that point, but in the direction opposite to the motion of the string at that point (see [11]).

The models in (1) are known as the classical wave equation with damping mechanisms when $\sigma = 1$, the classical beam equation with damping mechanisms when $\sigma = 2$ and the damped σ-evolution equations in other cases. All these models are known to be the most relevant models in real-world applications, which motivates us to investigate them using our new nonlocal damping $(-\Delta)^{-\delta}u_t(t,x)$.

Now, we will present some previous results to the following linear σ-evolution equations with different fractional damping mechanisms

$$\begin{cases} v_{tt}(t,x) + (-\Delta)^{\sigma}v(t,x) + (-\Delta)^{\gamma}v_t(t,x) = 0, & x \in \mathbb{R}^n,\ t > 0, \quad \sigma \geq 1,\ \ \gamma \in [0,\infty), \\ v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), & x \in \mathbb{R}^n,\ t = 0. \end{cases} \tag{2}$$

The types of damping in (2) are: frictional damping ($\gamma = 0$), effective-structural damping ($0 < \gamma < \frac{\sigma}{2}$), critical structural damping ($\gamma = \frac{\sigma}{2}$), non-effective structural damping ($\frac{\sigma}{2} < \gamma < \sigma$)), visco-elastic damping ($\gamma = \sigma$) and very strong damping ($\sigma < \gamma < \infty$). In several papers [2], [4], [5], [14] and reference therein, there are many interesting results concerning the energy estimates of solutions to (2). It has been shown in [5] that the damping $(-\Delta)^{\gamma}v_t(t,x)$ with $\gamma \in [0,\sigma]$ generates polynomial decay at low-frequency zone and exponential decay at high-frequency zone with standard regularity on the initial data, while in [1] the damping $(-\Delta)^{\gamma}v_t(t,x)$ with $\gamma \in (\sigma,\infty)$ generates polynomial decay at low-frequency zone together with another polynomial decay at high-frequency zone but with high (spatial) regularity on the initial data, this phenomenon is known as regularity-loss type and was discovered by Prof. Shuichi Kawashima and his collaborators in the study of Timoshenko and hyperbolic-elliptic systems, see [8] and [9].

To better understanding, we recall some theorems. In [4], the authors derived $(L^1 \cap L^2) - L^2$ energy estimates for model (2) with frictional damping:

Theorem 1.1. *[4] The solution $v(t,x)$ and its derivatives satisfy the following sharp $(L^1 \cap L^2) - L^2$ estimates:*

$$\|v(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4\sigma}}\|v_0\|_{L^1\cap L^2} + (1+t)^{-\frac{n}{4\sigma}}\|v_1\|_{L^1\cap L^2},$$

$$\|v_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4\sigma}-1}\|v_0\|_{L^1\cap H^\sigma} + (1+t)^{-\frac{n}{4\sigma}-1}\|v_1\|_{L^1\cap L^2},$$

$$\|(-\Delta)^{\sigma/2}v(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4\sigma}-\frac{1}{2}}\|v_0\|_{L^1\cap H^\sigma} + (1+t)^{-\frac{n}{4\sigma}-\frac{1}{2}}\|v_1\|_{L^1\cap L^2}.$$

For model (2) with effective-structural damping, we have the following sharp energy estimates derived from [5] and [2].

Theorem 1.2. *[5] Let $\gamma \in (0,\frac{\sigma}{2})$ and $n > 4\gamma$. The solution $v(t,x)$ and its derivatives satisfy the following sharp $(L^1 \cap L^2) - L^2$ estimates:*

$$\|v(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4(\sigma-\gamma)}}\|v_0\|_{L^1\cap L^2} + (1+t)^{-\frac{n-4\gamma}{4(\sigma-\gamma)}}\|v_1\|_{L^1\cap L^2},$$

$$\|v_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4(\sigma-\gamma)}-1}\|v_0\|_{L^1\cap H^\sigma} + (1+t)^{-\frac{n-4\gamma}{4(\sigma-\gamma)}-1}\|v_1\|_{L^1\cap L^2},$$

$$\|(-\Delta)^{\sigma/2}v(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n+2\sigma}{4(\sigma-\gamma)}}\|v_0\|_{L^1\cap H^\sigma} + (1+t)^{-\frac{n-4\delta+2\sigma}{4(\sigma-\gamma)}}\|v_1\|_{L^1\cap L^2}.$$

One can see that the restriction on the space dimension in the linear model depends only on the fractional power of structural damping term and not on the power of the dispersion term $(-\Delta)^\sigma u$.

To complete the picture, we present the following result about the effect of a very strong damping, this model has its originate form $\sigma = 1, \gamma = 2$ in the paper due to M. Ghisi, M. Gobbino, A. Haraux [7] and has been extended in [1], where the authors derived the following estimates:

Theorem 1.3. *[1] Assume $n \geq 1$, $0 < \sigma < \gamma$ and $l \geq 0$. If $v_0 \in H^{l+1} \cap L^1$, $v_1 \in H^l \cap L^1$, then it is true that*

$$\|v_t(t,\cdot)\|^2_{L^2} + \|(-\Delta)^{\sigma/2}v(t,\cdot)\|^2_{L^2} \lesssim (1+t)^{-\frac{n}{2\gamma}}\|v_1\|^2_{L^1} + (1+t)^{-\frac{n+2\sigma}{2\gamma}}\|v_0\|^2_{L^1} + (1+t)^{-\frac{l}{\gamma-\sigma}}\left(\|v_1\|^2_{H^l} + \|v_0\|^2_{H^{l+1}}\right).$$

Theorem 1.4. *[1] Assume $n > 2\sigma$, $0 < \sigma < \gamma$ and $l \geq \sigma$. If $v_0 \in H^l \cap L^1$, $v_1 \in H^{l-\sigma} \cap L^1$, then it is true that*

$$\|v(t,\cdot)\|^2_{L^2} \lesssim (1+t)^{-\frac{n-2\sigma}{2\gamma}}\|v_1\|^2_{L^1} + (1+t)^{-\frac{n}{2\gamma}}\|v_0\|^2_{L^1} + (1+t)^{-\frac{l}{\gamma-\sigma}}\left(\|v_1\|^2_{H^{l-\sigma}} + \|v_0\|^2_{H^l}\right).$$

One can see that the restriction on the space dimension in the linear model depends only on the fractional power σ of the dispersion term $(-\Delta)^\sigma u$ and not on the power γ of a very strong damping.

These results show that the decay rate related to the initial displacement v_0 is always better than that of the initial velocity v_1 for all positive powers of fractional damping, except for $\gamma = 0$, where they are equal. A natural question arises: does this phenomenon hold true when $\gamma < 0$?

Our first main goal of this paper is to show that this phenomenon is broken in our models (1), that is, the decay rate related to u_1 is always better than that of u_0 due to the special structure of the characteristics roots $\lambda_1(\xi), \lambda_2(\xi)$ defined below. Thus, the case $\gamma = 0$ can be considered as a new critical case between two completely different models.

Now, in the pioneering paper [13], the authors are the first who studied the interaction between frictional damping $u_t(t,x)$ and visco-elastic damping $-\Delta u_t(t,x)$ in the wave model. They proved that the frictional damping is more dominant than the visco-elastic one in terms of linear estimates in the low-frequency zone, while in the high-frequency zone, the regularity of initial data is influenced by the visco-elastic damping, see also [3]. Thereafter, R. Ikehata and S. Kitazaki have already published a result [12] that continues to attract regularity-loss type for wave equation with frictional damping and very strong damping, dealing with the following model

$$\begin{cases} v_{tt}(t,x) - \Delta v(t,x) + v_t(t,x) + (-\Delta)^\gamma v_t(t,x) = 0, & x \in \mathbb{R}^n,\ t > 0,\ \gamma > 1, \\ v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), & x \in \mathbb{R}^n,\ t = 0, \end{cases} \tag{3}$$

and they proved that the regularity loss still occurs even with the presence of frictional damping and that the so-called diffusion phenomenon dissapears.

The second main goal of this paper is to show that the phenomenon of regularity-loss type is broken in our model (1). More precisely, when $(a = 0, b = 1)$, this model verifies the regularity-loss type, but when $(a = 1, b = 1)$, this model it is not of regularity-loss type. Moreover, the model $(a = 1, b = 1)$ inherits the same decay rates of solution as the model $(a = 0, b = 1)$ and the same regularity of initial data as the model $(a = 1, b = 0)$.

Before we present our main results, we write $f \lesssim g$ when there exists a constant $c > 0$ such that $f \leq cg$ and $f \approx g$ when $g \lesssim f \lesssim g$. Next, for the Fourier transform of a function f we write $\hat{f}(\xi)$ instead of $\mathcal{F}(f)(\xi)$. Finally, $L^1(\mathbb{R}^n)$, $L^2(\mathbb{R}^n)$ and $H^l(\mathbb{R}^n)$ denote the usual Lebesgue, Hilbert and Sobolev spaces with the norms

$$\|f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f(x)|dx < \infty, \ \ \|f\|_{L^2(\mathbb{R}^n)} = \|f^2\|^{1/2}_{L^1(\mathbb{R}^n)} < \infty,$$

and

$$\|f\|_{H^l(\mathbb{R}^n)} = \|(1+|\cdot|^2)^{\frac{l}{2}}\hat{f}(\cdot)\|_{L^2(\mathbb{R}^n)} < \infty,$$

we will omit the notation $\mathbb{R}^n$ in all functional spaces and we write L^1, L^2 and H^l instead of $L^1(\mathbb{R}^n)$, $L^2(\mathbb{R}^n)$ and $H^l(\mathbb{R}^n)$.

2. Main Results

Let us define the energy function $E(u)(t)$ (or $E(\hat{u})(t)$) associated to models (1) as follows:

$$E(u)(t) = \frac{1}{2}\|u_t(t,\cdot)\|^2_{L^2} + \frac{1}{2}\|(-\Delta)^{\sigma/2}u(t,\cdot)\|^2_{L^2} = \frac{1}{2}\|\hat{u}_t(t,\cdot)\|^2_{L^2} + \frac{1}{2}\||\cdot|^{\sigma}\hat{u}(t,\cdot)\|^2_{L^2} = E(\hat{u})(t).$$

In model $(a=0, b=1)$, the term whose action may dissipate the energy $E(u)$ is $(-\Delta)^{-\delta}u_t$. In fact, for a fixed u we have,

$$\frac{dE(u)(t)}{dt} = -\|(-\Delta)^{-\delta/2}u_t(t,\cdot)\|^2_{L^2} = -\||\cdot|^{-\delta}\hat{u}_t(t,\cdot)\|_{L^2} = \frac{dE(\hat{u})(t)}{dt} \leq 0,$$

this indicates that for all $t>0$, the energy is a non-increasing function and tends to 0 as time tends to ∞, its decay rate is determined below.

One can use this energy method to prove the well-posedness of solutions to (1) in Sobolev space of fractional order similarly to [6, page 193] for classical wave model, or to [17, Theorem 4] and [16, Proposition 1.2] for damped wave models.

Our main results read as follows.

Theorem 2.1. *Let $u_0 \in H^s$ and $u_1 \in H^{s-\sigma}$ with nonnegative real number s. Then, the Cauchy problem (1) is well-posed in*

$$C([0,\infty), H^s) \cap C^1([0,\infty), H^{s-\sigma}).$$

Theorem 2.2. *Let us consider the Cauchy problem $(a=0, b=1)$ with $\sigma \geq 1$ and $2\delta \in (0,n)$. We assume that*

$$u_0 \in L^1 \cap H^l, \;\; u_1 \in L^1 \cap H^{l-\sigma}, \quad l \geq \sigma.$$

Then the solution u satisfies the estimates:

$$\|u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4(\sigma+\delta)}}\|u_0\|_{L^1} + (1+t)^{-\frac{n}{4(\sigma+\delta)}-\frac{\delta}{\sigma+\delta}}\|u_1\|_{L^1} + (1+t)^{-\frac{l}{2\delta}}\left(\||D|^l u_0\|_{L^2} + \||D|^{l-\sigma}u_1\|_{L^2}\right). \tag{4}$$

Theorem 2.3. *Let us consider the Cauchy problem $(a=0, b=1)$ with $\sigma \geq 1$ and $2\delta \in (0,n)$. We assume that*

$$u_0 \in L^1 \cap H^{l+\sigma}, \;\; u_1 \in L^1 \cap H^l, \quad l \geq 0.$$

Then the energy function $E(u)(t)$ satisfies the estimates:

$$\left\|(-\Delta)^{\sigma/2}u(t,\cdot)\right\|_{L^2} \lesssim (1+t)^{-\frac{n}{4(\sigma+\delta)}-\frac{\sigma}{2(\sigma+\delta)}}\|u_0\|_{L^1} + (1+t)^{-\frac{n}{4(\sigma+\delta)}-\frac{\sigma+2\delta}{2(\sigma+\delta)}}\|u_1\|_{L^1} + (1+t)^{-\frac{l}{2\delta}}\left(\||D|^{l+\sigma}u_0\|_{L^2} + \||D|^l u_1\|_{L^2}\right), \tag{5}$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4(\sigma+\delta)}-\frac{\sigma}{\sigma+\delta}}\|u_0\|_{L^1} + (1+t)^{-\frac{n}{4(\sigma+\delta)}-\frac{\sigma+2\delta}{\sigma+\delta}}\|u_1\|_{L^1} + (1+t)^{-\frac{l}{2\delta}}\left(\||D|^{l+\sigma}u_0\|_{L^2} + \||D|^l u_1\|_{L^2}\right). \tag{6}$$

Theorem 2.4. *Let us consider the Cauchy problem $(a=1, b=1)$ with $\sigma \geq 1$ and $2\delta \in (0,n)$. We assume that*

$$u_0 \in L^1 \cap H^{\sigma}, \;\; u_1 \in L^1 \cap L^2.$$

Then the solution u and its energy $E(u)(t)$ satisfy the estimates:

$$\|u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4(\sigma+\delta)}}\|u_0\|_{L^1\cap L^2} + (1+t)^{-\frac{n}{4(\sigma+\delta)}-\frac{\delta}{\sigma+\delta}}\|u_1\|_{L^1\cap L^2},$$

$$\left\|(-\Delta)^{\sigma/2}u(t,\cdot)\right\|_{L^2} \lesssim (1+t)^{-\frac{n}{4(\sigma+\delta)}-\frac{\sigma}{2(\sigma+\delta)}}\|u_0\|_{L^1\cap H^{\sigma}} + (1+t)^{-\frac{n}{4(\sigma+\delta)}-\frac{\sigma+2\delta}{2(\sigma+\delta)}}\|u_1\|_{L^1\cap L^2},$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4(\sigma+\delta)}-\frac{\sigma}{\sigma+\delta}}\|u_0\|_{L^1\cap H^{\sigma}} + (1+t)^{-\frac{n}{4(\sigma+\delta)}-\frac{\sigma+2\delta}{\sigma+\delta}}\|u_1\|_{L^1\cap L^2}.$$

Remark 2.5. *The estimates in the first two theorems show that the model* $(a = 0, b = 1)$ *is of the regularity-loss type due to the presence of the factor* $e^{-|\xi|^{-2\delta}t}$ *in the high frequency zone, which causes some polynomial decay rather than exponential decay, see Lemma 3.2.*

Remark 2.6. *Both models* $(a = 0, b = 1)$ *and (2) with* $\gamma > \sigma$ *are of regularity-loss type. According to Theorem 2.2, the restriction on the space dimension* $n > 2\delta$ *is caused by the Riesz potential, not by the singularity in the linear estimates. In contrast, Theorem 3 shows that the restriction* $n > 2\sigma$ *is caused by the singularity in the linear estimates.*

Remark 2.7. *If we formally set* $\gamma = -\delta$ *in (2), the estimates from Theorems 2.2 and 2.3 are not the same as in Theorem 1.2.*

Remark 2.8. *One can see that the decay rate corresponding to* u_1 *is always better than that of* u_0 *for all* $0 < \delta < n/2$*. This new phenomenon occurs due to the special structure of the solution and the characteristics roots (see the proof below). Since Duhamel's principle is always dependent on the kernel of* u_1 *and is used in the treatment of semilinear models, choosing* $u_1 = 0$ *is generally undesirable. Hence, we obtain the following double-decay rates for the model* $(a = 1, b = 1)$*:*

$$\|u(t,\cdot)\|_{L^2} \lesssim \begin{cases} (1+t)^{-\frac{n}{4(\sigma+\delta)}} \left(\|u_0\|_{L^1\cap L^2} + \|u_1\|_{L^1\cap L^2}\right), & \text{if } u_0 \neq 0,\ u_1 \neq 0, \\ (1+t)^{-\frac{n}{4(\sigma+\delta)}-\frac{\delta}{\sigma+\delta}} \|u_1\|_{L^1\cap L^2}, & \text{if } u_0 = 0,\ u_1 \neq 0. \end{cases}$$

Remark 2.9. *Some interactions between frictional and structural damping of negative order in* $(a = 1, b = 1)$ *are shown in the following points:*

- *the solution inherits the same decay rates as that to the problem* $(a = 0, b = 1)$*.*
- *the solution has the same regularity as that to the problem* $(a = 1, b = 0)$*. This is remarkable when compared to the results of Ikehata-Kitazaki [12].*
- *the solution and its energy satisfy better decay rates than those in Theorem 1.1, provided that*

$$u_0 = 0, \quad n < 4\sigma.$$

In this case, the structural damping of negative order enhance the damping effect in the model $(a = 1, b = 1)$ *without higher regularity on the data.*

We recall that the effect of double damping mechanism for the wave model originated in [13] and was extended to the σ*-evolution model in [3]. Hence, we can see that the structural damping of negative order in our model* $(a = 1, b = 1)$ *takes the same role as frictional damping in [3], and the frictional damping in* $(a = 1, b = 1)$ *plays the same role as visco-elastic damping.*

3. Proof of Theorems

The proof is straightforward and based on Fourier analysis, for example, see researches by R. Ikehata, M. Reissig, and their collaborators.

We begin by applying the spatial Fourier transform, one has

$$\begin{cases} \hat{u}_{tt}(t,\xi) + |\xi|^{2\sigma}\hat{u}(t,\xi) + |\xi|^{-2\delta}\hat{u}_t(t,\xi) = 0, \\ \hat{u}(0,\xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0,\xi) = \hat{u}_1(\xi), \end{cases} .$$

for any $\xi \in \mathbb{R}^n/\{0\}$, $t > 0$. The corresponding characteristic equation is:

$$\lambda^2 + |\xi|^{-2\delta}\lambda + |\xi|^{2\sigma} = 0,$$

where its roots are given by

$$\lambda_1(\xi) = -\frac{1}{2|\xi|^{2\delta}} + \frac{\sqrt{|\xi|^{-4\delta} - 4|\xi|^{2\sigma}}}{2},$$

$$\lambda_2(\xi) = -\frac{1}{2|\xi|^{2\delta}} - \frac{\sqrt{|\xi|^{-4\delta} - 4|\xi|^{2\sigma}}}{2},$$

in small frequency zone $\left\{\xi \in \mathbb{R}^n : 0 < |\xi| < \left(\frac{1}{2}\right)^{\frac{1}{\sigma+2\delta}}\right\}$, and

$$\lambda_1(\xi) = -\frac{1}{2|\xi|^{2\delta}} + i\frac{\sqrt{4|\xi|^{2\sigma} - |\xi|^{-4\delta}}}{2},$$

$$\lambda_2(\xi) = -\frac{1}{2|\xi|^{2\delta}} - i\frac{\sqrt{4|\xi|^{2\sigma} - |\xi|^{-4\delta}}}{2},$$

in high frequency zone $\left\{\xi \in \mathbb{R}^n : \left(\frac{1}{2}\right)^{\frac{1}{\sigma+2\delta}} < |\xi| < \infty\right\}$.

Using the theory of second order differential equation, we have the following explicit representation of the solution:

$$\hat{u}(t,\xi) = \frac{\lambda_1(\xi)e^{\lambda_2(\xi)t} - \lambda_2(\xi)e^{\lambda_1(\xi)t}}{\lambda_1(\xi) - \lambda_2(\xi)}\hat{u}_0(\xi) + \frac{e^{\lambda_1(\xi)t} - e^{\lambda_2(\xi)t}}{\lambda_1(\xi) - \lambda_2(\xi)}\hat{u}_1(\xi), \ \ \lambda_1(\xi) \neq \lambda_2(\xi), \tag{7}$$

$$\hat{u}(t,\xi) = (1 - \lambda_1(\xi)t)e^{\lambda_1(\xi)t}\hat{u}_0(\xi) + te^{\lambda_1(\xi)t}\hat{u}_1(\xi), \ \ \lambda_1(\xi) = \lambda_2(\xi). \tag{8}$$

We will state the following lemma to show the asymptotic behaviors of these roots at low and high frequency zones. These asymptotic behaviors play an important role in discovering some new phenomena in linear evolution equations.

Lemma 3.1. *It holds in the low-frequency zone $|\xi| \to 0$:*

- $\lambda_1(\xi), \ \lambda_2(\xi) \in \mathbb{R}_-, \quad \lambda_2(\xi) \leq \lambda_1(\xi)$,
- $\lambda_1(\xi) \to 0$, $\lambda_2(\xi) \to -\infty$ *as* $|\xi| \to 0$,
- $\lambda_1(\xi) \approx -|\xi|^{2(\sigma+\delta)}, \quad \lambda_2(\xi) \lesssim -c, \quad \frac{1}{|\lambda_1(\xi)-\lambda_2(\xi)|} \lesssim |\xi|^{2\delta}, \quad c > 0$,
- $\frac{|\lambda_1(\xi)|}{|\lambda_1(\xi)-\lambda_2(\xi)|} \lesssim |\xi|^{2(\sigma+2\delta)}, \ \frac{|\lambda_2(\xi)|}{|\lambda_1(\xi)-\lambda_2(\xi)|} \lesssim 1.$

It holds in the high-frequency zone $|\xi| \to \infty$:

- $\lambda_1(\xi), \ \lambda_2(\xi) \in \mathbb{C}$,
- $\lambda_1(\xi) = \overline{\lambda_2(\xi)}, \ \ \Re\lambda_1(\xi) = \Re\lambda_2(\xi) = -\frac{1}{|\xi|^{2\delta}}$,
- $|\lambda_1(\xi)|^2 = |\lambda_1(\xi)|^2 = |\xi|^{2\sigma}, \quad \frac{1}{|\lambda_1(\xi)-\lambda_2(\xi)|^2} \lesssim |\xi|^{-2\sigma}.$

We also need to use the following crucial estimates.

Lemma 3.2. *Let $c > 0$, $\alpha > -n$, $\beta > 0$ and $a > 0$. Then*

$$\int_{|\xi|\le c} |\xi|^{\alpha} e^{-a|\xi|^{\beta}t} d\xi \lesssim (1+t)^{-\frac{n+\alpha}{\beta}}, \quad \forall t \ge 0,$$

$$\sup_{|\xi|\ge 1} \left(\frac{e^{-a\frac{t}{|\xi|^{2\beta}}}}{|\xi|^{2l}} \right) \lesssim (1+t)^{-\frac{l}{\beta}}, \quad l \ge 0.$$

Proof. The proof of this lemma can be found in [12] or [15]. □

Now, we are in position to prove Theorem 2.2. Using the representation (7) and the first part of Lemma 3.1 we can estimate

$$\begin{aligned}
\|\hat{u}(t,\cdot)\|^2_{L^2(|\xi|\ll 1)} &\lesssim \|\hat{u_0}\|^2_{L^\infty} \int_{|\xi|\ll 1} \frac{|\lambda_1(\xi)|^2}{|\lambda_1(\xi)-\lambda_2(\xi)|^2} e^{-2ct} d\xi + \|\hat{u_0}\|^2_{L^\infty} \int_{|\xi|\ll 1} \frac{|\lambda_2(\xi)|^2}{|\lambda_1(\xi)-\lambda_2(\xi)|^2} e^{-2|\xi|^{2(\sigma+\delta)}t} d\xi \\
&\quad + \|\hat{u_1}\|^2_{L^\infty} \int_{|\xi|\ll 1} \frac{1}{|\lambda_1(\xi)-\lambda_2(\xi)|^2} e^{-2|\xi|^{2(\sigma+\delta)}t} d\xi + \|\hat{u_1}\|^2_{L^\infty} \int_{|\xi|\ll 1} \frac{1}{|\lambda_1(\xi)-\lambda_2(\xi)|^2} e^{-2ct} d\xi \\
&\lesssim \|\hat{u_0}\|^2_{L^\infty} \left(e^{-2ct} + (1+t)^{-\frac{n}{2(\sigma+\delta)}} \right) + \|\hat{u_1}\|^2_{L^\infty} \left(e^{-2ct} + (1+t)^{-\frac{n+4\delta}{2(\sigma+\delta)}} \right) \\
&\lesssim (1+t)^{-\frac{n}{2(\sigma+\delta)}} \|u_0\|^2_{L^1} + (1+t)^{-\frac{n+4\delta}{2(\sigma+\delta)}} \|u_1\|^2_{L^1}
\end{aligned}$$

where we used $\|fg\|_{L^2} \le \|f\|_{L^\infty} \|g\|_{L^2}$ and $\|\hat{f}\|_{L^\infty} \le \|f\|_{L^1}$.

In the high frequency zone, using again the representation (7) and the second part of Lemma 3.1, we can estimate

$$\begin{aligned}
\|\hat{u}(t,\cdot)\|^2_{L^2(|\xi|\gg 1)} &\lesssim \sup_{|\xi|\gg 1} \left(\frac{e^{-\frac{t}{|\xi|^{2\delta}}}}{|\xi|^{2l}} \right) \Big(\int_{|\xi|\gg 1} |\xi|^{2l} |\hat{u_0}(\xi)|^2 d\xi + \int_{|\xi|\gg 1} |\xi|^{2l-2\sigma} |\hat{u_1}(\xi)|^2 d\xi \Big) \\
&\lesssim (1+t)^{-\frac{l}{\delta}} \left(\|u_0\|^2_{H^l} + \|u_1\|^2_{H^{l-\sigma}} \right).
\end{aligned}$$

The estimate in the intermediate zone $c_1 < |\xi| < c_2$ leads to an exponential decay with L^2 regularity on the initial data. Combining the above estimates, the proof of Theorem 2.2 is completed. Similar procedures can be used to prove Theorem 2.3, so we omit its proof.

To prove Theorem 2.4, let us consider the second linear equation ($a = 1, b = 1$)

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^{\sigma} u(t,x) + u_t(t,x) + (-\Delta)^{-\delta} u_t(t,x) = 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \end{cases}$$

Applying the spatial Fourier transform again, one has

$$\begin{cases} \hat{u}_{tt}(t,\xi) + |\xi|^{2\sigma} \hat{u}(t,\xi) + \hat{u}_t(t,\xi) + |\xi|^{-2\delta} \hat{u}_t(t,\xi) = 0, \\ \hat{u}(0,\xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0,\xi) = \hat{u}_1(\xi), \end{cases}$$

for any $\xi \in \mathbb{R}^n/\{0\}$, $t > 0$. Now, the energy $E(\hat{u})(t)$ is a non-increasing function for $t > 0$, due to

$$\frac{dE(\hat{u})(t)}{dt} = -\|(1 + |\xi|^{-\delta}) \hat{u}_t(t,\cdot)\|_{L^2}.$$

The corresponding characteristic equation is:

$$\lambda^2 + (1 + |\xi|^{-2\delta})\lambda + |\xi|^{2\sigma} = 0.$$

Hence, the roots are

$$\lambda_1(\xi) = -\frac{1}{2}\left(1 + \frac{1}{|\xi|^{2\delta}}\right) + \frac{1}{2}\sqrt{(1 + |\xi|^{-2\delta})^2 - 4|\xi|^{2\sigma}},$$

$$\lambda_2(\xi) = -\frac{1}{2}\left(1 + \frac{1}{|\xi|^{2\delta}}\right) - \frac{1}{2}\sqrt{(1 + |\xi|^{-2\delta})^2 - 4|\xi|^{2\sigma}},$$

as $|\xi| \to 0$, and

$$\lambda_1(\xi) = -\frac{1}{2}\left(1 + \frac{1}{|\xi|^{2\delta}}\right) + \frac{i}{2}\sqrt{4|\xi|^{2\sigma} - (1 + |\xi|^{-2\delta})^2},$$

$$\lambda_2(\xi) = -\frac{1}{2}\left(1 + \frac{1}{|\xi|^{2\delta}}\right) - \frac{i}{2}\sqrt{4|\xi|^{2\sigma} - (1 + |\xi|^{-2\delta})^2},$$

as $|\xi| \to \infty$. It is worth noting that because of the following property

$$\lim_{|\xi|\to 0} \frac{\frac{1}{|\xi|^{2\delta}}}{1 + \frac{1}{|\xi|^{2\delta}}} = 1 \ \textit{and} \ \lim_{|\xi|\to\infty} \frac{1}{1 + \frac{1}{|\xi|^{2\delta}}} = 1,$$

the next lemma is expected.

Lemma 3.3. *It holds in the low frequency zone* $|\xi| \to 0$*:*

- $\lambda_1(\xi),\ \lambda_2(\xi) \in \mathbb{R}_-,\quad \lambda_2(\xi) \leq \lambda_1(\xi),$
- $\lambda_1(\xi) \to 0,\ \lambda_2(\xi) \to -\infty$ *as* $|\xi| \to 0,$
- $\lambda_1(\xi) \approx -|\xi|^{2(\sigma+\delta)},\quad \lambda_2(\xi) \lesssim -C,\quad \frac{1}{|\lambda_1(\xi)-\lambda_2(\xi)|} \lesssim |\xi|^{2\delta},\quad C > 0,$
- $\frac{|\lambda_1(\xi)|}{|\lambda_1(\xi)-\lambda_2(\xi)|} \lesssim |\xi|^{2(\sigma+2\delta)},\ \frac{|\lambda_2(\xi)|}{|\lambda_1(\xi)-\lambda_2(\xi)|} \lesssim 1.$

It holds in the high frequency zone $|\xi| \to \infty$*:*

- $\lambda_1(\xi),\ \lambda_2(\xi) \in \mathbb{C},$
- $\lambda_1(\xi) = \overline{\lambda_2(\xi)},\ \ \Re\lambda_1(\xi) = \Re\lambda_2(\xi) = -1 - \frac{1}{|\xi|^{2\delta}},$
- $|\lambda_1(\xi)|^2 = |\lambda_1(\xi)|^2 = |\xi|^{2\sigma},\quad \frac{1}{|\lambda_1(\xi)-\lambda_2(\xi)|^2} \lesssim |\xi|^{-2\sigma}.$

Obviously, the only difference between Lemma 3.1 and 3.3 is in the high-frequency zone, where the roots in the second lemma lead to exponential decay as in the following estimate

$$\begin{aligned}\|\hat{u}(t,\cdot)\|^2_{L^2(|\xi|\gg 1)} &\lesssim \sup_{|\xi|\gg 1}\left(e^{-\frac{t}{|\xi|^{2\delta}}}\right)e^{-ct}\Big(\int_{|\xi|\gg 1}|\hat{u_0}(\xi)|^2 d\xi + \int_{|\xi|\gg 1}|\xi|^{-2\sigma}|\hat{u_1}(\xi)|^2 d\xi\Big)\\ &\lesssim e^{-ct}\left(\|u_0\|^2_{L^2} + \|u_1\|^2_{L^2}\right).\end{aligned}$$

The estimate in low-frequency zone is the same as in Theorem 2.2. The proof of Theorem 2.4 is now completed.

Finally, to prove Theorem 2.1, we use the representation (7) and the second part of Lemma 3.1, we can estimate the solution as follows:

$$|\hat{u}(t,\xi)|^2 \lesssim |\hat{u}_0(\xi)|^2 + |\xi|^{-2\sigma}|\hat{u}_1(\xi)|^2,$$

for all ξ in high frequency zone, following the approach as in [6, page 193] for classical wave model, or in [17, Theorem 4] for wave equation with frictional and visco-elastic damping, we can reach the desired result.

4. Conclusion

Indeed, the idea of applying the Riesz potential of frictional damping is inspired by a symmetry property of the well-known structural damping of positive order, but its effects are not simply constructed from those of structural damping. See the differences between Theorems 1.2, 2.2, and 2.3.

We have now shown that the influence of the fractional damping mechanism is divided into three cases $\delta < 0$ (structural damping of negative order), $\delta = 0$ (frictional damping), and $\delta > 0$ (structural damping of positive order) based on their effects on energy estimates, with frictional damping being the critical case, where the decay rates coincide in both initial data.

In the future, it is reasonable to apply the Riesz potential of mass or rotational inertia in wave models, by including the following terms:

$$(-\Delta)^{-\delta_1}u_{tt}, \quad (-\Delta)^{-\delta_2}u, \quad \delta_1, \delta_2 \in (0, n/2),$$

and show their effects on energy estimates.

We have analyzed the linear equation and derived the energy estimates in this study, and we know that these energy estimates can be effectively used to prove the global (in time) existence of small data solutions and to find the critical exponent for the corresponding semi-linear models with power nonlinearity on the right-hand side, which is the subject of our next papers.

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