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Partial isometries, strongly EP elements and general solutions of some equations

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Abstract. In this article, we presented some necessary and sufficient conditions for elements in rings with involution to be a partial isometry, strongly EP element and generalized inverse by using solutions of equations.

1. Introduction

Throughout this article, R will be used to signify a unital ring with unit 1 and involution. An involution in R is an anti-isomorphism $* : R \to R, a \mapsto a^*$ of degree 2, satisfying

$$(a+b)^* = a^* + b^*, (ab)^* = b^*a^*, (a^*)^* = a,$$

for all $a,\ b \in R$.
An element a of a ring R is said to be group invertible if there exists $a^\# \in R$ such that

$$aa^\# = a^\# a,\ a = aa^\# a,\ a^\# = a^\# aa^\#.$$

The element $a^\#$ is called group inverse of a, which is uniquely determined by above equations (see [5,6,10]). Denote by $R^\#$ the set of all group invertible elements of R.

An element a^+ is said to be the Moore-Penrose inverse (or MP-inverse) of a [8-10], if it satisfies the following conditions:

$$(a^+a)^* = a^+a,\ (aa^+)^* = aa^+,\ a^+aa^+ = a^+,\ aa^+a = a.$$

If a^+ exists, then it is unique. We denote by R^+ the set of all Moore-Penrose invertible elements of R.

An element $a \in R$ is called partial isometry if $a = aa^*a$. Obviously, $a \in R^+$ is a partial isometry [3] if and only if $a^* = a^+$. We denote by R^{PI} the set of all partial isometries of R. An element $a \in R$ is called EP [1,10] if $a \in R^\# \cap R^+$ and $a^+ = a^\#$. We denote by R^{EP} the set of all EP elements of R. Let $a \in R^\# \cap R^+$, then a is said to be a strongly EP element if $a \in R^{EP}$ is partial isometry. Also we denote by R^{SEP} the set of all strongly EP elements of R.

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Several authors [1-4,6,7,9,10,13] have studied the characteristics of partial isometry, EP elements, and strongly EP elements in rings with involution.

Let $a \in R^{\#} \cap R^{+}$ and $\chi_a = \left\{a, a^{\#}, a^{+}, a^{*}, (a^{\#})^{*}, (a^{+})^{*}\right\}$. We identify and investigate new properties of several equations to be a partial isometry, strongly EP elements and generalized solutions of some equations.

2. Partial isometries and strongly EP elements

In this section, we establish and investigate new properties of partial isometries elements and strongly EP elements.

The following lemmas are well-known and frequently used in the rest of the article.

Lemma 2.1. *([1, Theorem 2.1]) Let $a \in R^{\#} \cap R^{+}$ and $n \in N$. Then $a \in R^{PI}$ if and only if one of the following equivalent conditions holds:*
(i) $a^n a^+ = a^n a^*$;
(ii) $a^+ a^n = a^* a^n$.

Lemma 2.2. *([3, Theorem 2.3]) Assume that $a \in R^{+}$. Then $a \in R^{SEP}$ if and only if $a \in R^{\#}$ and one of the following conditions holds:*
(i) $a^* = a^{\#}$;
(ii) $aa^* = a^+ a$;
(iii) $aa^* = aa^{\#}$;
(iv) $a^+ a^* = a^{\#} a^+$;
(v) $a^+ a^* = a^+ a^{\#}$;
(vi) $a^* a^+ = a^{\#} a^+$;
(vii) $a^* a^+ = a^+ a^{\#}$;
(viii) $a^* a^{\#} = a^{\#} a^+$;
(ix) $aa^* a^+ = a^+$;
(x) $aa^* a^+ = a^{\#}$;
(xi) $a^* a^{\#} = a^{\#} a^{\#}$;
(xii) $aa^+ a^* = a^+$;
(xiii) $a^* a^2 = a$;
(xiv) $a^2 a^* = a$;
(xv) $aa^+ a^* = a^{\#}$;
(xvi) $a^* a^+ a = a^{\#}$.

Lemma 2.3. *([11, 12]) Let $a \in R^{\#} \cap R^{+}$. Then*
1) $(a^+)^* aa^{\#} = (a^+)^* = a^{\#} a (a^+)^*$;
2) $(a^{\#})^* a^* a^+ = a^+ a^* (a^{\#})^* = a^+$.

The following lemma follows from [14, Corollary 2.10 and Lemma 2.11].

Lemma 2.4. *Let $a \in R^{\#} \cap R^{+}$.*
(1) If $a^+ a^+ a^* = a^+ a^+ a^+$, then $a \in R^{PI}$.
(2) If $a^* a^+ a^* = a^* a^+ a^+$, then $a \in R^{PI}$.
(3) If $a^+ a^* a^* = a^+ a^+ a^*$, then $a \in R^{PI}$.
(4) If $a^+ a^+ a^* a^+ = a^+ a^+ a^+ a^+$, then $a \in R^{PI}$.
(5) If $a^+ a^+ a^* a^* = a^+ a^+ a^+ a^*$, then $a \in R^{PI}$.

In ([4, Proposition 3.1]), we shown that if $a \in R^{\#} \cap R^{+}$, then $a \in R^{PI}$ if and only if there is at least one solution to the equation (2.1) in χ_a.

$$aa^* xa = xa. \tag{1}$$

Now, we modify the equation (2.1) as follows:

$$aa^+xa = x(a^+)^*. \tag{2}$$

Theorem 2.5. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if there is at least one solution to the equation (2.2) in χ_a.*
Proof. $' \Rightarrow'$ Suppose that $a \in R^{PI}$, then $a = (a^+)^*$, which implies that $x = (a^+)^*$ is a solution.
$' \Leftarrow'$ (1) If $x = a$, then $aa^+a^2 = a(a^+)^*$, that is,

$$a^2 = a(a^+)^*.$$

Post-multiply $a^2 = a(a^+)^*$ by a^*, this leads to $a^2a^* = a^2a^+$. By Lemma 2.1, $a \in R^{PI}$.
(2) If $x = a^\#$, then $aa^+a^\#a = a^\#(a^+)^*$, that is,

$$a^\#a = a^\#(a^+)^*.$$

Pre-multiply $a^\#a = a^\#(a^+)^*$ by a^2, we can get $a^2 = a(a^+)^*$. By (1), $a \in R^{PI}$.
(3) If $x = a^+$, then $aa^+a^+a = a^+(a^+)^*$. Post-multiply $aa^+a^+a = a^+(a^+)^*$ by a^*, this leads to

$$aa^+a^* = a^+.$$

By Lemma 2.2, $a \in R^{PI}$.
(4) If $x = a^*$, then $aa^+a^*a = a^*(a^+)^*$, that is,

$$aa^+a^*a = a^+a.$$

Post-multiply $aa^+a^*a = a^+a$ by a^+, we can get $aa^+a^* = a^+$. By Lemma 2.2, $a \in R^{PI}$.
(5) If $x = (a^\#)^*$, then $aa^+(a^\#)^*a = (a^\#)^*(a^+)^*$. Apply the involution, this leads to

$$a^*a^\#aa^+ = a^+a^\#.$$

Post-multiply $a^*a^\#aa^+ = a^+a^\#$ by a^2, we can get $a^*a = a^+a$. Hence, $a \in R^{PI}$ by [3, Theorem 2.1].
(6) If $x = (a^+)^*$, then $aa^+(a^+)^*a = (a^+)^*(a^+)^*$, that is,

$$(a^+)^*a = (a^+)^*(a^+)^*.$$

Apply the involution, this leads to $a^*a^+ = a^+a^+$. we can infer $a \in R^{PI}$ by [14, Corollary 2.10].

We modify the equation (2.2) as follows:

$$a^+axa = x(a^+)^*. \tag{3}$$

Theorem 2.6. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if there is at least one solution to the equation (2.3) in χ_a.*
Proof. $' \Rightarrow'$ Assume that $a \in R^{PI}$, then $a^* = a^+$, which implies that $x = a^*$ is a solution.
$' \Leftarrow'$ (1) If $x = a$, then $a^+a^3 = a(a^+)^*$. Pre-multiply $a^+a^3 = a(a^+)^*$ by $a^\#$, one has $a = (a^+)^*$. Hence, $a \in R^{PI}$.
(2) If $x = a^\#$, then $a^+a = a^\#(a^+)^*$. Pre-multiply $a^+a = a^\#(a^+)^*$ by a, one has $a = (a^+)^*$. Therefore, $a \in R^{PI}$.
(3) If $x = a^+$, then $a^+a = a^+(a^+)^*$. Post-multiply $a^+a = a^+(a^+)^*$ by a^*, one has $a^* = a^+$. Hence, $a \in R^{PI}$.
(4) If $x = a^*$, then $a^*a = a^+a$. Hence, $a \in R^{PI}$ by [3, Theorem 2.1].
(5) If $x = (a^\#)^*$, then $(a^\#)^*a = (a^\#)^*(a^+)^*$. Apply the involution, one obtains $a^*a^\# = a^+a^\#$. Therefore, $a \in R^{PI}$ by [3, Theorem 2.2].
(6) If $x = (a^+)^*$, then $a^+a(a^+)^*a = (a^+)^*(a^+)^*$. Apply the involution, one obtains $a^*a^+a^+a = a^+a^+$. Post-multiply $a^*a^+a^+a = a^+a^+$ by a^+a, one gets $a^*a^+a^+a = a^+a^+a^+a$. Apply the involution, one yields $a^+a(a^+)^*a = a^+a(a^+)^*(a^+)^*$. Pre-multiply by $a^\#a$, one gives $(a^+)^*a = (a^+)^*(a^+)^*$. Apply the involution, this leads to $a^*a^+ = a^+a^+$. Hence, $a \in R^{PI}$ by [2, Lemma 2.5].

We change the equation (2.3) as follows:

$$a^+axa + a^\# = x(a^+)^* + a^+. \tag{4}$$

Theorem 2.7. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if there is at least one solution to the equation (2.4) in χ_a.*

Proof. $' \Rightarrow '$ Suppose that $a \in R^{SEP}$, then $a^* = a^+ = a^{\#}$. Hence, $x = a$ is a solution.

$' \Leftarrow '$ (1) If $x = a$, then $a^+a^3 + a^{\#} = a(a^+)^* + a^+$. Pre-multiply and post-multiply $a^+a^3 + a^{\#} = a(a^+)^* + a^+$ by a, we have

$$a^4 = a^2(a^+)^*a.$$

Pre-multiply and post-multiply $a^4 = a^2(a^+)^*a$ by $a^{\#}$, one yields

$$a^2 = a(a^+)^*.$$

Hence, $a \in R^{PI}$ by Theorem 2.1, so $a = (a^+)^*$. Now we have

$$a^+a^3 + a^{\#} = a^2 + a^+.$$

Pre-multiply $a^+a^3 + a^{\#} = a^2 + a^+$ by a, we can deduce that $aa^{\#} = aa^+ = aa^*$. By Lemma 2.2, $a \in R^{SEP}$.

(2) If $x = a^{\#}$, then $a^+aa^{\#}a + a^{\#} = a^{\#}(a^+)^* + a^+$, that is,

$$a^+a + a^{\#} = a^{\#}(a^+)^* + a^+.$$

Pre-multiply and post-multiply $a^+a + a^{\#} = a^{\#}(a^+)^* + a^+$ by a, we have $a^2 = (a^+)^*a$. Hence, $a \in R^{PI}$ and so $a^{\#} = a^+$. Therefore, $a \in R^{SEP}$.

(3) If $x = a^+$, then $a^+aa^+a + a^{\#} = a^+(a^+)^* + a^+$, that is,

$$a^+a + a^{\#} = a^+(a^+)^* + a^+.$$

Pre-multiply and post-multiply $a^+a + a^{\#} = a^+(a^+)^* + a^+$ by a, this leads to $a^2 = (a^+)^*a$. Thus, $a \in R^{PI}$, which implies $a = (a^+)^*$. Hence, we obtain $a^{\#} = a^+$. Therefore, $a \in R^{SEP}$.

(4) If $x = a^*$, then $a^+aa^*a + a^{\#} = a^*(a^+)^* + a^+$, that is,

$$a^*a + a^{\#} = a^+a + a^+.$$

Pre-multiply and post-multiply $a^*a + a^{\#} = a^+a + a^+$ by a, this leads to $aa^*a^2 = a^2$. Post-multiply $aa^*a^2 = a^2$ by $a^{\#}$, we can deduce that $aa^*a = a$. Hence, $a \in R^{PI}$. It follows that $a^{\#} = a^+ = a^*$. Therefore, $a \in R^{SEP}$.

(5) If $x = (a^{\#})^*$, then $a^+a(a^{\#})^*a + a^{\#} = (a^{\#})^*(a^+)^* + a^+$, that is,

$$(a^{\#})^*a + a^{\#} = (a^{\#})^*(a^+)^* + a^+.$$

Pre-multiply and post-multiply $(a^{\#})^*a + a^{\#} = (a^{\#})^*(a^+)^* + a^+$ by a, we can infer

$$a(a^{\#})^*a^2 = a(a^{\#})^*(a^+)^*a.$$

Pre-multiply $a(a^{\#})^*a^2 = a(a^{\#})^*(a^+)^*a$ by a^+, one gets $(a^{\#})^*a^2 = (a^{\#})^*(a^+)^*a$. Apply the involution, one has

$$a^*a^*a^{\#} = a^*a^+a^{\#}.$$

Post-multiply $a^*a^*a^{\#} = a^*a^+a^{\#}$ by a^2a^+, one obtains $a^*a^* = a^*a^+$. This infers $a \in R^{PI}$ by [14, Corollary 2.10]. Since $a^* = a^+$, $(a^{\#})^*a + a^{\#} = (a^{\#})^*(a^*)^* + a^*$, this gives $a^{\#} = a^*$. By Lemma 2.2, $a \in R^{SEP}$.

(6) If $x = (a^+)^*$, then $a^+a(a^+)^*a + a^{\#} = (a^+)^*(a^+)^* + a^+$. Pre-multiply and post-multiply $a^+a(a^+)^*a + a^{\#} = (a^+)^*(a^+)^* + a^+$ by a, this leads to

$$a(a^+)^*a^2 = a(a^+)^*(a^+)^*a.$$

Pre-multiply and post-multiply $a(a^+)^*a^2 = a(a^+)^*(a^+)^*a$ by $a^{\#}$, we can deduce that $(a^+)^*a = (a^+)^*(a^+)^*$. Apply the involution, one has $a^+a^+ = a^*a^+$. Hence, $a \in R^{PI}$ by [14, Corollary 2.10]. Since

$$a^+a^3 + a^{\#} = a^2 + a^+.$$

Pre-multiply $a^+a^3 + a^{\#} = a^2 + a^+$ by a, one yields $aa^{\#} = aa^+ = aa^*$. By Lemma 2.2, $a \in R^{SEP}$.

Also revise the equation (2.1) as follows:

$$a^*axa = xa. \tag{5}$$

Theorem 2.8. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if there is at least one solution to the equation (2.5) in χ_a.*
Proof. $' \Rightarrow'$ Suppose that $a \in R^{PI}$, then $a^* = a^+$. Hence, we obtain $x = a^+$ is a solution.
$' \Leftarrow'$ (1) If $x = a$, then $a^*a^3 = a^2$. Post-multiply $a^*a^3 = a^2$ by $a^\#$, this leads to

$$a^*a^2 = a.$$

By Lemma 2.2, $a \in R^{PI}$.
(2) If $x = a^\#$, then $a^*aa^\#a = a^\#a$, that is,

$$a^*a = a^\#a.$$

By Lemma 2.2, $a \in R^{PI}$.
(3) If $x = a^+$, then $a^*aa^+a = a^+a$, that is,

$$a^*a = a^+a.$$

Therefore, $a \in R^{PI}$ by [3, Theorem 2.1] .
(4) If $x = a^*$, then $a^*aa^*a = a^*a$. Post-multiply $a^*aa^*a = a^*a$ by $a^+(a^+)^*$, this leads to

$$a^*a = a^+a.$$

Hence, we obtain $a \in R^{PI}$ by [3, Theorem 2.1].
(5) If $x = (a^\#)^*$, then $a^*a(a^\#)^*a = (a^\#)^*a$. Apply the involution, one has

$$a^*a^\#a^*a = a^*a^\#.$$

Pre-multiply $a^*a^\#a^*a = a^*a^\#$ by $(a^+)^*$, this leads to $a^\#a^*a = a^\#$. Hence, we obtain $a \in R^{PI}$ by [3, Theorem 2.2].
(6) If $x = (a^+)^*$, then $a^*a(a^+)^*a = (a^+)^*a$. Apply the involution, one gets

$$a^*a^+a^*a = a^*a^+.$$

Post-multiply $a^*a^+a^*a = a^*a^+$ by a^+, we get $a^*a^+a^* = a^*a^+a^+$. By Lemma 2.4, $a \in R^{PI}$.

Revising the equation (2.5) as follows:

$$a^*ax(a^+)^* = x(a^\#)^*. \tag{6}$$

Theorem 2.9. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if there is at least one solution to the equation (2.6) in χ_a.*
Proof. $' \Rightarrow'$ Suppose that $a \in R^{SEP}$, then $a^* = a^+ = a^\#$. Hence, $x = a$ is a solution.
$' \Leftarrow'$ (1) If $x = a$, then $a^*a^2(a^+)^* = a(a^\#)^*$. Apply the involution, one obtains

$$a^+a^*a^*a = a^\#a^*.$$

Post-multiply $a^+a^*a^*a = a^\#a^*$ by $(1 - a^+a)$, one gets $a^\#a^*(1 - a^+a) = 0$. Pre-multiply $a^\#a^*(1 - a^+a) = 0$ by a^+a^2, one has $a^* = a^*a^+a$. Apply the involution, one gets $a^+a^2 = a$. Hence, we obtain $a \in R^{EP}$ by [11, Lemma 2.1]. It follows that $a^*a^2(a^+)^* = a(a^+)^*$. Post-multiply by $a^*a^\#a$, we can infer $a^*a^2 = a$. Therefore, $a \in R^{SEP}$ by Lemma 2.2.
(2) If $x = a^\#$, then $a^*aa^\#(a^+)^* = a^\#(a^\#)^*$. Post-multiply $a^*aa^\#(a^+)^* = a^\#(a^\#)^*$ by a^*a^+, we can deduce that

$$a^*a^+ = a^\#a^+.$$

By Lemma 2.2, $a \in R^{SEP}$.
(3) If $x = a^+$, then $a^*aa^+(a^+)^* = a^+(a^\#)^*$, that is,

$$a^+a = a^+(a^\#)^*.$$

Post-multiply $a^+a = a^+(a^\#)^*$ *by* a^**, we can deduce that* $a^* = a^+$*. Hence, we obtain* $a \in R^{PI}$*. Since* $a^*a = a^*(a^\#)^*$*. Apply the involution, we can deduce that* $a^*a = a^\# a$*. By Theorem 2.4,* $a \in R^{SPE}$*.*
(4) If $x = a^*$*, then* $a^*aa^*(a^+)^* = a^*(a^\#)^*$*, that is,*

$$a^*a = a^*(a^\#)^*.$$

By (3) $a \in R^{SEP}$ *.*
(5) If $x = (a^\#)^*$*, then* $a^*a(a^\#)^*(a^+)^* = (a^\#)^*(a^\#)^*$*. Apply the involution, one gets*

$$a^+a^\#a^*a = a^\#a^\#.$$

Pre-multiply $a^+a^\#a^*a = a^\#a^\#$ *by* a^3*, this leads to* $aa^*a = a$*. Thus,* $a \in R^{PI}$*. Since*

$$a^*a(a^\#)^*(a^*)^* = (a^\#)^*(a^\#)^*.$$

Apply the involution, one obtains $a^*a^\#a^*a = a^\#a^\#$*. Pre-multiply* $a^*a^\#a^*a = a^\#a^\#$ *by* a^+a*, one has* $a^\#a^\# = a^+a^\#$*. Hence,* $a \in R^{EP}$*. Therefore,* $a \in R^{SEP}$*.*
(6) If $x = (a^+)^*$*, then* $a^*a(a^+)^*(a^+)^* = (a^+)^*(a^\#)^*$*. Apply the involution, one has*

$$a^+a^+a^*a = a^\#a^+.$$

Post-multiply $a^+a^+a^*a = a^\#a^+$ *by* $(1 - a^+a)$*, one gets* $a^\#a^+(1 - a^+a) = 0$*. Pre-multiply* $a^\#a^+(1 - a^+a) = 0$ *by* a^+a^2*, this leads to* $a^+(1 - a^+a) = 0$*. Hence, we obtain* $a \in R^{EP}$ *by [11, Lemma 2.1]. Post-multiply* $a^*a(a^+)^*(a^+)^* = (a^+)^*(a^+)^*$ *by* $a^*aa^\#a^*a$*, one obtains* $a^*a^2 = a$*. By Lemma 2.2,* $a \in R^{SEP}$*.*

We modify the equation (2.2) as follows:

$$x(a^+)^* = aa^+xaa^*(a^+)^*. \tag{7}$$

Now, we change the equation (2.7) as follows:

$$a^+axaa^*y = xy. \tag{8}$$

Theorem 2.10. *Let* $a \in R^\# \cap R^+$*. Then* $a \in R^{PI}$ *if and only if there is at least one solution to the equation (2.8) in* $\chi_a^2 = \{(x, y)|x, y \in \chi_a\}$*.*
Proof. *′ ⇒′ Suppose that* $a \in R^{PI}$*, then* $a^+ = a^*$*, we have* $(x, y) = (a^+, a)$ *is a solution.*
′ ⇐′ (1) If $(x, y) = (a, a)$*, then* $a^+a^3a^*a = a^2$*. Pre-multiply* $a^+a^3a^*a = a^2$ *by* $a^\#$*, this leads to* $aa^*a = a$*. Thus* $a \in R^{PI}$*.*
(2) If $(x, y) = (a^\#, a)$*, then* $a^+aa^\#aa^*a = a^\#a$*, that is,*

$$a^*a = a^\#a.$$

By Theorem 2.4, $a \in R^{PI}$*.*
(3) If $(x, y) = (a^+, a)$*, then* $a^+aa^+aa^*a = a^+a$*, that is,*

$$a^*a = a^+a.$$

Hence, we obtain $a \in R^{PI}$ *by [3, Theorem 2.1].*
(4) If $(x, y) = (a^*, a)$*, then* $a^+aa^*aa^*a = a^*a$*, that is,*

$$a^*aa^*a = a^*a.$$

By Theorem 2.4, $a \in R^{PI}$*.*
(5) If $(x, y) = ((a^\#)^*, a)$*, then* $a^+a(a^\#)^*aa^*a = (a^\#)^*a$*, that is,*

$$(a^\#)^*aa^*a = (a^\#)^*a.$$

Pre-multiply $(a^\#)^*aa^*a = (a^\#)^*a$ *by* aa^+a^**, this leads to* $aa^*a = a$*. Hence, we obtain* $a \in R^{PI}$*.*
(6) If $(x, y) = ((a^+)^*, a)$*, then* $a^+a(a^+)^*aa^*a = (a^+)^*a$*, Pre-multiply* $a^+a(a^+)^*aa^*a = (a^+)^*a$ *by* $a^\#aa^*a^\#a$*, this leads to*

$$aa^*a = a.$$

Hence, we obtain $a \in R^{PI}$.
(7) *If* $(x, y) = (a, a^{\#})$, *then* $a^+a^3a^*a^{\#} = aa^{\#}$. *Post-multiply* $a^+a^3a^*a^{\#} = aa^{\#}$ *by* a^2, *one has*

$$a^+a^3a^*a = a^2.$$

Pre-multiply $a^+a^3a^*a = a^2$ *by* $a^{\#}$, *this leads to* $aa^*a = a$. *Hence, we obtain* $a \in R^{PI}$.
(8) *If* $(x, y) = (a^{\#}, a^{\#})$, *then* $a^+aa^{\#}aa^*a^{\#} = a^{\#}a^{\#}$, *that is,*

$$a^*a^{\#} = a^{\#}a^{\#}.$$

By Lemma 2.2, $a \in R^{PI}$.
(9) *If* $(x, y) = (a^+, a^{\#})$, *then* $a^+aa^+aa^*a^{\#} = a^+a^{\#}$, *that is,*

$$a^*a^{\#} = a^+a^{\#}.$$

Hence, $a \in R^{PI}$ *by [3, Theorem 2.2].*
(10) *If* $(x, y) = (a^*, a^{\#})$, *then* $a^+aa^*aa^*a^{\#} = a^*a^{\#}$, *that is,*

$$a^*aa^*a^{\#} = a^*a^{\#}.$$

Post-multiply $a^*aa^*a^{\#} = a^*a^{\#}$ *by* a^2, *this leads to* $a^*aa^*a = a^*a$. *By Theorem 2.4,* $a \in R^{PI}$.
(11) *If* $(x, y) = ((a^{\#})^*, a^{\#})$, *then* $a^+a(a^{\#})^*aa^*a^{\#} = (a^{\#})^*a^{\#}$, *that is,*

$$(a^{\#})^*aa^*a^{\#} = (a^{\#})^*a^{\#}.$$

Post-multiply $(a^{\#})^*aa^*a^{\#} = (a^{\#})^*a^{\#}$ *by* a^2, *we can deduce that* $(a^{\#})^*aa^*a = (a^{\#})^*a$. *Apply the involution, one obtains* $a^*aa^*a^{\#} = a^*a^{\#}$. *Pre-multiply* $a^*aa^*a^{\#} = a^*a^{\#}$ *by* $(a^+)^*$, *we have that* $aa^*a^{\#} = a^{\#}$. *Hence,* $a \in R^{PI}$ *by [3, Theorem 2.2].*
(12) *If* $(x, y) = ((a^+)^*, a^{\#})$, *then* $a^+a(a^+)^*aa^*a^{\#} = (a^+)^*a^{\#}$. *Post-multiply* $a^+a(a^+)^*aa^*a^{\#} = (a^+)^*a^{\#}$ *by* a^2, *this leads to*

$$a^+a(a^+)^*aa^*a = (a^+)^*a.$$

Pre-multiply $a^+a(a^+)^*aa^*a = (a^+)^*a$ *by* $a^{\#}aa^*a^{\#}a$, *we get* $aa^*a = a$. *Hence, we obtain* $a \in R^{PI}$.
(13) *If* $(x, y) = (a, a^+)$, *then* $a^+a^3a^*a^+ = aa^+$. *Pre-multiply* $a^+a^3a^*a^+ = aa^+$ *by* a^+a, *one has* $aa^+ = a^+a^2a^+$. *So, we have*

$$a = aa^+a = a^+a^2a^+a = a^+a^2.$$

Hence, $a \in R^{EP}$ *by Lemma 2.1, so* $a^+ = a^{\#}$. *This gives* $a^2a^*a^{\#} = aa^{\#}$. *Pre-multiply by* $a^{\#}$, *one has* $aa^*a^{\#} = a^{\#}$. *Therefore,* $a \in R^{PI}$ *by [3, Theorem 2.2].*
(14) *If* $(x, y) = (a^{\#}, a^+)$, *then* $a^+aa^{\#}aa^*a^+ = a^{\#}a^+$, *that is,*

$$a^*a^+ = a^{\#}a^+.$$

By Lemma 2.2, $a \in R^{PI}$.
(15) *If* $(x, y) = (a^+, a^+)$, *then* $a^+aa^+aa^*a^+ = a^+a^+$, *that is,*

$$a^*a^+ = a^+a^+.$$

Therefore, $a \in R^{PI}$ *by [14, Corollary 2.10].*
(16) *If* $(x, y) = (a^*, a^+)$, *then* $a^+aa^*aa^*a^+ = a^*a^+$, *that is,*

$$a^*aa^*a^+ = a^*a^+.$$

Apply the involution, one has $(a^+)^*aa^*a = (a^+)^*a$. *Pre-multiply* $(a^+)^*aa^*a = (a^+)^*a$ *by* $a^{\#}aa^*$, *this leads to* $aa^*a = a$. *Hence, we obtain* $a \in R^{PI}$.
(17) *If* $(x, y) = ((a^{\#})^*, a^+)$, *then* $a^+a(a^{\#})^*aa^*a^+ = (a^{\#})^*a^+$, *that is,*

$$(a^{\#})^*aa^*a^+ = (a^{\#})^*a^+.$$

Apply the involution, one gets $(a^+)^*aa^*a^\# = (a^+)^*a^\#$. Pre-multiply $(a^+)^*aa^*a^\# = (a^+)^*a^\#$ by $aa^\#a^*$, we get $aa^*a^\# = a^\#$. Hence, we obtain $a \in R^{PI}$ by [3, Theorem 2.2].

(18) If $(x, y) = ((a^+)^*, a^+)$, then $a^+a(a^+)^*aa^*a^+ = (a^+)^*a^+$. Apply the involution, one has $(a^+)^*aa^*a^+a^+a = (a^+)^*a^+$. Pre-multiply by $a^+a^\#aa^*$, one gives $a^*a^+a^+a = a^+a^\#aa^+$. Apply the involution, one yields $a^+a(a^+)^*a = aa^+(a^\#)^*(a^+)^*$. Pre-multiply by $aa^\#$, this leads to $a^+a(a^+)^*a = aa^\#(a^+)^*a$. Post-multiply by $a^\#a^*$, one gets $a^+a^2a^+ = aa^+$. Hence, by (13) $a \in R^{EP}$. So, we have

$$a^\#a(a^\#)^*aa^*a^\# = (a^\#)^*a^\#.$$

Pre-multiply by $a^+a^*a^+a$, one obtains $a^*a^\# = a^+a^\#$. Therefore, $a \in R^{PI}$ by Lemma 2.2.

(19) If $(x, y) = (a, a^*)$, then $a^+a^3a^*a^* = aa^*$. Post-multiply $a^+a^3a^*a^* = aa^*$ by $(a^+)^*$, we can get

$$a^+a^3a^*a^+a = a.$$

Pre-multiply $a^+a^3a^*a^+a = a$ by $a^\#a$, one gives $a = a^2a^*a^+a = a^2a^*$. Therefore, $a \in R^{PI}$ by Lemma 2.2.

(20) If $(x, y) = (a^\#, a^*)$, then $a^+aa^\#aa^*a^* = a^\#a^*$, that is,

$$a^*a^* = a^\#a^*.$$

Post-multiply $a^*a^* = a^\#a^*$ by $(a^+)^*$, this leads to $a^*a^+a = a^\#$. By Lemma 2.2, $a \in R^{PI}$.

(21) If $(x, y) = (a^+, a^*)$, then $a^+aa^+aa^*a^* = a^+a^*$, that is,

$$a^*a^* = a^+a^*.$$

Therefore, $a \in R^{PI}$ by [14, Corollary 2.10].

(22) If $(x, y) = (a^*, a^*)$, then $a^+aa^*aa^*a^* = a^*a^*$, that is,

$$a^*aa^*a^* = a^*a^*.$$

Apply the involution, one has $a^2a^*a = a^2$. Pre-multiply $a^2a^*a = a^2$ by $a^\#$, we can get $aa^*a = a$. Hence, we obtain $a \in R^{PI}$.

(23) If $(x, y) = ((a^\#)^*, a^*)$, then $a^+a(a^\#)^*aa^*a^* = (a^\#)^*a^*$, that is,

$$(a^\#)^*aa^*a^* = (a^\#)^*a^*.$$

Apply the involution, one gets $a^2a^*a^\# = aa^\#$. Pre-multiply $a^2a^*a^\# = aa^\#$ by $a^+a^\#$, this gives $a^*a^\# = a^+a^\#$. Hence, we obtain $a \in R^{PI}$ by [3, Theorem 2.2].

(24) If $(x, y) = ((a^+)^*, a^*)$, then $a^+a(a^+)^*aa^*a^* = (a^+)^*a^*$. Apply the involution, one obtains $a^2a^*a^+a^+a = aa^+$. Post-multiply by $a^\#a$, one has $aa^+ = aa^+a^\#a = a^\#a$. Hence, $a \in R^{EP}$ by [11, Lemma 2.1]. So, we have

$$a^\#a(a^\#)^*aa^*a^* = (a^\#)^*a^*.$$

Pre-multiply by a^+a, one gets $(a^\#)^*aa^*a^* = (a^\#)^*a^*$. Apply the involution, one obtains $a^2a^*a^\# = aa^\#$. Pre-multiply by $a^\#$, one yields $aa^*a^\# = a^\#$. Therefore, $a \in R^{PI}$ by [3, Theorem 2.2].

(25) If $(x, y) = (a, (a^\#)^*)$, then

$$a^+a^3a^*(a^\#)^* = a(a^\#)^*.$$

Post-multiply a^+, one has $a^+a^3a^+ = a(a^\#)^*a^+$. Apply the involution, one gives $aa^+a^*a^+a = (a^+)^*a^\#a^*$. Pre-multiply by a^*, one gets $a^*a^*a^+a = a^+aa^\#a^*$. Post-multiply by aa^+, one obtains

$$a^*a^*a^+a = a^*a^*a^+a^2a^+.$$

Apply the involution, one yields $a^+a^3 = aa^+a^+a^3$. Post-multiply by $(a^\#)^2$, this leads to $a^+a = aa^+a^+a$. Hence, $a \in R^{EP}$. So, we have

$$a^+a^3 = a(a^+)^*.$$

Pre-multiply by $a^{\#}$*, this gives* $a = (a^+)^*$*. Therefore* $a \in R^{PI}$*.*
(26) If $(x, y) = (a^{\#}, (a^{\#})^*)$*, then* $a^+aa^{\#}aa^*(a^{\#})^* = a^{\#}(a^{\#})^*$*, that is,*

$$a^*(a^{\#})^* = a^{\#}(a^{\#})^*.$$

Post-multiply $a^*(a^{\#})^* = a^{\#}(a^{\#})^*$ *by* a^*a^+*, this leads to* $a^*a^+ = a^{\#}a^+$*. By Lemma 2.2,* $a \in R^{PI}$*.*
(27) If $(x, y) = (a^+, (a^{\#})^*)$*, then* $a^+aa^+aa^*(a^{\#})^* = a^+(a^{\#})^*$*, that is,*

$$a^*(a^{\#})^* = a^+(a^{\#})^*.$$

Post-multiply $a^*(a^{\#})^* = a^+(a^{\#})^*$ *by* a^*a^+*, this leads to* $a^*a^+ = a^+a^+$*. Hence, we obtain* $a \in R^{PI}$ *by [14, Corollary 2.10].*
(28) If $(x, y) = (a^*, (a^{\#})^*)$*, then* $a^+aa^*aa^*(a^{\#})^* = a^*(a^{\#})^*$*, that is,*

$$a^*aa^*(a^{\#})^* = a^*(a^{\#})^*.$$

Apply the involution, one gets $a^{\#}aa^*a = a^{\#}a$*. Pre-multiply* $a^{\#}aa^*a = a^{\#}a$ *by* a*, this leads to* $aa^*a = a$*. Thus,* $a \in R^{PI}$*.*
(29) If $(x, y) = ((a^{\#})^*, (a^{\#})^*)$*, then* $a^+a(a^{\#})^*aa^*(a^{\#})^* = (a^{\#})^*(a^{\#})^*$*, that is,*

$$(a^{\#})^*aa^*(a^{\#})^* = (a^{\#})^*(a^{\#})^*.$$

Apply the involution, one obtains $a^{\#}aa^*a^{\#} = a^{\#}a^{\#}$*. Pre-multiply* $a^{\#}aa^*a^{\#} = a^{\#}a^{\#}$ *by* a*, this leads to* $aa^*a^{\#} = a^{\#}$*. Hence, we obtain* $a \in R^{PI}$ *by [3, Theorem 2.2].*
(30) If $(x, y) = ((a^+)^*, (a^{\#})^*)$*, then*

$$a^+a(a^+)^*aa^*(a^{\#})^* = (a^+)^*(a^{\#})^*.$$

Post-multiply by a^*a^+*, this leads to* $a^+a(a^+)^*aa^*a^+ = (a^+)^*a^+$*. Pre-multiply by* $a^{\#}a$*, one has* $a^+a(a^+)^*aa^*a^+ = (a^+)^*aa^*a^+$*. Apply the involution, one gets* $(a^+)^*aa^*a^+a^+a = (a^+)^*aa^*a^+$*. Pre-multiply by* $a^+a^{\#}aa^*$*, one yields* $a^*a^+a^+a = a^*a^+$*. Apply the involution, one obtains* $a^+a(a^+)^*a = (a^+)^*a$*. Post-multiply by* $a^{\#}a^*$*, one gives* $a^+a^2a^+ = a^+a$*. Hence,* $a \in R^{EP}$*. So, we have*

$$a^+a(a^+)^*aa^*(a^+)^* = (a^+)^*(a^+)^*,$$

that is, $a^+a(a^+)^*a = (a^+)^*(a^+)^*$*. Apply the involution, this gives* $a^*a^+a^+a = a^+a^+$*. Post-multiply by* a^+*, we have* $a^*a^+a^+ = a^+a^+a^+$*. Therefore,* $a \in R^{PI}$ *by Lemma 2.4.*
(31) If $(x, y) = (a, (a^+)^*)$*, then* $a^+a^3a^*(a^+)^* = a(a^+)^*$*, that is,*

$$a^+a^3 = a(a^+)^*.$$

Hence, we get $a \in R^{PI}$ *by (25).*
(32) If $(x, y) = (a^{\#}, (a^+)^*)$*, then* $a^+aa^{\#}aa^*(a^+)^* = a^{\#}(a^+)^*$*, that is,*

$$a^+a = a^{\#}(a^+)^*.$$

Post-multiply $a^+a = a^{\#}(a^+)^*$ *by* a^**, this leads to* $a^* = a^{\#}aa^+$*. Pre-multiply* $a^* = a^{\#}aa^+$ *by* a*, we get* $aa^* = aa^+$*. Thus,* $a \in R^{PI}$ *by [3, Theorem 2.1].*
(33) If $(x, y) = (a^+, (a^+)^*)$*, then* $a^+aa^+aa^*(a^+)^* = a^+(a^+)^*$*, that is,*

$$a^+a = a^+(a^+)^*.$$

Post-multiply $a^+a = a^+(a^+)^*$ *by* a^**, we have* $a^* = a^+$*. Hence, we obtain* $a \in R^{PI}$*.*
(34) If $(x, y) = (a^*, (a^+)^*)$*, then* $a^+aa^*aa^*(a^+)^* = a^*(a^+)^*$*, that is,*

$$a^*a = a^+a.$$

Hence, we obtain $a \in R^{PI}$ *by [3, Theorem 2.1].*
(35) If $(x, y) = ((a^{\#})^*, (a^+)^*)$*, then* $a^+a(a^{\#})^*aa^*(a^+)^* = (a^{\#})^*(a^+)^*$*, that is*

$$(a^{\#})^*a = (a^{\#})^*(a^+)^*.$$

Apply the above involution, this leads to $a^*a^\# = a^+a^\#$. Thus, $a \in R^{PI}$ by [3, Theorem 2.2].
(36) If $(x, y) = ((a^+)^*, (a^+)^*)$, then $a^+a(a^+)^*aa^*(a^+)^* = (a^+)^*(a^+)^*$, that is

$$a^+a(a^+)^*a = (a^+)^*(a^+)^*.$$

Pre-multiply $a^+a(a^+)^*a = (a^+)^*(a^+)^*$ by $aa^*a^\#a$, this leads to $a^2 = a(a^+)^*$. Hence, we obtain $a \in R^{PI}$ by Theorem 2.1.

Revised the equation (2.8) as follows:

$$axa^*y + a^\# = xy + a^+. \tag{9}$$

Theorem 2.11. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if there is at least one solution to the equation (2.9) in χ_a^2.*
Proof. $'\Rightarrow'$ Suppose that $a \in R^{SEP}$, then $a^+ = a^* = a^\#$, we have $(x, y) = (a, a^+)$ is a solution.
$'\Leftarrow'$ (1) If $(x, y) = (a, a)$, then $a^2a^*a + a^\# = a^2 + a^+$. Pre-multiply and post-multiply $a^2a^*a + a^\# = a^2 + a^+$ by a, one gets

$$a^3a^*a^2 = a^4.$$

Pre-multiply $a^3a^*a^2 = a^4$ by $a^\#a^+$, this leads to $aa^*a^2 = a^2$. Post-multiply by $a^\#$, one obtains $aa^*a = a$ Hence, $a \in R^{PI}$. Since $a^2a^+a + a^\# = a^2 + a^+$, this gives $a^\# = a^+ = a^*$. Therefore, $a \in R^{SEP}$.
(2) If $(x, y) = (a^\#, a)$, then $aa^\#a^*a + a^\# = a^\#a + a^+$. Pre-multiply and post-multiply $aa^\#a^*a + a^\# = a^\#a + a^+$ by a, one gets

$$aa^*a^2 = a^2.$$

Pre-multiply $aa^*a^2 = a^2$ by a^+, this leads to $a^*a^2 = a^+a^2$. By Lemma 2.1, $a \in R^{PI}$. It follows from $aa^\#a^*a + a^\# = a^\#a + a^+$ that $aa^\#a^+a + a^\# = a^\#a + a^+$. This gives $a^\# = a^+$ because $a^\#a^+a = a^\#$. Therefore, we obtain $a \in R^{SEP}$.
(3) If $(x, y) = (a^+, a)$, then $aa^+a^*a + a^\# = a^+a + a^+$. Post-multiply by a^+a, one has $a^+ = a^+a^+a$. Hence, we obtain $a \in R^{EP}$ by [11, Lemma 2.1]. So, we have

$$aa^+a^*a = a^+a.$$

Post-multiply by a^+, one gets $aa^+a^* = a^+$. Therefore, $a \in R^{SEP}$ by Lemma 2.2.
(4) If $(x, y) = (a^*, a)$, then $aa^*a^*a + a^\# = a^*a + a^+$. Post-multiply by a^+a, one has $a^+ = a^+a^+a$. Hence, $a \in R^{EP}$ by [11, Lemma 2.1]. This gives

$$aa^*a^*a = a^*a.$$

Post-multiply by a^+, one gets $aa^*a^* = a^*$. Apply the involution, one obtains $a^2a^* = a$. Therefore, $a \in R^{SEP}$ by Lemma 2.2.
(5) If $(x, y) = ((a^\#)^*, a)$, then $a(a^\#)^*a^*a + a^\# = (a^\#)^*a + a^+$. Post-multiply by a^+a, one yields $a^+ = a^+a^+a$. Hence, one obtains $a \in R^{EP}$ by [11, Lemma 2.1]. So, one has

$$a(a^\#)^*a^*a = (a^\#)^*a.$$

Post-multiply by a^+, one obtains $a(a^\#)^*a^* = (a^\#)^*$. Apply the involution, one gets $aa^\#a^* = a^\#$. Pre-multiply by a, one gives $aa^* = aa^\#$. Therefore, $a \in R^{SEP}$ by Lemma 2.2.
(6) If $(x, y) = ((a^+)^*, a)$, then $a(a^+)^*a^*a + a^\# = (a^+)^*a + a^+$, that is, $a^2 + a^\# = (a^+)^*a + a^+$. Pre-multiply and post-multiply $a^2 + a^\# = (a^+)^*a + a^+$ by a, one gets

$$a^4 = a(a^+)^*a^2.$$

Post-multiply $a^4 = a(a^+)^*a^2$ by $(a^\#)^2a^*$, this leads to $a^2a^* = a^2a^+$. By Lemma 2.1, $a \in R^{PI}$. It follows that $a^2 + a^\# = (a^*)^*a + a^*$. Post-multiply by a^+a, one obtains $a^* = a^*a^+a$. hence $a = a^+a^2$ and so $a \in R^{EP}$. Therefore, $a \in R^{SEP}$.
(7) If $(x, y) = (a, a^\#)$, then $a^2a^*a^\# + a^\# = aa^\# + a^+$. Pre-multiply and post-multiply by a, one gets

$$a^3a^*a^\#a = a^2.$$

Post-multiply $a^3a^*a^\#a = a^2$ by aa^+, this leads to $a^3a^* = a^3a^+$. By Lemma 2.1, $a \in R^{PI}$. It follows that $a^2a^+a^\# + a^\# = aa^\# + a^+$. we can infer $a^\# = a^+$. Hence, we obtain $a \in R^{EP}$. Therefore, $a \in R^{SEP}$.

(8) If $(x, y) = (a^\#, a^\#)$, then $aa^\#a^*a^\# + a^\# = a^\#a^\# + a^+$. Pre-multiply and post-multiply $aa^\#a^*a^\# + a^\# = a^\#a^\# + a^+$ by a, one gets

$$aa^*a^\#a = a^\#a.$$

Post-multiply $aa^*a^\#a = a^\#a$ by a, this leads to $aa^*a = a$. Hence, we obtain $a \in R^{PI}$. This gives $aa^\#a^+a^\# + a^\# = a^\#a^\# + a^+$. Pre-multiply $aa^\#a^+a^\# + a^\# = a^\#a^\# + a^+$ by a^2, one obtains $a = a^2a^+$. Therefore, $a \in R^{SEP}$ by [11, Lemma 2.1].
(9) If $(x, y) = (a^+, a^\#)$, then $aa^+a^*a^\# + a^\# = a^+a^\# + a^+$. Post-multiply by a^+a, one yields $a^+ = a^+a^+a$. Hence, $a \in R^{EP}$ by [11, Lemma 2.1]. It follows that

$$aa^+a^*a^\# = a^+a^\#.$$

Post-multiply by a^2a^+, one has $aa^+a^* = a^+$. Therefore, $a \in R^{SEP}$ by Lemma 2.2.
(10) If $(x, y) = (a^*, a^\#)$, then $aa^*a^*a^\# + a^\# = a^*a^\# + a^+$. Post-multiply by a^+a, one gets $a^+ = a^+a^+a$. Hence, $a \in R^{EP}$ by [11, Lemma 2.1]. It follows that

$$aa^*a^*a^\# = a^*a^\#.$$

Post-multiply by a^2a^+, one obtains $aa^*a^* = a^*$. Apply the involution, one gives $a^2a^* = a$. Therefore, $a \in R^{SEP}$ by Lemma 2.2.
(11) If $(x, y) = ((a^\#)^*, a^\#)$, then $a(a^\#)^*a^*a^\# + a^\# = (a^\#)^*a^\# + a^+$. Post-multiply by a^+a, one obtains $a^+ = a^+a^+a$. Thus, $a \in R^{EP}$ by [11, Lemma 2.1]. This leads to

$$a(a^\#)^*a^*a^\# = (a^\#)^*a^\#.$$

Noting that $a^+a(a^\#)^* = (a^\#)^*$. Pre-multiply the above equality by a^+, one gets $(a^\#)^*a^*a^\# = a^+(a^\#)^*a^\#$. Post-multiply by a^2a^+, one yields $(a^\#)^*a^* = a^+(a^\#)^*$. Apply the involution, one gives $aa^\# = a^\#(a^+)^*$. Post-multiply by a^*, one obtains $aa^\#a^* = a^\#aa^+$. Pre-multiply by a, one has $aa^* = aa^+$. Hence, we obtain $a \in R^{PI}$ by [3, Theorem 2.1]. Therefore, $a \in R^{SEP}$.
(12) If $(x, y) = ((a^+)^*, a^\#)$, then $a(a^+)^*a^*a^\# + a^\# = (a^+)^*a^\# + a^+$, that is,

$$aa^\# + a^\# = (a^+)^*a^\# + a^+.$$

Pre-multiply and post-multiply $aa^\# + a^\# = (a^+)^*a^\# + a^+$ by a, one gets $a^2 = a(a^+)^*a^\#a$. Post-multiply $a^2 = a(a^+)^*a^\#a$ by $aa^\#a^*$, one yields $a^2a^* = a^2a^+$. By Lemma 2.1, $a \in R^{PI}$. This gives $aa^\# + a^\# = (a^+)^+a^\# + a^+$, that is, $a^\# = a^+$. Hence, we obtain $a \in R^{EP}$. Therefore, $a \in R^{SEP}$.
(13) If $(x, y) = (a, a^+)$, then $a^2a^*a^+ + a^\# = aa^+ + a^+$. Post-multiply by aa^+, one has $a^\# = a^\#aa^+$. Pre-multiply by a, one gets $aa^\# = aa^+$. Hence, we obtain $a \in R^{EP}$ by [11, Lemma 2.1], so $a^+ = a^\#$. It follows that

$$a^2a^*a^\# = aa^\#.$$

Post-multiply by a^2a^+, one yields $a^2a^* = a^2a^+$. Thus, $a \in R^{PI}$ by Lemma 2.1. Therefore, $a \in R^{SEP}$.
(14) If $(x, y) = (a^\#, a^+)$, then $aa^\#a^*a^+ + a^\# = a^\#a^+ + a^+$. Post-multiply by aa^+, one has $a^\# = a^\#aa^+$. Hence, we obtain $a \in R^{EP}$ by (13). It follows that

$$aa^\#a^*a^\# = a^\#a^\#.$$

Pre-multiply by a, one yields $aa^*a^\# = a^\#$. Thus, $a \in R^{PI}$ by Lemma 2.1. Therefore, $a \in R^{SEP}$.
(15) If $(x, y) = (a^+, a^+)$, then $aa^+a^*a^+ + a^\# = a^+a^+ + a^+$. Post-multiply by aa^+, one has $a^\# = a^\#aa^+$. Hence, we obtain $a \in R^{EP}$ by (13). It follows that

$$aa^+a^*a^+ = a^+a^+.$$

Since $a \in R^{EP}$, $aa^\#a^*a^\# = a^\#$. Hence, $a \in R^{SEP}$ by (14).
(16) If $(x, y) = (a^*, a^+)$, then $aa^*a^*a^+ + a^\# = a^*a^+ + a^+$. Post-multiply by aa^+, one has $a^\# = a^\#aa^+$. Hence, we obtain $a \in R^{EP}$ by (13). It follows that

$$aa^*a^*a^+ = a^*a^+.$$

Pre-multiply by a^+, one yields $a^*a^*a^+ = a^+a^*a^+$. Apply the involution, one gets $(a^+)^*a^2 = (a^+)^*a(a^+)^*$. Pre-multiply by $a^\#a^*$, this leads to $a = (a^+)^*$. Thus, $a \in R^{PI}$. Therefore, $a \in R^{EP}$.
(17) If $(x, y) = ((a^\#)^*, a^+)$, then $a(a^\#)^*a^*a^+ + a^\# = (a^\#)^*a^+ + a^+$. Post-multiply by aa^+, one obtains $a^\# = a^\#aa^+$. Hence, we obtain $a \in R^{EP}$ by (13). So, we have

$$a(a^\#)^*a^*a^+ = (a^\#)^*a^+.$$

Pre-multiply by a^+*, one gets* $(a^\#)^*a^*a^+ = a^+(a^\#)^*a^+$*. Apply the involution, one gives* $(a^+)^*aa^\# = (a^+)^*a^\#(a^+)^*$*. Pre-multiply by* aa^**, this leads to* $a = aa^\#(a^+)^*$*. Post-multiply by* a^**, which implies that* $aa^* = aa^+$*. Thus,* $a \in R^{PI}$ *by [3, Theorem 2.1]. Therefore,* $a \in R^{SEP}$*.*
(18) If $(x, y) = ((a^+)^*, a^+)$*, then* $a(a^+)^*a^*a^+ + a^\# = (a^+)^*a^+ + a^+$*. Post-multiply by* aa^+*, one has* $a^\# = a^\#aa^+$*. Hence, we obtain* $a \in R^{EP}$ *by (13). It follows that* $a^2a^+a^+ = (a^+)^*a^+$*, that is,*

$$aa^\# = (a^\#)^*a^\#.$$

Post-multiply by a^2*, one has* $a^2 = (a^\#)^*a$*. Apply the involution, one gives* $a^*a^* = a^*a^\#$*. Pre-multiply by* $(a^+)^*$*, one yields* $aa^+a^* = a^\#$*. Therefore,* $a \in R^{SEP}$ *by Lemma 2.2.*
(19) If $(x, y) = (a, a^*)$*, then* $a^2a^*a^* + a^\# = aa^* + a^+$*. Post-multiply by* aa^+*, one has* $a^\# = a^\#aa^+$*. Hence, we obtain* $a \in R^{EP}$ *by (13). It follows that*

$$a^2a^*a^* = aa^*.$$

Post-multiply by $(a^+)^*$*, one gives* $a = a^2a^*a^+a = a^2a^*$*. Therefore,* $a \in R^{SEP}$ *by Lemma 2.2.*
(20) If $(x, y) = (a^\#, a^*)$*, then* $aa^\#a^*a^* + a^\# = a^\#a^* + a^+$*. Post-multiply by* aa^+*, one has* $a^\# = a^\#aa^+$*. Hence, we obtain* $a \in R^{EP}$ *by (13). It follows that*

$$aa^+a^*a^* = a^+a^*.$$

Pre-multiply by a^+*, one yields* $a^+a^*a^* = a^+a^+a^*$*. Hence,* $a \in R^{PI}$ *by Lemma 2.4. Therefore,* $a \in R^{SEP}$*.*
(21) If $(x, y) = (a^+, a^*)$*, then* $aa^+a^*a^* + a^\# = a^+a^* + a^+$*. Post-multiply by* aa^+*, one has* $a^\# = a^\#aa^+$*. Hence, we obtain* $a \in R^{EP}$ *by (13). It follows that*

$$aa^+a^*a^* = a^+a^*.$$

Therefore, $a \in R^{SEP}$ *by (20).*
(22) If $(x, y) = (a^*, a^*)$*, then* $aa^*a^*a^* + a^\# = a^*a^* + a^+$*. Post-multiply by* aa^+*, one has* $a^\# = a^\#aa^+$*. Hence, we obtain* $a \in R^{EP}$ *by (13). It follows that*

$$aa^*a^*a^* = a^*a^*.$$

Apply the involution, one obtains $a^3a^* = a^2$*. Pre-multiply by* $a^\#$*, this leads to* $a^2a^* = a$*. Therefore,* $a \in R^{SEP}$*.*
(23) If $(x, y) = ((a^\#)^*, a^*)$*, then* $a(a^\#)^*a^*a^* + a^\# = (a^\#)^*a^* + a^+$*. Post-multiply by* aa^+*, one obtains* $a^\# = a^\#aa^+$*. Hence, we obtain* $a \in R^{EP}$ *by (13). So, we have*

$$a(a^\#)^*a^*a^* = (a^\#)^*a^*.$$

Apply the involution, one gives $aa^* = aa^\#$*. Therefore,* $a \in R^{SEP}$ *by [3, Theorem 2.3].*
(24) If $(x, y) = ((a^+)^*, a^*)$*, then* $a(a^+)^*a^*a^* + a^\# = (a^+)^*a^* + a^+$*. Post-multiply by* aa^+*, one obtains* $a^\# = a^\#aa^+$*. Hence, we obtain* $a \in R^{EP}$ *by (13). So, we have* $a(a^+)^*a^*a^* = (a^+)^*a^*$*, that is,*

$$a(a^\#)^*a^*a^* = (a^\#)^*a^*.$$

Apply the involution, one gets $aa^* = aa^\#$*. Therefore,* $a \in R^{SEP}$ *by [3, Theorem 2.3].*
(25) If $(x, y) = (a, (a^\#)^*)$*, then* $a^2a^*(a^\#)^* + a^\# = a(a^\#)^* + a^+$*. Post-multiply by* aa^+*, one obtains* $a^\# = a^\#aa^+$*. Hence, we obtain* $a \in R^{EP}$ *by (13). So, we have* $a^2a^*(a^+)^* = a(a^+)^*$*, that is,*

$$a^2 = a(a^+)^*.$$

Pre-multiply by $a^\#$*, one has* $a = (a^+)^*$*. Hence, we obtain* $a \in R^{PI}$*. Therefore,* $a \in R^{SEP}$*.*
(26) If $(x, y) = (a^\#, (a^\#)^*)$*, then* $aa^\#a^*(a^\#)^* + a^\# = a^\#(a^\#)^* + a^+$*. Post-multiply by* aa^+*, one obtains* $a^\# = a^\#aa^+$*. Hence, we obtain* $a \in R^{EP}$ *by (13). So, we have* $aa^+a^*(a^+)^* = a^+(a^+)^*$*, that is,*

$$aa^+a^+a = a^+(a^+)^*.$$

Post-multiply by a^**, one has* $aa^+a^* = a^+$*. Therefore,* $a \in R^{SEP}$ *by Lemma 2.2.*
(27) If $(x, y) = (a^+, (a^\#)^*)$*, then* $aa^+a^*(a^\#)^* + a^\# = a^+(a^\#)^* + a^+$*, that is,* $aa^+ + a^\# = a^+(a^\#)^* + a^+$*. Post-multiply by* aa^+*, one obtains* $a^\# = a^\#aa^+$*. Hence, we obtain* $a \in R^{EP}$ *by (13). So, we have*

$$aa^+ = a^+(a^+)^*.$$

Post-multiply by a^**, one has* $aa^+a^* = a^+$*. Therefore,* $a \in R^{SEP}$ *.*
(28) If $(x, y) = (a^*, (a^\#)^*)$*, then* $aa^*a^*(a^\#)^* + a^\# = a^*(a^\#)^* + a^+$*. Post-multiply by* aa^+*, one obtains* $a^\# = a^\# aa^+$*. Hence, we obtain* $a \in R^{EP}$ *by (13). So, we have*

$$aa^*a^*(a^\#)^* = a^*(a^\#)^*.$$

Apply the involution, one has $aa^* = a^\# a$*. Hence,* $a \in R^{SEP}$ *by Lemma 2.1.*
(29) If $(x, y) = ((a^\#)^*, (a^\#)^*)$*, then* $a(a^\#)^*a^*(a^\#)^* + a^\# = (a^\#)^*(a^\#)^* + a^+$*. Post-multiply by* aa^+*, one obtains* $a^\# = a^\# aa^+$*. Hence, we obtain* $a \in R^{EP}$ *by (13). It follows that* $a(a^\#)^*a^*(a^\#)^* = (a^\#)^*(a^\#)^*$*, that is,*

$$a(a^\#)^* = (a^\#)^*(a^\#)^*.$$

Apply the involution, one has $a^\# a^* = a^\# a^\#$*. Pre-multiply by* a^2*, one gets* $aa^* = aa^\#$*. Therefore,* $a \in R^{SEP}$ *by Lemma 2.1.*
(30) If $(x, y) = ((a^+)^*, (a^\#)^*)$*, then* $a(a^+)^*a^*(a^\#)^* + a^\# = (a^+)^*(a^\#)^* + a^+$*. Post-multiply by* aa^+*, one obtains* $a^\# = a^\# aa^+$*. Hence, we obtain* $a \in R^{EP}$ *by (13). It follows that* $a(a^+)^*a^*(a^+)^* = (a^+)^*(a^+)^*$*, that is,*

$$a(a^+)^* = (a^+)^*(a^+)^*.$$

Apply the involution, one has $a^+a^* = a^+a^+$*. Thus,* $a \in R^{PI}$ *by [12, Lemma 4.2]. Therefore,* $a \in R^{SEP}$*.*
(31) If $(x, y) = (a, (a^+)^*)$*, then* $a^2a^*(a^+)^* + a^\# = a(a^+)^* + a^+$*, that is,* $a^2 + a^\# = a(a^+)^* + a^+$*. Pre-multiply and post-multiply* $a^2 + a^\# = a(a^+)^* + a^+$ *by* a*, one gets*

$$a^4 = a^2(a^+)^*a.$$

Pre-multiply and post-multiply $a^4 = a^2(a^+)^*a$ *by* $a^\#$*, this leads to* $a^2 = a(a^+)^*$*. Hence* $a \in R^{PI}$*. It follows that* $a^2 + a^\# = a(a^+)^+ + a^+$*, we can infer* $a^\# = a^+$*. Thus,* $a \in R^{EP}$*. Therefore,* $a \in R^{SEP}$*.*
(32) If $(x, y) = (a^\#, (a^+)^*)$*, then* $aa^\# a^*(a^+)^* + a^\# = a^\#(a^+)^* + a^+$*, that is,* $aa^\# + a^\# = a^\#(a^+)^* + a^+$*. Pre-multiply and post-multiply* $aa^\# + a^\# = a^\#(a^+)^* + a^+$ *by* a*, one gets*

$$a^2 = (a^+)^*a.$$

Thus, $a \in R^{PI}$*, and so* $a = (a^+)^*$*. It follows that* $a^\# = a^+ = a^*$*. Hence,* $a \in R^{SEP}$*.*
(33) If $(x, y) = (a^+, (a^+)^*)$*, then* $aa^+a^*(a^+)^* + a^\# = a^+(a^+)^* + a^+$*, that is,* $aa^+a^+a + a^\# = a^+(a^+)^* + a^+$*. Post-multiply by* $aa^\#$*, one obtains* $a^+ = a^+aa^\#$*. Pre-multiply by* a*, one yields* $aa^+ = aa^\#$*. Hence,* $a \in R^{EP}$ *by [11, Lemma 2.1]. It follows that*

$$aa^+a^+a = a^+(a^+)^*.$$

Therefore, $a \in R^{SEP}$ *by (26).*
(34) If $(x, y) = (a^*, (a^+)^*)$*, then* $aa^*a^*(a^+)^* + a^\# = a^*(a^+)^* + a^+$*, that is,* $aa^*a^+a + a^\# = a^+a + a^+$*. Post-multiply by* $aa^\#$*, one obtains* $a^+ = a^+aa^\#$*. Hence, we obtain* $a \in R^{EP}$ *by (33). It follows that*

$$aa^*a^+a = a^+a.$$

Post-multiply by a^+*, one has* $aa^*a^+ = a^+$*. Therefore,* $a \in R^{SEP}$ *by Lemma 2.2.*
(35) If $(x, y) = ((a^\#)^*, (a^+)^*)$*, then* $a(a^\#)^*a^*(a^+)^* + a^\# = (a^\#)^*(a^+)^* + a^+$*, that is,* $a(a^\#)^*a^+a + a^\# = (a^\#)^*(a^+)^* + a^+$*. Post-multiply by* $aa^\#$*, one obtains* $a^+ = a^+aa^\#$*. Hence, we obtain* $a \in R^{EP}$ *by (33). It follows that*

$$a(a^\#)^*a^+a = (a^\#)^*(a^+)^*.$$

Apply the involution, one yields $a^+aa^\# a^* = a^+a^\#$*. Pre-multiply by* a^2*, one has* $aa^* = aa^\#$*. Therefore,* $a \in R^{SEP}$ *by Lemma 2.2.*
(36) If $(x, y) = ((a^+)^*, (a^+)^*)$*, then* $a(a^+)^*a^*(a^+)^* + a^\# = (a^+)^*(a^+)^* + a^+$*, that is,* $a(a^+)^* + a^\# = (a^+)^*(a^+)^* + a^+$*. Post-multiply by* $aa^\#$*, one obtains* $a^+ = a^+aa^\#$*. Hence, we obtain* $a \in R^{EP}$ *by (33). It follows that*

$$a(a^+)^* = (a^+)^*(a^+)^*.$$

Therefore, $a \in R^{SEP}$ *by (30).*

We modify the equation (2.9) as follows:

$$aa^+xa^*y = a^+xy. \tag{10}$$

From the equation (2.10), we can obtain the following characterization of SEP element which proof is routine.

Theorem 2.12. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if there is at least one solution to the equation (2.10) in χ_a^2.*

3. General solutions of some equations

In this section, we established some equations and discussing the general solutions of these equations. The equation (2.3) can be extended to:

$$a^+axa = y(a^+)^*. \tag{11}$$

Theorem 3.1. *Let $a \in R^\# \cap R^+$. Then the general solution to equation (3.11) is given by*

$$\begin{cases} x = pa^+ + u - a^+auaa^+ \\ y = a^+apa^* + v - vaa^+ \end{cases}, where \;\; p, u, v \in R. \tag{12}$$

Proof. First, we show that the (3.12) is the solution to the equation (3.11).
In fact, we have

$$a^+a(pa^+ + u - a^+auaa^+)a = a^+apa^+a + a^+aua - a^+aa^+auaa^+a = a^+apa^+a,$$

$$(a^+apa^* + v - vaa^+)(a^+)^* = a^+apa^*(a^+)^* + v(a^+)^* - vaa^+(a^+)^* = a^+apa^+a.$$

Next, we show that all solutions to equation (3.11) can be written in (3.12).
Assume that $x = x_0$, $y = y_0$ is a solution of the equation (3.11), then

$$a^+ax_0a = y_0(a^+)^*.$$

Choose $u = x_0$, $v = y_0$ and $p = y_0(a^+)^*$.

$$pa^+ + u - a^+auaa^+ = (y_0(a^+)^*)a^+ + x_0 - a^+ax_0aa^+ = (a^+ax_0a)a^+ + x_0 - a^+ax_0aa^+ = x_0,$$

$$a^+apa^* + v - vaa^+ = a^+a(a^+ax_0a)a^* + y_0 - y_0aa^+ = (a^+ax_0a)a^* + y_0 - y_0aa^+ = y_0(a^+)^*a^* + y_0 - y_0aa^+ = y_0.$$

Hence, the general solution of (3.11) is given by (3.12).

Theorem 3.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if the general solution to equation (3.11) is given by*

$$\begin{cases} x = pa^* + u - a^+auaa^+ \\ y = a^+apa^* + v - vaa^+ \end{cases}, where \;\; p, u, v \in R. \tag{13}$$

Proof. $'\Rightarrow'$ Suppose that $a \in R^{PI}$, then $a^+ = a^*$. As a result, the general solution (3.12) of equation (3.11) equals (3.13).
$'\Leftarrow'$ If (3.13) is the general solution of (3.11), then

$$a^+a(pa^* + u - a^+auaa^+)a = a^+apa^*a + a^+aua - a^+aa^+auaa^+a = a^+apa^*a,$$

$$(a^+apa^* + v - vaa^+)(a^+)^* = a^+apa^*(a^+)^* + v(a^+)^* - vaa^+(a^+)^* = a^+apa^*(a^+)^*.$$

Therefore, $a^+apa^*a = a^+apa^+a$, for each $p \in R$. Choose $p = 1$ in particular, we get $a^*a = a^+a$. Hence, $a \in R^{PI}$ by [3, Theorem 2.1].

Theorem 3.3. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution to equation (3.14) is given by (3.13).*

$$a^{+}axaa^{+} = yaa^{+}. \tag{14}$$

Proof. *It's similar to Theorem 3.1.*

Theorem 3.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if equation (3.11) has the same solution as equation (3.14).*
Proof. *$' \Rightarrow'$ Suppose that $a \in R^{PI}$, then $a^{+} = a^{*}$. Hence, the solution of equation (3.11) has the same solution as equation (3.14) by Theorem 3.2 and Theorem 3.3.*
$' \Leftarrow'$ If the solution of equation (3.11), has the same the solution as equation (3.14), then

$$a^{+}a(pa^{+} + u - a^{+}auaa^{+})aa^{+} = a^{+}apa^{+} = (a^{+}apa^{*} + v - vaa^{+})aa^{+} = a^{+}apa^{*},$$

for each $p \in R$. Choose $p = 1$ in particular, this gives $a^{+} = a^{*}$. Hence, $a \in R^{PI}$.

We also have the following theorem, which is related to Theorem 3.1.

Theorem 3.5. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution to equation (3.11) is given by*

$$\begin{cases} x = p(a^{+})^{*}a^{+} + u - a^{+}auaa^{+} \\ \quad\quad y = a^{+}ap + v - vaa^{+} \end{cases}, \text{where } p, u, v \in R. \tag{15}$$

Theorem 3.6. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution to equation (3.11) is given by*

$$\begin{cases} \quad x = p + u - a^{+}auaa^{+} \\ y = a^{+}apaa^{*} + v - vaa^{+} \end{cases}, \text{where } p, u, v \in R. \tag{16}$$

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