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Characterizations and representations for the $\mathfrak{m}$-core-EP inverse

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Abstract. In this paper, we provide some characterizations and representations for the $\mathfrak{m}$-core-EP inverse. We give a relationship between the $\mathfrak{m}$-core-EP inverse and an invertible bordered matrix. Also, some characterizations for the $\mathfrak{m}$-core-EP inverse as an {2}-inverse with prescribed range and null space are presented. The Cramer's rule for the solution of a singular equation $Ax = b$ is also given. Perturbation bounds related with the $\mathfrak{m}$-core-EP inverse are estimated. Furthermore, the successive matrix squaring algorithm for computing the $\mathfrak{m}$-core-EP inverse is constructed. Finally, we show that the $\mathfrak{m}$-core-EP inverse can be used in solving appropriate systems of linear equations.

1. Introduction

Throughout this paper, we denote the set of all $n \times n$ complex matrices by $\mathbb{C}_{n,n}$. Let A^*, $\mathcal{N}(A)$, $\mathcal{R}(A)$, $\|A\|$, ρ, $\mathcal{M}$ and $\mathrm{rk}(A)$ represent the conjugate transpose, the null space, the range space (column space), the spectrum norm, the spectral radius, the Minskowski space and the rank, respectively, of A. The smallest nonnegative integer k, which satisfies $\mathrm{rk}\left(A^{k+1}\right) = \mathrm{rk}\left(A^k\right)$, is called the index of A and is denoted by $\mathrm{Ind}(A)$. In particular, if $\mathrm{Ind}(A) = 1$, that is,

$$\mathbb{C}_n^{\mathrm{CM}} = \left\{A \mid A \in \mathbb{C}_{n,n},\ \mathrm{rk}(A^2) = \mathrm{rk}(A)\right\}.$$

Let $\mathbb{C}_n$ be the space of complex n-tuples, we shall index the components of a complex vector in $\mathbb{C}_n$ from 0 to $n-1$, that is, $u = (u_0, u_1, u_2, \cdots u_{n-1})$. Let G be the Minkowski metric tensor defined by

$$Gu = (u_0, -u_1, -u_2, \cdots, -u_{n-1}).$$

Moreover, the Minkowski metric tensor G can be written as

$$G = \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix}, \quad G = G^* \ \ and \ \ G^2 = I_n. \tag{1.1}$$

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In [2], Minkowski inner product on $\mathbb{C}_n$ is defined by $(u, v) = [u, Gv]$, where $[.,.]$ denotes the conventional Hilbert (unitary) inner product. A space with Minkowski inner product is called a Minkowski space and denoted as $\mathcal{M}$. For $A \in \mathbb{C}_{n,n}$, $x, y \in \mathbb{C}_n$ in Minskowski space, by applying (1.1), the Minskowski conjugate matrix $A^{\sim}$ of A can be defined as follows

$$\begin{aligned}(Ax, y) &= [Ax, Gy] = [x, A^*Gy] \\ &= [x, G(GA^*G)y] \\ &= [x, GA^{\sim}y] = (x, A^{\sim}y)\end{aligned}$$

where $A^{\sim} = GA^*G$ (see [2]).

In 2000, Meenakshi [3] defined the generalized inverse in $\mathcal{M}$. For $A \in \mathbb{C}_{n,n}$, the Minkowski inverse $A^{\mathfrak{m}}$ of A is the unique matrix $X \in \mathbb{C}_{n,n}$ satisfying the following four equations:

$$AXA = A, \ \ XAX = X, \ \ (AX)^{\sim} = AX, \ \ (XA)^{\sim} = XA.$$

For $A \in \mathbb{C}_{n,n}$, the Minkowski inverse $A^{\mathfrak{m}}$ of A exists if and only if

$$\text{rk}(A) = \text{rk}\,(A^{\sim}A) = \text{rk}\,(AA^{\sim})\,.$$

In [4, 5], Kılıçman and Al-Zhour studied the weighed Minkowski inverse in $\mathcal{M}$. More propeties of the Minkowski inverse can be seen in [6, 7].

In 2019, Wang, Li and Liu [22] defined the $\mathfrak{m}$-core inverse in $\mathcal{M}$. For $A \in \mathbb{C}_n^{\text{CM}}$, the $\mathfrak{m}$-core inverse $A^{\textcircled{m}}$ of A is the unique matrix $X \in \mathbb{C}_{n,n}$ satisfying the following three equations:

$$AXA = A, \ \ AX^2 = X, \ \ (AX)^{\sim} = AX. \tag{1.2}$$

For $A \in \mathbb{C}_n^{\text{CM}}$, A is $\mathfrak{m}$-core invertible if and only if

$$\text{rk}(A) = \text{rk}(A^{\sim}A).$$

In recent years, the core-EP inverse was studied in numerous papers. For $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$, the core-EP inverse $A^{\textcircled{\dagger}}$ of A is the unique matrix $X \in \mathbb{C}_{n,n}$ satisfying the following four equations [8]:

$$XA^{k+1} = A^k, \ \ XAX = X, \ \ (AX)^* = AX, \ \ \mathcal{R}(X) \subseteq \mathcal{R}\left(A^k\right).$$

In [9], Ferreyra, Levis and Thome generalize the core-EP inverse to rectangular matrices. In [12], Ma and Stanimirović studied perturbations and SMS algorithm of the core-EP inverse. In [13], Mosić, Stanimirović and Katsikis applied the core-EP inverses to study some constrained matrix approximation problems. In [14], Gao, Chen and Patrício studied continuity of the core-EP inverse and its applications to semistable matrices. More propeties of the core-EP inverse can be seen in [10, 11, 16–18].

In 2021, Wang, Wu and Liu [23] generalize the core-EP inverse to Minkowski space, and defined the $\mathfrak{m}$-core-EP inverse in $\mathcal{M}$. For $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$, the $\mathfrak{m}$-core-EP inverse $A^{\textcircled{E}}$ of A is the unique matrix $X \in \mathbb{C}_{n,n}$ satisfying the following four equations:

$$XA^{k+1} = A^k, \ \ XAX = X, \ \ (AX)^{\sim} = AX, \ \ \mathcal{R}(X) \subseteq \mathcal{R}\left(A^k\right). \tag{1.3}$$

For $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$, A is $\mathfrak{m}$-core-EP invertible if and only if

$$\text{rk}(A^k) = \text{rk}((A^k)^{\sim}A^k).$$

Motivated by recent researches about the core-EP inverse, we give some characterizations and representations for the $\mathfrak{m}$-core-EP inverse. The main structures of this paper are as follows

(1) Some characterizations for the $\mathfrak{m}$-core-EP inverse is investigated.

(2) A Cramer's rule for the solution of a singular equation $Ax = b$ is given.

(3) Perturbation bounds for the $\mathfrak{m}$-core-EP inverse are established.

(4) A successive matrix squaring (SMS) algorithm for the $\mathfrak{m}$-core-EP inverse is proposed.

(5) Applications of the $\mathfrak{m}$-core-EP inverse in solving linear equations.

2. Preliminaries

In this paper, we mainly use the core-EP decomposition. The core-EP decomposition is defined as follows

Lemma 2.1 ([15], core-EP decomposition). *Let* $A \in \mathbb{C}_{n,n}$ *with* $\mathrm{Ind}(A) = k$ *and* $\mathrm{rk}(A^k) = r$. *Then*

$$A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*, \tag{2.1}$$

where $U \in \mathbb{C}_{n,n}$ *is unitary,* $T \in \mathbb{C}_{r,r}$ *is nonsingular,* $S \in \mathbb{C}_{r,n-r}$, $N \in \mathbb{C}_{n-r,n-r}$ *is nilpotent, and* $N^k = 0$.

Let A be as in (2.1), then

$$A^{\textcircled{\dagger}} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*, \tag{2.2}$$

$$A^k = U\begin{bmatrix} T^k & \widehat{T} \\ 0 & 0 \end{bmatrix}U^*, \;\; A^{k+1} = U\begin{bmatrix} T^{k+1} & \overline{T} \\ 0 & 0 \end{bmatrix}U^*, \tag{2.3}$$

where $\widehat{T} = T^{k-1}S + T^{k-2}SN + \cdots + TSN^{k-2} + SN^{k-1}$, and $\overline{T} = T^kS + T^{k-1}SN + \cdots + TSN^{k-1}$. It is easy to get $T^{-1}\overline{T} = \widehat{T}$.

Let U be as in (2.1). Denote

$$U^*GU = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}, \tag{2.4}$$

where $G_1 \in \mathbb{C}_{r,r}$.

Lemma 2.2 ([23]). *Let* $A \in \mathbb{C}_{n,n}$ *with* $\mathrm{Ind}(A) = k$, $\mathrm{rk}\left(A^k\right) = \mathrm{rk}\left(\left(A^k\right)^{\sim} A^k\right) = r$ *if and only if* $G_1 \in \mathbb{C}_{r,r}$ *is invertible.*

Lemma 2.3 ([23]). *Let* A *be as in (2.1),* $\mathrm{rk}\left(A^k\right) = \mathrm{rk}\left(\left(A^k\right)^{\sim} A^k\right) = r$. *Then*

$$A^{\textcircled{E}} = U\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*G. \tag{2.5}$$

Lemma 2.4 ([19]). *Let* $A \in \mathbb{C}_{n,n}$ *and* $M \in \mathbb{C}_{2n,2n}$ *partitioned as* $M = \begin{bmatrix} A & AT \\ SA & B \end{bmatrix}$. *Then*

$$\mathrm{rk}(M) = \mathrm{rk}(A) + \mathrm{rk}(B - SAT).$$

Lemma 2.5 ([1]). *Let* $A \in \mathbb{C}_{n,n}$ *with* $\mathrm{Ind}(A) = k$, *the Drazin inverse* A^D *of* A *is the unique matrix* $X \in \mathbb{C}_{n,n}$ *satisfying the following three equations:*

$$A^kXA = A^k, \;\; XAX = X, \;\; AX = XA.$$

Lemma 2.6 ([23]). *Let* $A \in \mathbb{C}_{n,n}$ *with* $\mathrm{Ind}(A) = k$, $\mathrm{rk}\left(A^k\right) = \mathrm{rk}\left(\left(A^k\right)^{\sim} A^k\right) = r$. *Then*

$$A^{\textcircled{E}} = A^kA^D\left(A^k\right)^{\textcircled{m}}.$$

Lemma 2.7 ([1]). *Let* E *and* F *be complementary subspaces of* $\mathbb{C}_n$, $P_{E,F}$ *represents the projector on the subspace* E *along the subspace* F *and* $M \in \mathbb{C}_{n,n}$. *Then*

(i) $P_{E,F}M = M \Leftrightarrow \mathcal{R}(M) \subseteq E$;
(ii) $MP_{E,F} = M \Leftrightarrow F \subseteq \mathcal{N}(M)$.

3. Some characterizations for the $\mathfrak{m}$-core-EP inverse

It is obvious that if A is an invertible matrix, then $X = A^{-1}$ is the unique matrix satisfy following rank equality

$$\mathrm{rk}\left(\begin{bmatrix} A & I \\ I & X \end{bmatrix}\right) = \mathrm{rk}(A).$$

In this section, by applying the $\mathfrak{m}$-core-EP inverse $A^{\textcircled{\tiny E}}$ of A, we give an analogous result.

Theorem 3.1. *Let $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}(A) = k$, $\mathrm{rk}\left(A^k\right) = \mathrm{rk}\left(\left(A^k\right)^{\sim} A^k\right) = r$. Then there exist a unique matrix X such that*

$$(A^{k+1})^{\sim}AX = 0,\ XA^k = 0,\ X^2 = X,\ \mathrm{rk}(X) = n - r, \tag{3.1}$$

a unique matrix Y such that

$$YA^k = 0,\ Y^2 = Y,\ Y = Y^{\sim},\ \mathrm{rk}(Y) = n - r, \tag{3.2}$$

and a unique matrix Z such that

$$\mathrm{rk}\left(\begin{bmatrix} A & I_n - Y \\ I_n - X & Z \end{bmatrix}\right) = \mathrm{rk}(A). \tag{3.3}$$

The matrix Z is the $\mathfrak{m}$-core-EP inverse $A^{\textcircled{\tiny E}}$ of A. Furthermore, we have

$$X = I_n - A^{\textcircled{\tiny E}}A,\ \ Y = I_n - AA^{\textcircled{\tiny E}}.$$

Proof. Let A be as in (2.1). It is easy to verify that

$$X = U\begin{bmatrix} 0 & -T^{-1}(S + G_1^{-1}G_2N) \\ 0 & I_{n-r} \end{bmatrix}U^* \tag{3.4}$$

satisfies condition (3.1). By applying (2.5), we obtain $X = I_n - A^{\textcircled{\tiny E}}A$. Next, we verify the uniqueness of X. Firstly, suppose that both X and X_1 satisfy (3.1). Let $X_1 = UX_0U^*$, and X_0 can be denoted by

$$X_0 = \begin{bmatrix} E & F \\ K & H \end{bmatrix}, \tag{3.5}$$

where $E \in \mathbb{C}_{r,r}$. By applying $X_1A^k = 0$, (3.5) and (2.3), we obtain

$$\begin{bmatrix} E & F \\ K & H \end{bmatrix}\begin{bmatrix} T^k & \widehat{T} \\ 0 & 0 \end{bmatrix} = 0.$$

Therefore, $E = 0$ and $K = 0$. Moreover, it follows from (3.1) that $\mathrm{rk}(X_1) = n - r$ and $X_1^2 = X_1$, and it is easy to obtain that $\mathrm{rk}(H) = n - r$, $H^2 = H$ and $F = FH$. Therefore, H is invertible and $H = I_{n-r}$.

Besides, by using (3.1), we have

$$\begin{aligned}(A^{k+1})^{\sim}AX_1 &= GU\begin{bmatrix} (T^{k+1})^* & 0 \\ \overline{T}^* & 0 \end{bmatrix}U^*GU\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*U\begin{bmatrix} 0 & F \\ 0 & I_{n-r} \end{bmatrix}U^* \\ &= GU\begin{bmatrix} 0 & (T^{k+1})^*G_1TF + (T^{k+1})^*G_1S + (T^{k+1})^*G_2N \\ 0 & \overline{T}^*G_1TF + \overline{T}^*G_1S + \overline{T}^*G_2N \end{bmatrix}U^* = 0.\end{aligned}$$

Since G_1 and T are invertible, by using $(T^{k+1})^*G_1TF + (T^{k+1})^*G_1S + (T^{k+1})^*G_2N = 0$, we obtain $F = -T^{-1}(S + G_1^{-1}G_2N)$. Thus, $X_1 = X$.

In a similar way, we can also verify (3.2), where Y can be denoted by

$$Y = U\begin{bmatrix} 0 & -G_1^{-1}G_2 \\ 0 & I_{n-r} \end{bmatrix}U^*.$$

By applying (2.1) and (2.5), then

$$\begin{aligned} I_n - AA^{\textcircled{\dagger}} &= I_n - U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*U\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*G \\ &= I_n - U\begin{bmatrix} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*G = I_n - U\begin{bmatrix} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*GUU^* \\ &= I_n - U\begin{bmatrix} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}U^* = I_n - U\begin{bmatrix} I_r & G_1^{-1}G_2 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} 0 & -G_1^{-1}G_2 \\ 0 & I_{n-r} \end{bmatrix}U^* = Y. \end{aligned}$$

The matrices $X = I_n - A^{\textcircled{\dagger}}A$ and $Y = I_n - AA^{\textcircled{\dagger}}$ satisfy

$$\begin{bmatrix} A & I_n - Y \\ I_n - X & Z \end{bmatrix} = \begin{bmatrix} A & AA^{\textcircled{\dagger}} \\ A^{\textcircled{\dagger}}A & Z \end{bmatrix}.$$

By applying Lemma 2.4 and (3.3), we get

$$rk(Z - A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}) = 0,$$

which is equivalent to $Z = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$. The above proof is completed. □

In the following, by using $X = I_n - A^{\textcircled{\dagger}}A$ and $Y = I_n - AA^{\textcircled{\dagger}}$, we obtain another representation of the $\mathfrak{m}$-core-EP inverse.

Theorem 3.2. *Let A be as in Theorem 3.1. Then*

$$A^{\textcircled{\dagger}} = (A - X)^{-1}(I_n - Y) = (A + X)^{-1}(I_n - Y), \tag{3.6}$$

where $X = I_n - A^{\textcircled{\dagger}}A$, $Y = I_n - AA^{\textcircled{\dagger}}$.

Proof. Let A be of the form (2.1), by using (3.4), we obtain

$$\begin{aligned} A - X &= U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^* - U\begin{bmatrix} 0 & -T^{-1}(S + G_1^{-1}G_2N) \\ 0 & I_{n-r} \end{bmatrix}U^* \\ &= U\begin{bmatrix} T & S + T^{-1}(S + G_1^{-1}G_2N) \\ 0 & N - I_{n-r} \end{bmatrix}U^*. \end{aligned}$$

Since T and $N - I_{n-r}$ are invertible, we have

$$(A - X)^{-1} = U\begin{bmatrix} T^{-1} & -T^{-1}[S + T^{-1}(S + G_1^{-1}G_2N)](N - I_{n-r})^{-1} \\ 0 & (N - I_{n-r})^{-1} \end{bmatrix}U^*$$

and

$$\begin{aligned} (A - X)^{-1}(I_n - Y) &= U\begin{bmatrix} T^{-1} & -T^{-1}[S+T^{-1}(S+G_1^{-1}G_2N)](N-I_{n-r})^{-1} \\ 0 & (N-I_{n-r})^{-1} \end{bmatrix}U^*U\begin{bmatrix} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*G \\ &= U\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*G = A^{\textcircled{\dagger}}. \end{aligned}$$

In a similar way, we can also obtain the equality $A^{\textcircled{E}} = (A+X)^{-1}(I_n - Y)$. Then

$$A^{\textcircled{E}} = (A-X)^{-1}(I_n - Y) = (A+X)^{-1}(I_n - Y),$$

which prove the representation (3.6). □

In the following, we take an example to verify the results of Theorem 3.1.

Example 3.3. *Let*

$$A = \begin{bmatrix} 0 & 4 & -1 \\ -1 & 3 & -1 \\ -2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

with $rk(A) = 2$ *and* $\mathrm{Ind}(A) = 2$. *The* $A^{\textcircled{E}}$ *is denoted by*

$$\begin{aligned} A^{\textcircled{E}} &= U \begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G \\ &= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} & \frac{2}{3} & -\frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{4}{3} & -\frac{2}{3} & \frac{4}{3} \end{bmatrix}. \end{aligned}$$

The block matrix

$$B = \begin{bmatrix} A & I_3 - Y \\ I_3 - X & Z \end{bmatrix} = \begin{bmatrix} A & AA^{\textcircled{E}} \\ A^{\textcircled{E}}A & A^{\textcircled{E}} \end{bmatrix} = \begin{bmatrix} 0 & 4 & -1 & 2 & -\frac{2}{3} & \frac{2}{3} \\ -1 & 3 & -1 & 1 & -\frac{1}{3} & \frac{1}{3} \\ -2 & -2 & 0 & -2 & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{9} & \frac{14}{9} & -\frac{4}{9} & \frac{2}{3} & -\frac{2}{9} & \frac{2}{9} \\ -\frac{1}{9} & \frac{7}{9} & -\frac{2}{9} & \frac{1}{3} & -\frac{1}{9} & \frac{1}{9} \\ \frac{2}{9} & -\frac{14}{9} & \frac{4}{9} & -\frac{2}{3} & \frac{2}{9} & -\frac{2}{9} \end{bmatrix}$$

satisfies $rk(B) = rk(A) = 2$. *Furthermore,*

$$X = I_3 - A^{\textcircled{E}}A = \begin{bmatrix} \frac{11}{9} & -\frac{14}{9} & \frac{4}{9} \\ \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ -\frac{2}{9} & \frac{14}{9} & \frac{5}{9} \end{bmatrix}$$

and

$$Y = I_3 - AA^{\textcircled{E}} = \begin{bmatrix} -1 & \frac{2}{3} & -\frac{2}{3} \\ -1 & \frac{4}{3} & -\frac{1}{3} \\ 2 & -\frac{2}{3} & \frac{5}{3} \end{bmatrix}$$

satisfy (3.1) and (3.2), respectively.

In the following, we give characterizations for the $\mathfrak{m}$-core-EP inverse as an {2}-inverse with prescribed range and null space.

Lemma 3.4 ([1], Theorem 14, p.72). *Let* $A \in \mathbb{C}_{n,n}$ *with* $\mathrm{rk}(A) = t$. *Let* T *be a subspace of* $\mathbb{C}_n$ *of dimension* $s \leq t$, *and let* S *be a subspace of* $\mathbb{C}_n$ *of dimension* $n - s$. X *is a {2}-inverse of* A *with prescribed range* T *and null space* S *if*

$$XAX = X, \ \mathcal{R}(X) = T, \ \mathcal{N}(X) = S.$$

In this case, X *is unique and denoted by* $A^{(2)}_{T,S}$.

Theorem 3.5. *Let A be as in Theorem 3.1. Then*

$$\mathcal{R}(AA^{\circledE}) = \mathcal{R}(A^k), \ \ \mathcal{N}(AA^{\circledE}) = \mathcal{N}((A^k)^{\sim}).$$

Furthermore, $AA^{\circledE} = P_{\mathcal{R}(AA^{\circledE}),\mathcal{N}(AA^{\circledE})} = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^{\sim})}$.

Proof. Let A be of the form (2.1), by applying Lemma 2.6, we obtian $\mathcal{R}(AA^{\circledE}) \subseteq \mathcal{R}(A^k)$. Next, we just need to verify $\mathcal{R}(A^k) \subseteq \mathcal{R}(AA^{\circledE})$. By applying (1.3), we known that $A^{\circledE}A^{k+1} = A^k$. Premultiplying both sides of equality with A, we obtain $AA^{\circledE}A^{k+1} = A^{k+1}$. Therefore, $\mathcal{R}(A^{k+1}) \subseteq \mathcal{R}(AA^{\circledE})$. Since $\mathrm{Ind}(A) = k$, we obtain $\mathcal{R}(A^k) = \mathcal{R}(A^{k+1}) \subseteq \mathcal{R}(AA^{\circledE})$. From above, $\mathcal{R}(A^k) = \mathcal{R}(AA^{\circledE})$.

In the following, we verify $\mathcal{N}(AA^{\circledE}) = \mathcal{N}((A^k)^{\sim})$. Let any $x \in \mathcal{N}(AA^{\circledE})$, that is, $AA^{\circledE}x = 0$. Denote

$$U^*Gx = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$\begin{aligned} AA^{\circledE}x &= U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*Gx \\ &= U\begin{bmatrix} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = U\begin{bmatrix} G_1^{-1}x_1 \\ 0 \end{bmatrix}. \end{aligned}$$

Since $AA^{\circledE}x = 0$, and G_1 is invertible, we obtain $x_1 = 0$, that is, $x = GU\begin{bmatrix} 0 \\ x_2 \end{bmatrix}$, where $x_2 \in \mathbb{C}_{n-r,1}$ is arbitrary.

In a similar way, let any $y \in \mathcal{N}((A^k)^{\sim})$, that is, $(A^k)^{\sim}y = 0$. Denote

$$U^*Gy = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

$$\begin{aligned} (A^k)^{\sim}y &= GU\begin{bmatrix} (T^k)^* & 0 \\ \widehat{T}^* & 0 \end{bmatrix}U^*Gy \\ &= GU\begin{bmatrix} (T^k)^* & 0 \\ \widehat{T}^* & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = GU\begin{bmatrix} (T^k)^*y_1 \\ \widehat{T}^*y_1 \end{bmatrix}. \end{aligned}$$

Since $(A^k)^{\sim}y = 0$, and T is invertible, we obtain $y_1 = 0$, that is, $y = GU\begin{bmatrix} 0 \\ y_2 \end{bmatrix}$, where $y_2 \in \mathbb{C}_{n-r,1}$ is arbitrary. Hence $AA^{\circledE}x = 0$ and $(A^k)^{\sim}y = 0$ are the same solution, we obtain $\mathcal{N}(AA^{\circledE}) = \mathcal{N}((A^k)^{\sim})$.

Furthermore, $AA^{\circledE}$ is idempotent matrix, then $AA^{\circledE}$ is projection operator, that is, $AA^{\circledE} = P_{\mathcal{R}(AA^{\circledE}),\mathcal{N}(AA^{\circledE})} = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^{\sim})}$. □

Theorem 3.6. *Let A be as in Theorem 3.1. Then*

$$A^{\circledE} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^{\sim})}. \tag{3.7}$$

Proof. By applying Lemma 2.6, we obtian $\mathcal{R}(A^{\circledE}) \subseteq \mathcal{R}(A^k)$. From (1.3), we known that $A^{\circledE}A^{k+1} = A^k$, that is, $\mathcal{R}(A^k) \subseteq \mathcal{R}(A^{\circledE})$. Therefore, $\mathcal{R}(A^{\circledE}) = \mathcal{R}(A^k)$. By applying (1.3), we have $\mathcal{N}(AA^{\circledE}) \subseteq \mathcal{N}(A^{\circledE}AA^{\circledE}) = \mathcal{N}(A^{\circledE}) \subseteq \mathcal{N}(AA^{\circledE})$. Therefore, $\mathcal{N}(A^{\circledE}) = \mathcal{N}(AA^{\circledE}) = \mathcal{N}((A^k)^{\sim})$. From (1.3), we obtain $A^{\circledE}AA^{\circledE} = A^{\circledE}$. Hence, by applying Lemma 3.4, we have (3.7). □

4. The Cramer's rule for the solution of a singular equation $Ax = b$

In the following, we study the relationship between the $\mathfrak{m}$-core-EP inverse $A^{Ⓔ}$ and an invertible bordered matrix. The relationship is derived by the Cramer's rule.

Theorem 4.1. *Let A be as in Theorem 3.1. Let B and C^* be full column rank matrices such that*

$$\mathcal{N}((A^k)^\sim) = \mathcal{R}(B), \ \ \mathcal{R}(A^k) = \mathcal{N}(C).$$

Then the bordered matrix

$$\mathcal{A} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$

is invertible and

$$\mathcal{A}^{-1} = \begin{bmatrix} A^{Ⓔ} & (I_n - A^{Ⓔ}A)C^\dagger \\ B^\dagger(I_n - AA^{Ⓔ}) & B^\dagger(AA^{Ⓔ}A - A)C^\dagger \end{bmatrix}.$$

Proof. Let

$$Z = \begin{bmatrix} A^{Ⓔ} & (I_n - A^{Ⓔ}A)C^\dagger \\ B^\dagger(I_n - AA^{Ⓔ}) & B^\dagger(AA^{Ⓔ}A - A)C^\dagger \end{bmatrix}.$$

By applying Lemma 2.6, we have

$$CA^{Ⓔ} = CA^kA^D(A^k)^{ⓜ} = 0.$$

By applying Theorem 3.5 and Lemma 2.7, we have

$$\begin{aligned} BB^\dagger(I_n - AA^{Ⓔ}) =& BB^\dagger P_{\mathcal{N}((A^k)^\sim),\mathcal{R}(A^k)} \\ =& P_{\mathcal{N}((A^k)^\sim),\mathcal{R}(A^k)} = I_n - AA^{Ⓔ}. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{A}Z &= \begin{bmatrix} AA^{Ⓔ} + BB^\dagger(I_n - AA^{Ⓔ}) & A(I_n - A^{Ⓔ}A)C^\dagger - BB^\dagger(I_n - AA^{Ⓔ})AC^\dagger \\ CA^{Ⓔ} & C(I_n - A^{Ⓔ}A)C^\dagger \end{bmatrix} \\ &= \begin{bmatrix} AA^{Ⓔ} + I_n - AA^{Ⓔ} & A(I_n - A^{Ⓔ}A)C^\dagger - (I_n - AA^{Ⓔ})AC^\dagger \\ CA^{Ⓔ} & CC^\dagger - CA^{Ⓔ}AC^\dagger \end{bmatrix} \\ &= \begin{bmatrix} I_n & A(I_n - A^{Ⓔ}A)C^\dagger - (I_n - AA^{Ⓔ})AC^\dagger \\ 0 & CC^\dagger \end{bmatrix} \\ &= \begin{bmatrix} I_n & 0 \\ 0 & I_{n-r} \end{bmatrix} = I_{2n-r}. \end{aligned}$$

In a similar way, we can verify $Z\mathcal{A} = I_{2n-r}$. So, $\mathcal{A}$ is invertible with $Z = \mathcal{A}^{-1}$. □

Theorem 4.2. *Let A be as in Theorem 3.1. Let B and C^* be full column rank matrices such that*

$$\mathcal{N}((A^k)^\sim) = \mathcal{R}(B), \ \ \mathcal{R}(A^k) = \mathcal{N}(C).$$

If $b \in \mathcal{R}(A^k)$, then the unique solution $x = A^{Ⓔ}b$ of a singular linear equation $Ax = b$ is denoted by

$$x_j = det\begin{bmatrix} A(j \longrightarrow b) & B \\ C(j \longrightarrow 0) & 0 \end{bmatrix} / det\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, j = 1, 2, \cdots, n.$$

Proof. Since $x = A^{\textcircled{\tiny E}}b \in \mathcal{R}(A^k)$ and $\mathcal{R}(A^k) = \mathcal{N}(C)$, we obtain $Cx = 0$. The solution of $Ax = b$ satisfies:

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

Applying Theorem 4.1, we obtain the unique solution $x = A^{\textcircled{\tiny E}}b$ of the singular linear equation $Ax = b$ is given by

$$x_j = det\begin{bmatrix} A(j \longrightarrow b) & B \\ C(j \longrightarrow 0) & 0 \end{bmatrix}/det\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, j = 1, 2, \cdots, n.$$

□

5. Perturbations of the $\mathfrak{m}$-core-EP inverse

In this section, we investigate perturbation bounds for the $\mathfrak{m}$-core-EP inverse.

Lemma 5.1 ([21]). *Let $A \in \mathbb{C}_{n,n}$ with $\|A\| \leq 1$. Then $I_n + A$ is nonsingular and*

$$\|(I_n + A)^{-1}\| \leq (1 - \|A\|)^{-1}.$$

Theorem 5.2. *Let A be as in Theorem 3.1, $B = A + E \in \mathbb{C}_{n,n}$. If the perturbation E satisfies $AA^{\textcircled{\tiny E}}E = E$ and $\|A^{\textcircled{\tiny E}}E\| < 1$, then*

$$B^{\textcircled{\tiny E}} = (I_n + A^{\textcircled{\tiny E}}E)^{-1}A^{\textcircled{\tiny E}} = A^{\textcircled{\tiny E}}(I_n + EA^{\textcircled{\tiny E}})^{-1}, \ BB^{\textcircled{\tiny E}} = AA^{\textcircled{\tiny E}}.$$

Furthermore,

$$\frac{\|A^{\textcircled{\tiny E}}\|}{1 + \|A^{\textcircled{\tiny E}}E\|} \leq \|B^{\textcircled{\tiny E}}\| \leq \frac{\|A^{\textcircled{\tiny E}}\|}{1 - \|A^{\textcircled{\tiny E}}E\|},$$
$$\frac{\|B^{\textcircled{\tiny E}} - A^{\textcircled{\tiny E}}\|}{\|A^{\textcircled{\tiny E}}\|} \leq \frac{\|A^{\textcircled{\tiny E}}E\|}{1 - \|A^{\textcircled{\tiny E}}E\|}.$$

Proof. Let A be of the form (2.1). Suppose that the perturbation E can be expressed as

$$E = U\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}U^*.$$

Since E satisfies $AA^{\textcircled{\tiny E}}E = E$, then

$$\begin{aligned} AA^{\textcircled{\tiny E}}E &= U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}U^* \\ &= U\begin{bmatrix} E_{11} + G_1^{-1}G_2E_{21} & E_{12} + G_1^{-1}G_2E_{22} \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}U^*. \end{aligned}$$

Then, we have $E_{21} = 0$ and $E_{22} = 0$, and

$$E = U\begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix}U^*, \ B = A + E = U\begin{bmatrix} T + E_{11} & S + E_{12} \\ 0 & N \end{bmatrix}U^*. \tag{5.1}$$

By applying (2.5) and (5.1), we have

$$\begin{aligned} I_n + A^{\textcircled{\tiny E}}E &= I_n + U\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}\begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} I_r + T^{-1}E_{11} & T^{-1}E_{12} \\ 0 & I_{n-r} \end{bmatrix}U^*. \end{aligned}$$

Since $\|A^{\textcircled{E}}E\| < 1$, we known that $I_n + A^{\textcircled{E}}E$ and $T + E_{11}$ are invertible. Then, we have

$$(I_n + A^{\textcircled{E}}E)^{-1} = U\begin{bmatrix} (T+E_{11})^{-1}T & -(T+E_{11})^{-1}E_{12} \\ 0 & I_{n-r} \end{bmatrix}U^*. \tag{5.2}$$

Since $T + E_{11}$ is invertible, by applying (5.1), we obtain $\mathrm{rk}\left(B^k\right) = \mathrm{rk}\left(\left(B^k\right)^{\sim} B^k\right) = r$. It follows from Lemma 2.3 that the $\mathfrak{m}$-core-EP inverse of B exists. Then,

$$B^{\textcircled{E}} = U\begin{bmatrix} (T+E_{11})^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*G. \tag{5.3}$$

By applying (2.5) and (5.2), we have

$$\begin{aligned}(I_n + A^{\textcircled{E}}E)^{-1}A^{\textcircled{E}} &= U\begin{bmatrix} (T+E_{11})^{-1}T & -(T+E_{11})^{-1}E_{12} \\ 0 & I_{n-r} \end{bmatrix}U^*U\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*G \\ &= U\begin{bmatrix} (T+E_{11})^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*G. \end{aligned} \tag{5.4}$$

From (5.3) and (5.4), we obtain $B^{\textcircled{E}} = (I_n + A^{\textcircled{E}}E)^{-1}A^{\textcircled{E}}$.

On the other hand,

$$\begin{aligned} I_n + EA^{\textcircled{E}} &= I_n + U\begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix}U^*U\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*G \\ &= I_n + U\begin{bmatrix} E_{11}T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*G = I_n + U\begin{bmatrix} E_{11}T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*GUU^* \\ &= I_n + U\begin{bmatrix} E_{11}T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}U^* = I_n + U\begin{bmatrix} E_{11}T^{-1} & E_{11}T^{-1}G_1^{-1}G_2 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} I_r + E_{11}T^{-1} & E_{11}T^{-1}G_1^{-1}G_2 \\ 0 & I_{n-r} \end{bmatrix}U^*, \end{aligned}$$

$$(I_n + EA^{\textcircled{E}})^{-1} = U\begin{bmatrix} T(T+E_{11})^{-1} & -T(T+E_{11})^{-1}E_{11}T^{-1}G_1^{-1}G_2 \\ 0 & I_{n-r} \end{bmatrix}U^*. \tag{5.5}$$

By applying (2.5), (5.3) and (5.5), we have

$$\begin{aligned} A^{\textcircled{E}}(I_n + EA^{\textcircled{E}})^{-1} &= U\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*GU\begin{bmatrix} T(T+E_{11})^{-1} & -T(T+E_{11})^{-1}E_{11}T^{-1}G_1^{-1}G_2 \\ 0 & I_{n-r} \end{bmatrix}U^* \\ &= U\begin{bmatrix} (T+E_{11})^{-1} & -(T+E_{11})^{-1}E_{11}T^{-1}G_1^{-1}G_2 + T^{-1}G_1^{-1}G_2 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (T+E_{11})^{-1} & (T+E_{11})^{-1}[-E_{11}T^{-1}G_1^{-1}G_2 + (T+E_{11})T^{-1}G_1^{-1}G_2] \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (T+E_{11})^{-1} & (T+E_{11})^{-1}[(-E_{11}+T+E_{11})T^{-1}G_1^{-1}G_2] \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (T+E_{11})^{-1} & (T+E_{11})^{-1}G_1^{-1}G_2 \\ 0 & 0 \end{bmatrix}U^*, \end{aligned} \tag{5.6}$$

$$\begin{aligned} B^{\circledR} &= U\begin{bmatrix} (T+E_{11})^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G \\ &= U\begin{bmatrix} (T+E_{11})^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G U U^* \\ &= U\begin{bmatrix} (T+E_{11})^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} U^* \\ &= U\begin{bmatrix} (T+E_{11})^{-1} & (T+E_{11})^{-1}G_1^{-1}G_2 \\ 0 & 0 \end{bmatrix} U^*. \end{aligned} \tag{5.7}$$

From (5.6) and (5.7), we obtain $B^{\circledR} = A^{\circledR}(I_n + EA^{\circledR})^{-1}$.

By applying (2.1), (2.5), (5.1) and (5.3), we have

$$\begin{aligned} BB^{\circledR} &= U\begin{bmatrix} T+E_{11} & S+E_{12} \\ 0 & N \end{bmatrix} U^* U \begin{bmatrix} (T+E_{11})^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G \\ &= U\begin{bmatrix} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G, \end{aligned}$$

$$\begin{aligned} AA^{\circledR} &= U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^* U \begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G \\ &= U\begin{bmatrix} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G. \end{aligned}$$

Therefore, $BB^{\circledR} = AA^{\circledR}$.

Next, we prove the perturbation bound inequality. From Lemma 5.1, we obtain

$$\|(I_n + A^{\circledR}E)^{-1}\| \leq \frac{1}{1 - \|A^{\circledR}E\|}. \tag{5.8}$$

By applying $B^{\circledR} = (I_n + A^{\circledR}E)^{-1}A^{\circledR}$ and (5.8), we obtain

$$\|B^{\circledR}\| \leq \frac{\|A^{\circledR}\|}{1 - \|A^{\circledR}E\|}. \tag{5.9}$$

By applying (2.5), (5.1) and (5.3), we have

$$\begin{aligned} B^{\circledR} - A^{\circledR} &= U\begin{bmatrix} (T+E_{11})^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G - U\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G \\ &= U\begin{bmatrix} (T+E_{11})^{-1}G_1^{-1} - T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G \\ &= U\begin{bmatrix} [I_r - T^{-1}(T+E_{11})](T+E_{11})^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G \\ &= U\begin{bmatrix} -T^{-1}E_{11}(T+E_{11})^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G, \end{aligned}$$

$$
\begin{aligned}
A^{\textcircled{E}}EB^{\textcircled{E}} &= U\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*GU \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix} U^*U \begin{bmatrix} (T+E_{11})^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*G \\
&= U\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (T+E_{11})^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*G \\
&= U\begin{bmatrix} T^{-1}E_{11} & T^{-1}E_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (T+E_{11})^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*G \\
&= U\begin{bmatrix} T^{-1}E_{11}(T+E_{11})^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*G.
\end{aligned}
$$

From above, we obtain

$$B^{\textcircled{E}} - A^{\textcircled{E}} = -A^{\textcircled{E}}EB^{\textcircled{E}}, \tag{5.10}$$

that is, $(I_n + A^{\textcircled{E}}E)B^{\textcircled{E}} = A^{\textcircled{E}}$. Therefore,

$$\|A^{\textcircled{E}}\| \leq (1 + \|A^{\textcircled{E}}E\|)\|B^{\textcircled{E}}\|,$$

$$\frac{\|A^{\textcircled{E}}\|}{1 + \|A^{\textcircled{E}}E\|} \leq \|B^{\textcircled{E}}\|. \tag{5.11}$$

From (5.9) and (5.11), we have

$$\frac{\|A^{\textcircled{E}}\|}{1 + \|A^{\textcircled{E}}E\|} \leq \|B^{\textcircled{E}}\| \leq \frac{\|A^{\textcircled{E}}\|}{1 - \|A^{\textcircled{E}}E\|}.$$

By applying $B^{\textcircled{E}} = (I_n + A^{\textcircled{E}}E)^{-1}A^{\textcircled{E}}$ and (5.10), we obtain

$$\frac{\|B^{\textcircled{E}} - A^{\textcircled{E}}\|}{\|A^{\textcircled{E}}\|} \leq \frac{\|A^{\textcircled{E}}E\|}{1 - \|A^{\textcircled{E}}E\|}.$$

□

Corollary 5.3. *Let A be as in Theorem 3.1, $B = A + E \in \mathbb{C}_{n,n}$. If the perturbation E satisfies $\mathcal{R}(E) \subseteq \mathcal{R}(A^k)$ and $\|A^{\textcircled{E}}E\| < 1$, then*

$$B^{\textcircled{E}} = (I_n + A^{\textcircled{E}}E)^{-1}A^{\textcircled{E}} = A^{\textcircled{E}}(I_n + EA^{\textcircled{E}})^{-1}, \ BB^{\textcircled{E}} = AA^{\textcircled{E}}. \tag{5.12}$$

Furthermore,

$$\frac{\|A^{\textcircled{E}}\|}{1 + \|A^{\textcircled{E}}E\|} \leq \|B^{\textcircled{E}}\| \leq \frac{\|A^{\textcircled{E}}\|}{1 - \|A^{\textcircled{E}}E\|}, \tag{5.13}$$

$$\frac{\|B^{\textcircled{E}} - A^{\textcircled{E}}\|}{\|A^{\textcircled{E}}\|} \leq \frac{\|A^{\textcircled{E}}E\|}{1 - \|A^{\textcircled{E}}E\|}. \tag{5.14}$$

Proof. By applying Lemma 2.7 and Theorem 3.5, we known that $\mathcal{R}(E) \subseteq \mathcal{R}(A^k)$ is equivalent to $AA^{\textcircled{E}}E = E$. Next, similar to the proof of the Theorem 5.2, we obtain (5.12), (5.13) and (5.14). □

Corollary 5.4. *Let A be as in Theorem 3.1, $B = A + E \in \mathbb{C}_{n,n}$. If the perturbation E satisfies $A^{\textcircled{E}}AE = E$ and $\|A^{\textcircled{E}}E\| < 1$, then*

$$B^{\textcircled{E}} = (I_n + A^{\textcircled{E}}E)^{-1}A^{\textcircled{E}} = A^{\textcircled{E}}(I_n + EA^{\textcircled{E}})^{-1}, \ BB^{\textcircled{E}} = AA^{\textcircled{E}}. \tag{5.15}$$

Furthermore,

$$\frac{\|A^{\textcircled{E}}\|}{1 + \|A^{\textcircled{E}}E\|} \leq \|B^{\textcircled{E}}\| \leq \frac{\|A^{\textcircled{E}}\|}{1 - \|A^{\textcircled{E}}E\|}, \tag{5.16}$$

$$\frac{\|B^{\textcircled{E}} - A^{\textcircled{E}}\|}{\|A^{\textcircled{E}}\|} \leq \frac{\|A^{\textcircled{E}}E\|}{1 - \|A^{\textcircled{E}}E\|}. \tag{5.17}$$

Proof. Let A be of the form (2.1). Assume that the perturbation E can be denoted by

$$E = U\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} U^*.$$

Since E satisfies $A^{\textcircled{E}}AE = E$, we have

$$\begin{aligned} A^{\textcircled{E}}AE &= U\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} U^* \\ &= U\begin{bmatrix} E_{11} + (T^{-1}S + T^{-1}G_1^{-1}G_2N)E_{21} & E_{12} + (T^{-1}S + T^{-1}G_1^{-1}G_2N)E_{22} \\ 0 & 0 \end{bmatrix} U^* \\ &= U\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} U^*. \end{aligned}$$

Then, we get $E_{21} = 0$ and $E_{22} = 0$. Next, similar to the proof of the Theorem 5.2, we obtain (5.15), (5.16) and (5.17). □

6. Successive matrix squaring algorithm for computing the $\mathfrak{m}$-core-EP inverse

In this section, we give successive matrix squaring (SMS) algorithm for computing the $\mathfrak{m}$-core-EP inverse $A^{\textcircled{E}}$. By applying Lemma 2.5, Lemma 2.6 and (1.2), we have

$$\begin{aligned} A^k(A^k)^{\sim}AA^{\textcircled{E}} &= A^k(A^k)^{\sim}AA^kA^D(A^k)^{\textcircled{m}} = A^k(A^k)^{\sim}A^{k+1}A^D(A^k)^{\textcircled{m}} \\ &= A^k(A^k)^{\sim}A^k(A^k)^{\textcircled{m}} = A^k(A^k)^{\sim}(A^k(A^k)^{\textcircled{m}})^{\sim} \\ &= A^k(A^k(A^k)^{\textcircled{m}}A^k)^{\sim} = A^k(A^k)^{\sim}. \end{aligned}$$

Then

$$A^{\textcircled{E}} = A^{\textcircled{E}} - \beta(A^k(A^k)^{\sim}AA^{\textcircled{E}} - A^k(A^k)^{\sim}) = (I_n - \beta A^k(A^k)^{\sim}A)A^{\textcircled{E}} + \beta A^k(A^k)^{\sim}.$$

Set

$$P = I_n - \beta A^k(A^k)^{\sim}A, \;\; Q = \beta A^k(A^k)^{\sim}, \;\; \beta > 0. \tag{6.1}$$

The iterative scheme for finding the $\mathfrak{m}$-core-EP inverse $A^{\textcircled{E}}$ will be given by [20]:

$$X_1 = Q = \beta A^k(A^k)^{\sim}, \;\; X_{m+1} = PX_m + Q, \quad m \in \mathbb{N}. \tag{6.2}$$

Taking

$$T = \begin{bmatrix} P & Q \\ 0 & I_n \end{bmatrix} \;\; and \;\; T^m = \begin{bmatrix} P^m & \Sigma_{i=0}^{m-1}P^iQ \\ 0 & I_n \end{bmatrix},$$

X_m is the top right block of T^m, i.e. $X_m = \Sigma_{i=0}^{m-1}P^iQ$. Notice that

$$T^{2^m} = \begin{bmatrix} P^{2^m} & \Sigma_{i=0}^{2^m-1}P^iQ \\ 0 & I_n \end{bmatrix}.$$

Applying [20], we have the following Theorem 6.1.

Theorem 6.1. *Let A be as in Theorem 3.1. The approximations*

$$X_{2^m} = \sum_{i=0}^{2^m-1} (I_n - \beta A^k (A^k)^{\sim} A)^i \beta A^k (A^k)^{\sim} \tag{6.3}$$

determined by the SMS algorithm

$$T_0 = T, \ \ T_{i+1} = T_i^2, \quad i = \overline{0, m-1},$$

converges in the matrix norm $\|\cdot\|$ to the m*-core-EP inverse $A^{\circledE}$ if spectral radius $\rho(I_n - AX_1) < 1$. In the case of convergence, we have the error estimates,*

$$\|A^{\circledE} - X_{2^m}\| \leq \|A^{\circledE}\| \|(I_n - AX_1)^{2^m}\|.$$

Moreover,

$$\lim_{m\to\infty} sup \sqrt[2^m]{\|A^{\circledE} - X_{2^m}\|} \leq \rho(I_n - AX_1).$$

Proof. By applying $A^{\circledE} A^{k+1} = A^k$ and (6.3), we have

$$A^{\circledE} A X_{2^m} = X_{2^m}.$$

Next, we verify following equality

$$I_n - AX_m = (I_n - AX_1)^m. \tag{6.4}$$

By the mathematical induction, for $m = 1$, the equality (6.4) is true. Assume that it holds for $m = k-1$. Next, we just need to prove that it holds for $m = k$. From (6.1), it is easy to obtain $P = I_n - QA$, and by applying (6.2), we have

$$\begin{aligned} I_n - AX_k &= I_n - A(PX_{k-1} + Q) \\ &= I_n - A(I_n - QA)X_{k-1} - AQ \\ &= I_n - AX_{k-1} + AQAX_{k-1} - AQ \\ &= I_n - AX_{k-1} - AQ(I_n - AX_{k-1}) \\ &= (I_n - AQ)(I_n - AX_{k-1}) \\ &= (I_n - AX_1)(I_n - AX_1)^{k-1} \\ &= (I_n - AX_1)^k. \end{aligned}$$

Therefore,

$$\begin{aligned} \|A^{\circledE} - X_{2^m}\| &= \|A^{\circledE} - A^{\circledE} A X_{2^m}\| \\ &= \|A^{\circledE}(I_n - AX_{2^m})\| \\ &\leq \|A^{\circledE}\| \|I_n - AX_{2^m}\| \\ &\leq \|A^{\circledE}\| \|(I_n - AX_1)^{2^m}\| \end{aligned}$$

and

$$\lim_{m\to\infty} sup \sqrt[2^m]{\|A^{\circledE} - X_{2^m}\|} \leq \lim_{m\to\infty} \sqrt[2^m]{\|A^{\circledE}\| \|(I_n - AX_1)^{2^m}\|} = \rho(I_n - AX_1).$$

In the last equality, we use the fact that $\lim_{n\to\infty} \|B^n\|^{1/n} = \rho(B)$ for any square matrix B and any norm. It completes the proof. □

Remark 6.2. *In order to achieve the convergence criterion $\rho(I_n - AX_1) = \rho(I_n - \beta A^{k+1}(A^k)^{\sim}) < 1$, the parameter β can be chosen as an arbitrary real number satisfying $\max_{1\leq i\leq s} |1 - \beta\lambda_i| < 1$, where $\lambda_i (i = 1, ..., s)$ are the nonzero eigenvalues of $A^{k+1}(A^k)^{\sim}$.*

Example 6.3. *Let* $A = \begin{bmatrix} 0 & \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{2}{3} & 0 \end{bmatrix}$ *with* $\mathrm{Ind}(A) = 2$. *Taking*

$$Q = \beta A^2(A^2)^{\sim}, \quad P = I_3 - \beta A^2(A^2)^{\sim} A, \quad \beta = 1/2.$$

The eigenvalues λ_i *of* $A^3(A^2)^{\sim}$ *are* $\{0, 0, 0.642\}$. *The nonzero eigenvalues* λ_i *satisfies*

$$\max_i |1 - \beta\lambda_i| = 1 - 0.642/2 = 0.679 < 1.$$

An approximation of $A^{\textcircled{E}}$ *can be generated from the upper right corner of the 29th approximation* $(T^2)^{29}$ *of the SMS algorithm, equal to*

$$\left[\begin{array}{ccc|ccc} -1 & 0.6667 & -0.6667 & -4 & 2 & -4 \\ -1 & 1.3333 & -0.3333 & -2 & 1 & -2 \\ 2 & -0.6667 & 1.6667 & 4 & -2 & 4 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

which gives

$$A^{\textcircled{E}} = \begin{bmatrix} -4 & 2 & -4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}.$$

7. Applications of the $\mathfrak{m}$-core-EP inverse

In this section, we apply the $\mathfrak{m}$-core-EP inverse to solve systems of linear equations. If the system is consisten, we give the general solution of linear equations in the Minkowski space. If the system is inconsisten, we give the least squares solution of constrained systems of linear equations.

In the following, we consider the general solutions of matrix equation in the Minkowski space

$$(A^k)^{\sim} A x = (A^k)^{\sim} b, \; b \in \mathbb{C}_{n,1}, \tag{7.1}$$

where $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}(A) = k$, $\mathrm{rk}\left(A^k\right) = \mathrm{rk}\left(\left(A^k\right)^{\sim} A^k\right) = r$.

Theorem 7.1. *Then the equation (7.1) is consistent and its general solution is*

$$x = A^{\textcircled{E}} b + (I_n - A^{\textcircled{E}} A)\, y, \tag{7.2}$$

for arbitrary $y \in \mathbb{C}_{n,1}$.

Proof. Let A be of the form (2.1), $A^{\textcircled{E}}$ and U^*GU are given in (2.4) and (2.5), respectively. By applying Lemma 2.2, we obtain G_1 is invertible. Denote

$$U^*x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; U^*Gb = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ and } A^{\textcircled{E}} b = U \begin{bmatrix} T^{-1}G_1^{-1}b_1 \\ 0 \end{bmatrix}, \tag{7.3}$$

where b_1, x_1 and $T^{-1}G_1^{-1}b_1 \in \mathbb{C}_{r,1}$. By using (2.1) and (2.4), we have

$$\begin{aligned} &(A^k)^{\sim} A x - (A^k)^{\sim} b \\ &= GU \begin{bmatrix} (T^k)^* & 0 \\ \widehat{T}^* & 0 \end{bmatrix} U^*GU \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*x - GU \begin{bmatrix} (T^k)^* & 0 \\ \widehat{T}^* & 0 \end{bmatrix} U^*Gb \\ &= GU \begin{bmatrix} (T^k)^* & 0 \\ \widehat{T}^* & 0 \end{bmatrix} \left(\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) \\ &= GU \begin{bmatrix} (T^k)^*G_1Tx_1 + (T^k)^*G_1Sx_2 + (T^k)^*G_2Nx_2 - (T^k)^*b_1 \\ \widehat{T}^*G_1Tx_1 + \widehat{T}^*G_1Sx_2 + \widehat{T}^*G_2Nx_2 - \widehat{T}^*b_1 \end{bmatrix}. \end{aligned} \tag{7.4}$$

Since G_1 and T are invertible, we obtain

$$x_1 = T^{-1}G_1^{-1}b_1 - T^{-1}(S + G_1^{-1}G_2N)x_2$$

such that

$$(T^k)^*G_1Tx_1 + (T^k)^*G_1Sx_2 + (T^k)^*G_2Nx_2 - (T^k)^*b_1 = 0$$

and

$$\widehat{T}^*G_1Tx_1 + \widehat{T}^*G_1Sx_2 + \widehat{T}^*G_2Nx_2 - \widehat{T}^*b_1 = 0,$$

that is, there exists x such that $(A^k)^{\sim}Ax = (A^k)^{\sim}b$. Hence, we obtain the equation (7.1) is consistent.

By using (7.3) and (7.4), we obtain

$$x = U\begin{bmatrix} T^{-1}G_1^{-1}b_1 - T^{-1}(S + G_1^{-1}G_2N)x_2 \\ x_2 \end{bmatrix}, \tag{7.5}$$

for arbitrary $x_2 \in \mathbb{C}_{n-r,1}$. Applying (2.1) and (2.5), it is easy to get

$$I_n - A^{\circledast}A = U\begin{bmatrix} 0 & -T^{-1}(S + G_1^{-1}G_2N) \\ 0 & I_{n-r} \end{bmatrix}U^*. \tag{7.6}$$

Therefore, by applying (7.3), (7.5) and (7.6), we obtain

$$\begin{aligned} x &= U\begin{bmatrix} T^{-1}G_1^{-1}b_1 \\ 0 \end{bmatrix} + U\begin{bmatrix} -T^{-1}(S + G_1^{-1}G_2N)x_2 \\ x_2 \end{bmatrix} \\ &= A^{\circledast}b + (I_n - A^{\circledast}A)y, \end{aligned}$$

where $x_2 \in \mathbb{C}_{n-r,1}$ and $y \in \mathbb{C}_{n,1}$ are arbitrary. Hence, we obtain the general solution (7.2). □

In [13], Mosić et al. considered constrained matrix minimization problem in the Euclidean norm:

$$\min_{x\in\mathcal{R}(A^k)} \|Ax - b\|_2,$$

where $A \in \mathbb{C}_{n,n}$ with $\operatorname{Ind}(A) = k$, and $b \in \mathbb{C}_{n,1}$. Futhermore, the least squares solution of the constrained system can be expressed as $A^{\circledast}b$.

In the following, we seek for the least squares solution of the problem

$$\min_{x\in\mathcal{R}(A^k)} \|(AA^{\circledast})^{\sim}Ax - b\|_2,$$

where $A \in \mathbb{C}_{n,n}$ with $\operatorname{Ind}(A) = k$, $\operatorname{rk}\left(A^k\right) = \operatorname{rk}\left(\left(A^k\right)^{\sim} A^k\right)$, and $b \in \mathbb{C}_{n,1}$.

Theorem 7.2. *Let A be as in Theorem 3.1. Then*

$$\min_{x\in\mathcal{R}(A^k)} \|(AA^{\circledast})^{\sim}Ax - b\|_2 = \|(I_n - AA^{\circledast})Gb\|_2. \tag{7.7}$$

Furthermore,

$$x = A^{\circledast}b \tag{7.8}$$

is the unique solution of (7.7).

Proof. Let A, $A^{\textcircled{\dagger}}$, $A^{\textcircled{E}}$ and U^*GU be of the form (2.1), (2.2), (2.5) and (2.4), respectively. By applying Lemma 2.2, we obtain G_1 is invertible. Since $x \in \mathcal{R}(A^k)$, there exists $y \in \mathbb{C}_{n,1}$ such that $x = A^k y$. Denote

$$U^*y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \ U^*Gb = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ and } A^{\textcircled{E}}b = U\begin{bmatrix} T^{-1}G_1^{-1}b_1 \\ 0 \end{bmatrix}, \tag{7.9}$$

where $y_1 \in \mathbb{C}_{r,1}$ and $b_1 \in \mathbb{C}_{r,1}$. Since G is unitary, we have

$$\begin{aligned} \|(AA^{\textcircled{\dagger}})^{\sim}Ax - b\|_2 &= \|AA^{\textcircled{\dagger}}GA^{k+1}y - Gb\|_2 \\ &= \left\| \begin{bmatrix} G_1T^{k+1}y_1 + G_1\overline{T}y_2 - b_1 \\ -b_2 \end{bmatrix} \right\|_2 \\ &= \left\|G_1T^{k+1}y_1 + G_1\overline{T}y_2 - b_1\right\|_2 + \|b_2\|_2 . \end{aligned} \tag{7.10}$$

Since G_1 and T are invertible, we have $\min\limits_{y_1,y_2} \left\|G_1T^{k+1}y_1 + G_1\overline{T}y_2 - b_1\right\|_2 = 0$, when

$$y_1 = (T^{k+1})^{-1}G_1^{-1}b_1 - (T^{k+1})^{-1}\overline{T}y_2. \tag{7.11}$$

Therefore, applying (7.10) and (7.11), we have

$$\min_{x\in\mathcal{R}(A^k)} \left\|(AA^{\textcircled{\dagger}})^{\sim}Ax - b\right\|_2 = \|b_2\|_2 .$$

On the other hand, by applying (2.1), (2.2) and (7.9), we obtain $\|(I_n - AA^{\textcircled{\dagger}})Gb\|_2 = b_2$. Therefore, we have (7.7). Applying (7.9) and (7.11), we obtain

$$\begin{aligned} x = A^ky &= U\begin{bmatrix} T^k & \widehat{T} \\ 0 & 0 \end{bmatrix}U^*y = U\begin{bmatrix} T^ky_1 + \widehat{T}y_2 \\ 0 \end{bmatrix} \\ &= U\begin{bmatrix} T^{-1}G_1^{-1}b_1 \\ 0 \end{bmatrix} = A^{\textcircled{E}}b, \end{aligned}$$

that is, (7.8) is the unique solution of (7.7). □

Conclusion

In this paper, we present characterizations and representations for the m-core-EP inverse. For Cramer's rule, perturbation bounds and SMS iterative algorithm are also studied. Moreover, the m-core-EP inverse can be used to solve linear equations. We believe that the research on the m-core-EP inverse will be popularized in the next years.

Some possibilities for further research are given as follows

1. New iterative algorithms and splitting methods for computing the m-core-EP inverse.
2. In addition, we can further generalize the m-core-EP inverse to tensors.

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