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Some results on μ-deferred double ideal statistically convergent sequences in credibility space

Binod Chandra Tripathy[a], **Ömer Kişi**[b], **Birojit Das**[c,*]

[a] *Department of Mathematics, Tripura University, India*
[b] *Department of Mathematics, Bartin University, 74100, Bartın, Turkey*
[c] *Department of Mathematics (AIASK), Amity University, Kolkata*

Abstract. The main purpose of this paper is to investigate different types of deferred convergent double sequences of fuzzy variables by using the notions of ideal convergence and μ-density in a given credibility space. We also establish a number of instances to illustrate the newly introduced notions in the same environment. In this study, we also present important findings that reveal the connections between these concepts.

1. Introduction

Fuzzy theory has made considerable progress on the mathematical underpinnings of fuzzy set theory, which was initialed by Zadeh [50]. Since then fuzzy theory is utilized to a wide range of realworld challenges. Many researchers studied possibility theory in this regard, for example, Dubois and Prade [15], Nahmias [40] . A fuzzy variable is a function that maps from a credibility space to a collection of real values. The convergence of fuzzy variables is an important component of credibility theory, which may be used to real-world engineering and financial challenges. Kaufmann [21] put forward the notions of fuzzy variables, possibility distributions, and membership functions. Possibility measure is a key notion in possibility theory, but it is not self-dual. It is generally described as a supremum preserving set function on the power set of a nonempty set. Since a self-dual measure is essential in both theory and practice, Liu and Liu [32] developed a self-duality measure called the credibility measure (Cr). It has certain essential characteristics with the possibility measure and this measure serves as a substitute for possibility measure in the fuzzy world. Particularly, since Liu began his examination of credibility theory, several specific contents have been investigated (see (Li and Liu [26]; Li and Liu [27]; Liu [29]; Liu [30])). In view of this fact Liu [31] proposed four types of convergence concepts for sequence of fuzzy variables in respect of credibility theory, viz.: convergence in credibility, convergence nearly certainly, convergence in mean, and convergence in distribution. Jiang [19] and Ma [34] have looked at a number of convergence characteristics of the credibility distribution for fuzzy variables based on the credibility theory. Wang and Liu [48] explored the connections between mean convergence, credibility convergence, almost uniform

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* Corresponding author: Birojit Das
Email addresses: tripathybc@gmail.com (Binod Chandra Tripathy), okisi@bartin.edu.tr (Ömer Kişi), dasbirojit@gmail.com (Birojit Das)

convergence, distribution convergence, and almost surely convergence. In addition, a number of academics highlighted convergence concepts in classical measure theory, credibility theory, and probability theory and looked at how these related to one another. Readers who are interested can look up Chen et al. [4], Lin [28], Liu and Wang [33], and You [49].

Statistical convergence was first introduced by Fast [17]. In order to extend the concept of statistical convergence, Kostyrko et al. [25] and Savaş and Das [43], respectively, developed the concepts of $\mathcal{I}$-convergence and $\mathcal{I}$-statistical convergence. Studies on several types of "statistical convergence, $\mathcal{I}$-convergence, and $\mathcal{I}$-statistical convergence" for sequences are also being made (see, for example, [2, 3, 12, 13, 18, 20, 35–39, 42, 47]). Savaş and Das [44] presented $\mathcal{I}$-statistically pre-Cauchy sequences and characterized the notion to some extent. Then, Et et al. [16] put forward the notion of μ-deferred statistically convergent functions by utilizing concept of μ-deferred density identified on $([1,\infty), \mathcal{L}, \mu)$, where $\mathcal{L}$ is the sigma algebra of subsets of $[1,\infty)$ and μ is the sigma finite measure on $\mathcal{L}$ with $\mu([1,\infty)) = \infty$. The idea of μ-deferred density determined on $(\mathbb{N}, \mathcal{L}, \mu)$, where $\mathcal{L}$ is the sigma algebra of subsets of $\mathbb{N}$ and μ is the sigma finite measure on $\mathcal{L}$ with $\mu(\mathbb{N}) = \infty$, was also used by Khan et al. [23] to study the idea of μ-deferred $\mathcal{I}$-statistically convergence for sequences. Furthermore, deferred Cesaro mean and statistical convergence by considering double sequences was studied by Dagadur and Sezgek [5].
Credibility theory and uncertainty theory are two concepts both of which are applied to study characteristics of sequence spaces. For credibility theory to be applied sequences of fuzzy variables are considered, on the other hand in case of uncertain sequence spaces uncertain variable are considered. Although both the theories are different but the convergence of sequences in both cases are being explored in same aspects. It is pertinent to mention here that a fuzzy variable is a function from a possibility space to the set of real numbers [40], whereas an uncertain variable is a function from the uncertainty space to the set of real numbers [29]. Convergence of complex uncertain single, double, triple sequences in an uncertainty space was investigated by Das et al. [8], Datta and Tripathy [14], and Tripathy and Nath [46]. Some other notable works of different convergences for sequence of complex uncertain variables may be seen in [6, 7, 9–11]. The deferred Cesaro means for real sequences is first introduced and studied by Agnew [1] back in the year 1932. Khan et al. [22] investigated μ-deferred $\mathcal{I}$-statistical convergence for complex uncertain sequence and they also demonstrated some significant results.

The definitions and properties needed in this paper are introduced and established in Khan et al. [22], Khan et al. [24], Li and Liu [26], Lin [28], Liu [31], Nath et al. [41], Savaş et al. [45], and Wang and Liu [48].

The results of the research article are presented sequentially in the following manner: Section 1 of the introduction includes a discussion of the literature review. The principal discoveries are then illustrated in Section 2. We have explored μ-deferred $\mathcal{I}_2$-statistical convergence, and $\mathcal{I}_2$-deferred strongly Cesàro summability of double sequence of fuzzy variable and developed essential features of these concepts in credibility. Finally, in the Conclusion section we made a few concluding remarks on the whole research outcomes and made a comment on the future scopes of the outputs.

2. Main Results

Throughout the article, we consider $p = \{p(m)\}$, $q = \{q(m)\}$, $r = \{r(n)\}$ and $s = \{s(n)\}$ to be sequences of non-negative integers satisfying the following conditions:

$$\begin{aligned} &p(m) < q(m),\ \lim_{m\to\infty} q(m) = \infty; \\ &r(n) < s(n),\ \lim_{n\to\infty} s(n) = \infty; \end{aligned} \tag{1}$$

and

$$q(m) - p(m) = \psi(m),\ s(n) - r(n) = \varkappa(n).$$

and $I^*_{p,q}(m) = [p_m, q_m] \cap \mathbb{N}$, $I^*_{r,s}(n) = [r_n, s_n] \cap \mathbb{N}$.

Furthermore, we consider (p,q) and (r,s) as two pairs of sequences which satisfies (1) and we also let $\mathcal{I}_2$ be a non-trivial strong admissible ideal of $\mathbb{N} \times \mathbb{N} = \mathbb{N}^2$. Let $X = \mathbb{N}^2$ and $\mathcal{L}$ be a sigma algebra of the subsets of X and μ be a sigma finite measure on $\mathcal{L}$ such that $\mu(X) = \infty$. Measure of any subset T of X which is in $\mathcal{L}$ will be indicated by $\mu(T) := |T|$.

The μ-deferred density of $T \subset \mathbb{N}^2$ is denoted by

$$_{\mu}D(T) = \lim_{m,n\to\infty} \frac{\left|\left\{(k,l) \in T : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n)\right\}\right|}{\left|I^*_{p,q}(m)\right| \left|I^*_{r,s}(n)\right|}$$

provided that the limit exists, where $I^*_{p,q}(m) = [p_m, q_m] \cap \mathbb{N}$ and $I^*_{r,s}(n) = [r_n, s_n] \cap \mathbb{N}$ and the vertical bar means the cardinality of the enclosed set. Keep in mind that the cardinality of the set T as it exists in natural or deferred density is not what is being shown by $|T|$ throughout the article; rather, it is the μ-measure of the set $T \subset \mathbb{N}^2$.

Definition 2.1. *Suppose $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ be a credibility space. Let $\{\varpi_{kl}\}$ be double sequence of fuzzy variable and ϖ be another fuzzy variable. Then, the double sequence is said to be $\mathcal{I}_2$-deferred strongly Cesàro summable with respect to almost surely in the credibility space if for any preassigned positive real $\varepsilon > 0$, there exists a $T \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{T\} = 1$ such that the set*

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\psi(m)\varkappa(n)} \sum_{k=p(m)+1}^{q(m)} \sum_{l=r(n)+1}^{s(n)} |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\} \in \mathcal{I}_2, \ \forall\theta \in T.$$

We denote the limit by $DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.}) - \lim(\varpi_{kl}) = \varpi$. The collection of all $\mathcal{I}_2$-deferred strongly summable double sequence with respect to almost surely is expressed by $DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.})$.

Definition 2.2. *Suppose $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ be a credibility space and $\{\varpi_{kl}\}$ be a double sequence of fuzzy variables. If $\forall\varepsilon, \delta > 0$, there exists a $T \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{T\} = 1$ and $\forall\theta \in T$, the set*

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I^*_{p,q}(m)\right| \left|I^*_{r,s}(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \ \& \ |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}_2,$$

then $\{\varpi_{kl}\}$ is called μ-deferred $\mathcal{I}_2$-statistically convergent almost surely in credibility space to the fuzzy variable ϖ and $_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.}) - \lim\{\varpi_{kl}\} = \varpi$. We indicate set of all μ-deferred $\mathcal{I}_2$-statistically convergent fuzzy variable double sequences with respect to almost surely by $_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.})$.

Example 2.3. *Assume the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to be $\theta_1, \theta_2, \theta_3, \ldots$ with the credibility measure is defined as follows:*

$$\mathrm{Cr}(T) = \begin{cases} \sup_{\theta_{k+l}\in T} \frac{k+l}{2(k+l)+1}, & \text{if } \sup_{\theta_{k+l}\in T} \frac{k+l}{2(k+l)+1} < \frac{1}{2}; \\ 1 - \sup_{\theta_{k+l}\in T^c} \frac{k+l}{2(k+l)+1}, & \text{if } \sup_{\theta_{k+l}\in T^c} \frac{k+l}{2(k+l)+1} < \frac{1}{2}; \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Moreover, consider the fuzzy variables $\{\varpi_{kl}\}$ $(k, l = 1, 2, ...)$ which are defined by

$$\varpi_{kl}(\theta) = \begin{cases} k+l, & \text{if } \theta = \theta_{k+l}; \\ 0, & \text{otherwise,} \end{cases}$$

for $k, l \in \mathbb{N}$ and ϖ be another fuzzy variable which is equivalent to zero function, i.e $\varpi \equiv 0$.
Obviously, by Definition 2.1 and Definition 2.2, fuzzy variable double sequence (ϖ_{kl}) is $\mathcal{I}_2$-deferred strongly Cesàro summable as well as μ-deferred $\mathcal{I}_2$-statistically convergent almost surely to ϖ.

Definition 2.4. *The double sequence $\{\varpi_{kl}\}$ of fuzzy variables is said to be $\mathcal{I}_2$-deferred strongly Cesàro summable in credibility to the fuzzy variable ϖ if for all $\varepsilon > 0$ and $\delta > 0$, the following set*

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\psi(m)\varkappa(n)} \sum_{k=p(m)+1}^{q(m)} \sum_{l=r(n)+1}^{s(n)} \mathrm{Cr}\left(\|\varpi_{kl} - \varpi\| \geq \varepsilon\right) \geq \delta\right\} \in \mathcal{I}_2.$$

We denote it as $DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}}) - \lim \varpi_{kl} = \varpi$. The space of all $\mathcal{I}_2$-deferred strongly Cesàro summable fuzzy variable double sequences in credibility is denoted by $DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}})$.

Definition 2.5. *The double sequence $\{\varpi_{kl}\}$ of fuzzy variables is said to be μ-deferred $\mathcal{I}_2$-statistically convergent in credibility to the fuzzy variable ϖ if for all $\varepsilon, \delta > 0$, and $\zeta > 0$, the set*

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|}\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \;\&\; \mathrm{Cr}\left\{\|\varpi_{kl} - \varpi\| \geq \varepsilon\right\} \geq \delta\right\}\right| \geq \zeta\right\} \in \mathcal{I}_2.$$

We denote it as ${}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}}) - \lim\{\varpi_{kl}\} = \varpi$. The family of all μ-deferred $\mathcal{I}_2$-statistically convergent fuzzy variable double sequence in credibility is denoted by ${}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}})$.

Example 2.6. *Assume the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to be $\theta_1, \theta_2, \theta_3, \ldots$ with*

$$\mathrm{Cr}(T) = \begin{cases} \sup_{\theta_{k+l} \in T} \frac{1}{k+l+1}, & \textit{if } \sup_{\theta_{k+l} \in T} \frac{1}{k+l+1} < \frac{1}{2}; \\ 1 - \sup_{\theta_{k+l} \in T^c} \frac{1}{k+l+1}, & \textit{if } \sup_{\theta_{k+l} \in T^c} \frac{1}{k+l+1} < \frac{1}{2}, \\ \frac{1}{2}, & \textit{otherwise}, \end{cases}$$

and the fuzzy variable double sequence $\{\varpi_{kl}\}$ be defined by

$$\varpi_{kl}(\theta) = \begin{cases} k + l + 1, & \textit{if } \theta = \theta_{k+l}; \\ 0, & \textit{otherwise}, \end{cases}$$

for $k, l \in \mathbb{N}$ and $\varpi \equiv 0$.
The fuzzy variable double sequence (ϖ_{kl}) is $\mathcal{I}_2$-deferred strongly Cesàro summable as well as μ-deferred $\mathcal{I}_2$-statistically convergent in credibility to ϖ.

Remark 2.7. *μ-deferred $\mathcal{I}_2$-statistical convergence almost surely does not imply μ-deferred $\mathcal{I}_2$-statistically convergence in credibility.*

Example 2.8. *For example, consider the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to a infinite discrete set $\{\theta_1, \theta_2, ...\}$, for which the credibility measure is defined as $\mathrm{Cr}\{\theta_1\} = 1$ and $\mathrm{Cr}\{\theta_t\} = (t-1)/t$ for $t = 2, 3, ...$ and the double sequence $\{\varpi_{kl}\}$ of fuzzy variables is defined by*

$$\varpi_{kl}(\theta_t) = \begin{cases} k + l, & \textit{if } t = k + l \\ 0, & \textit{otherwise}, \end{cases}$$

for $k, l = 1, 2, ...$ and another fuzzy variable ϖ by $\varpi(\theta) = 0$ for all $\theta \in \Theta$. Here, the double sequence $\{\varpi_{kl}\}$ μ-deferred $\mathcal{I}_2$-statistically converges in almost surely to ϖ.
But, for any small number $\varepsilon, \delta > 0$ and $\zeta \in \left(0, \frac{1}{2}\right)$, the double sequence $\{\varpi_{kl}\}$ is not μ-deferred $\mathcal{I}_2$-statistically convergence in credibility. Also,

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|}\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \;\&\; \mathrm{Cr}\left\{\|\varpi_{kl} - \varpi\| \geq \varepsilon\right\} \geq \delta\right\}\right| \geq \zeta\right\} \notin \mathcal{I}_2.$$

That is to say, the double sequence $\{\varpi_{kl}\}$ does not converge in credibility to ϖ.

Remark 2.9. *Converse part of the Remark 2.7 is not true, in general. A suitable example supporting this claim is placed in the following.*

Example 2.10. *Consider the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to be an infinite discrete set $\{\theta_1, \theta_2, ...\}$, for which the credibility measure is defined as $\mathrm{Cr}\{\theta_t\} = 1/t$ for $t = 1, 2, ...$ and fuzzy variable double sequences $\{\varpi_{kl}\}$ are defined by*

$$\varpi_{kl}(\theta_t) = \begin{cases} (t+1)/t, & \text{if } t = k+l, k+l+1, k+l+2, ...; \\ 0, & \text{elsewhere,} \end{cases} \tag{2}$$

for $k, l = 1, 2, ...$ and ϖ be a fuzzy variable defined by $\varpi(\theta) = 0$ for all $\theta \in \Theta$. For any small number $\varepsilon, \delta > 0$ and $\zeta \in \left[\frac{1}{2}, 1\right)$,

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I^*_{p,q}(m)\right| \left|I^*_{r,s}(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \;\&\; \mathrm{Cr}\{\|\varpi_{kl} - \varpi\| \geq \varepsilon\} \geq \delta\right\}\right| \geq \zeta\right\} \in \mathcal{I}_2$$

from which we can say that that $\{\varpi_{kl}\}$ μ-deferred $\mathcal{I}_2$-statistically converges in credibility to ϖ. But, it is obvious that $\varpi_{kl} \overset{{}_\mu DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.})}{\nrightarrow} \varpi$.

Definition 2.11. *The double sequence (ϖ_{kl}) of fuzzy variables is said to be $\mathcal{I}_2$-deferred strongly Cesàro summable in mean to the fuzzy variable ϖ if for all $\varepsilon > 0$, the following set*

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\psi(m)\varkappa(n)} \sum_{k=p(m)+1}^{q(m)} \sum_{l=r(n)+1}^{s(n)} E\left[\|\varpi_{kl}(\theta) - \varpi(\theta)\|\right] \geq \varepsilon\right\} \in \mathcal{I}_2.$$

We write it as $DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_E) - \lim \varpi_{kl} = \varpi$. The family of all $\mathcal{I}_2$-deferred strongly Cesàro summable fuzzy variable double sequences in mean is denoted by $DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_E)$.

Definition 2.12. *Let $\{\varpi_{kl}\}$ be a double sequence of fuzzy variables with finite expected values defined on $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$. For $\forall \varepsilon, \delta > 0$, if the set*

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I^*_{p,q}(m)\right| \left|I^*_{r,s}(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \;\&\; E\left[\|\varpi_{kl} - \varpi\|\right] \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}_2,$$

then, the double sequence $\{\varpi_{kl}\}$ is μ-deferred $\mathcal{I}_2$-statistically convergent in mean to the fuzzy variable ϖ and we write ${}_\mu DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_E) - \lim\{\varpi_{kl}\} = \varpi$.

Example 2.13. *Consider an infinite point set credibility space $(\mathcal{Y}, B, \mathrm{Cr})$, where $\mathcal{Y} = \{\theta_1, \theta_2, \theta_3, \ldots\}$, B is the Borel algebra defined on $\mathcal{Y}$ and take the well-known Lebesgue measure as the credibility measure. Then, there exist integers y_1, y_2 such that $k = 2^{y_1} + p$, $l = 2^{y_2} + p$, p being an integer between 0 and $\min\{2^{y_1}, 2^{y_2}\} - 1$. Now, for any $k, l \in \mathbb{Z}$, we define the double sequence $\{\varpi_{kl}\}$ of fuzzy variables by*

$$\varpi_{kl}(\theta) = \begin{cases} 1, & \text{if } \frac{p}{2^{y_1+y_2}} \leq \theta \leq \frac{p+1}{2^{y_1+y_2}}; \\ 0, & \text{otherwise,} \end{cases}$$

and ϖ be a fuzzy variable defined by $\varpi(\theta) = 0$ for all $\theta \in \mathcal{Y}$. Calculation to this fuzzy variable double sequence proves that it is $\mathcal{I}_2$-deferred strongly Cesàro summable as well as μ-deferred $\mathcal{I}_2$-statistically convergent in mean to the fuzzy variable ϖ.

Remark 2.14. *μ-deferred $\mathcal{I}_2$-statistically convergence in mean does not imply μ-deferred $\mathcal{I}_2$-statistically convergence almost surely.*

Example 2.15. *Consider the fuzzy variable double sequence taken in (2) which does not μ-deferred $\mathcal{I}_2$-statistically converge almost surely to the fuzzy variable ϖ. But*

$$E\left[\|\varpi_{kl}-\varpi\|\right]=\frac{(k+l)+1}{2\,(k+l)^2}\to 0, \text{as } k,l\to\infty.$$

Thus, for each $\varepsilon,\delta>0$, we obtain

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|}\left|\left\{(k,l)\in\mathbb{N}^2:k\in I^*_{p,q}(m),l\in I^*_{r,s}(n)\ \&\ E\left[\|\varpi_{kl}-\varpi\|\right]\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}_2$$

which proves that the double sequence $\{\varpi_{kl}\}$ μ-deferred $\mathcal{I}_2$-statistically converges in mean to the fuzzy variable ϖ.

Remark 2.16. *μ-deferred $\mathcal{I}_2$-statistically convergence almost surely does not imply μ-deferred $\mathcal{I}_2$-statistically convergence in mean. A suitable example supporting this claim is placed in the following.*

Example 2.17. *Consider the credibility space as taken in Example 2.10 and define a double sequence $\{\varpi_{kl}\}$ in that space as follows:*

$$\varpi_{kl}\left(\theta_t\right)=\begin{cases}k+l, & \text{if } t=k+l\\ 0, & \text{if not}\end{cases} \tag{3}$$

for $k,l=1,2,...$ and also take ϖ to be a fuzzy variable defined by $\varpi(\theta)=0$ for all $\theta\in\Theta$. Here, the double sequence $\{\varpi_{kl}\}$ μ-deferred $\mathcal{I}_2$-statistically converges almost surely to the fuzzy variable ϖ.
However, for all $\varepsilon>0$ and for any $\delta\in\left(0,\frac{1}{2}\right)$,

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|}\left|\left\{(k,l)\in\mathbb{N}^2:k\in I^*_{p,q}(m),l\in I^*_{r,s}(n)\ \&\ E\left[\|\varpi_{kl}-\varpi\|\right]\geq\varepsilon\right\}\right|\geq\delta\right\}\notin\mathcal{I}_2.$$

Consequently, the double sequence $\{\varpi_{kl}\}$ does not μ-deferred $\mathcal{I}_2$-statistically converge in mean to the fuzzy variable ϖ.

Remark 2.18. *μ-deferred $\mathcal{I}_2$-statistically convergence in credibility does not imply μ-deferred $\mathcal{I}_2$-statistically convergence in mean.*

Example 2.19. *Consider the fuzzy variable sequence defined by (3) which does not μ-deferred $\mathcal{I}_2$-statistically converge in mean to μ. But, for any small number $\varepsilon,\delta>0$ and $\zeta\in\left[\frac{1}{2},1\right)$, we get*

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|}\left|\left\{(k,l)\in\mathbb{N}^2:k\in I^*_{p,q}(m),l\in I^*_{r,s}(n)\ \&\ \mathrm{Cr}\left\{\|\varpi_{kl}-\varpi\|\geq\varepsilon\right\}\geq\delta\right\}\right|\geq\zeta\right\}\in\mathcal{I}_2$$

for each $\varepsilon,\delta>0$ and $\zeta>0$. Therefore, the double sequence $\{\varpi_{kl}\}$ μ-deferred $\mathcal{I}_2$-statistically converges in credibility to ϖ.

Definition 2.20. *Let φ_{kl} and φ be the credibility distributions of fuzzy variables ϖ_{kl},ϖ respectively. The double sequence $\{\varpi_{kl}\}$ of fuzzy variables is $\mathcal{I}_2$-deferred strongly Cesàro summable in distribution to the fuzzy variable ϖ provided that for all $\varepsilon>0$, the following set*

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{\psi(m)\,\varkappa(n)}\sum_{k=p(m)+1}^{q(m)}\sum_{l=r(n)+1}^{s(n)}\left|\varphi_{kl}(x)-\varphi(x)\right|\geq\varepsilon\right\}\in\mathcal{I}_2$$

for all points x at which φ is continuous. We write it as $DC^{\mathcal{I}_2}_{(\psi,\varkappa)}\left(\Theta_\varphi\right)-\lim\varpi_{kl}=\varpi$. The space of all $\mathcal{I}_2$-deferred strongly Cesàro summable fuzzy variable double sequences in distribution is demonstrated by $DC^{\mathcal{I}_2}_{(\psi,\varkappa)}\left(\Theta_\varphi\right)$.

Definition 2.21. *Let $\{\varpi_{kl}\}$ be a double sequence of fuzzy variables. Let φ_{kl} and φ are credibility distributions for fuzzy variables ϖ_{kl} $(k,l=1,2,3,\ldots)$ and ϖ, respectively. For $\varepsilon,\delta>0$ and $\forall x$, (x being the points where φ is continuous), if the set*

$$\left\{(m,n)\in\mathbb{N}^2 : \frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|}\left|\left\{(k,l)\in\mathbb{N}^2 : k\in I^*_{p,q}(m), l\in I^*_{r,s}(n)\ \&\ |\varphi_{kl}(x)-\varphi(x)|\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}_2,$$

then $\{\varpi_{kl}\}$ is called μ-deferred $\mathcal{I}_2$ statistically convergent in distribution to the fuzzy variable ϖ and we write ${}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}\left(\Theta_{\varphi}\right)-\lim\{\varpi_{kl}\}=\varpi$.

Definition 2.22. *The double sequence $\{\varpi_{kl}\}$ of fuzzy variables is called $\mathcal{I}_2$-deferred strongly Cesàro summable w.r.t. uniformly almost surely (u.a.s.) to the fuzzy variable ϖ if $\forall\varepsilon>0,\exists\delta>0$ and a sequence $\left(E_j\right)$ with*

$$\left\{(m,n)\in\mathbb{N}^2 : \tfrac{1}{\psi(m)\varkappa(n)}\sum_{k=p(m)+1}^{q(m)}\sum_{l=r(n)+1}^{s(n)}\left|\mathrm{Cr}\left(E_j\right)-0\right|\geq\varepsilon\right\}\in\mathcal{I}_2$$

i.e., $DC^{\mathcal{I}_2}_{\psi,\varkappa}-\lim\mathrm{Cr}\left(E_j\right)=0$ *such that*

$$\left\{(m,n)\in\mathbb{N}^2 : \tfrac{1}{\psi(m)\varkappa(n)}\sum_{k=p(m)+1}^{q(m)}\sum_{l=r(n)+1}^{s(n)}|\varpi_{kl}(\theta)-\varpi(\theta)|\geq\delta\right\}\in\mathcal{I}_2,\quad\forall\theta\in\Theta-\left(E_j\right).$$

We write it as $DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{u.a.s.})-\lim\varpi_{kl}=\varpi$. The space of all $\mathcal{I}_2$-deferred strongly Cesàro summable fuzzy variable double sequences with respect to uniformly almost surely (u.a.s.) and it is denoted by $DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{u.a.s.})$.

Definition 2.23. *Let $\{\varpi_{kl}\}$ be a double sequence of fuzzy variables. If $\forall\alpha,\beta>0$, $\exists$ a sequence of events $\{F_t\}\in\mathcal{P}(\Theta)$ with $\mathrm{Cr}\{F_t\}$ converges to 0 so that the set*

$$\left\{(m,n)\in\mathbb{N}^2 : \tfrac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|}\left|\left\{(k,l)\in\mathbb{N}^2 : k\in I^*_{p,q}(m), l\in I^*_{r,s}(n)\quad \&\ |\varpi_{kl}(\theta)-\varpi(\theta)|\geq\alpha\}\right|\geq\beta\right\}\in\mathcal{I}_2,$$

$\forall\theta\in\Theta-\{F_t\}$.

In this case, we say $\{\varpi_{kl}\}$ is μ-deferred $\mathcal{I}_2$-statistically convergent uniformly almost surely (u.a.s.) in credibility space to the fuzzy variable ϖ and ${}_{\mu}DS^{\mathcal{I}_2}(\Theta_{u.a.s.})-\lim\{\varpi_{kl}\}=\varpi$.

Definition 2.24. *Let $\{\varpi_{kl}\}$ be a double sequence of fuzzy variables. We say $\{\varpi_{kl}\}$ is bounded in credibility if $\forall\varepsilon>0,\exists K>0$ such that*

$$\mathrm{Cr}\left(\|\varpi_{kl}\|\geq K\right)<\varepsilon.$$

We denote the set of such type of fuzzy double sequences by $\ell^2_\infty(\Theta_{\mathrm{Cr}})$.

Theorem 2.25. *${}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}})\cap\ell^2_\infty(\Theta_{\mathrm{Cr}})$ is closed set of $\ell^2_\infty(\Theta_{\mathrm{Cr}})$.*

Proof. Let $\varpi^{ij}=\left\{\varpi^{ij}_{kl}\right\}_{i,j\in\mathbb{N}}\in{}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}})\cap\ell^2_\infty(\Theta_{\mathrm{Cr}})$ be a μ-deferred $\mathcal{I}_2$-statistically convergent double sequence and converges to $\varpi=\{\varpi_{kl}\}\in\ell^2_\infty(\Theta_{\mathrm{Cr}})$. Assume ${}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}})-\lim\left\{\varpi^{ij}\right\}=y_{ij}$ for all $i,j\in\mathbb{N}$. Take a double sequence $\left\{\alpha_{ij}\right\}$ such that $\alpha_{ij}=\frac{\alpha}{2^{i+j}}$. So, for any $\alpha>0$, we have $\left\{\alpha_{ij}\right\}\to0$.
Therefore, for any given $\zeta>0$, there exists i,j such that

$$\mathrm{Cr}\left\{\left\|\varpi-\varpi^{ij}\right\|\geq\zeta\right\}<\frac{\alpha_{ij}}{4}.$$

Assume $\beta \in (0,1)$ and then, we have

$$K = \Big\{(m,n) \in \mathbb{N}^2 : \tfrac{1}{|I^*_{p,q}(m)||I^*_{r,s}(n)|} \Big|\Big\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \\ \& \operatorname{Cr}\Big\{\Big\|\varpi^{ij}_{kl} - y_{ij}\Big\| \geq \zeta\Big\} \geq \tfrac{\alpha_{ij}}{4}\Big\}\Big| < \tfrac{\beta}{3}\Big\} \in \mathcal{F}(\mathcal{I}_2),$$

and

$$L = \Big\{(m,n) \in \mathbb{N}^2 : \tfrac{1}{|I^*_{p,q}(m)||I^*_{r,s}(n)|} \Big|\Big\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \\ \& \operatorname{Cr}\Big\{\|\varpi^{de}_{kl} - y_{de}\| \geq \zeta\Big\} \geq \tfrac{\alpha_{de}}{4}\Big\}\Big| < \tfrac{\beta}{3}\Big\} \in \mathcal{F}(\mathcal{I}_2).$$

So, $K \cap L \in \mathcal{F}(\mathcal{I}_2)$. Hence, $K \cap L$ can not be empty set.
Let $(m,n) \in K \cap L$ be arbiitrary.
Then, we have

$$\frac{\Big|\Big\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \text{ and } \operatorname{Cr}\Big\{\Big\|\varpi^{ij}_{kl} - y_{ij}\Big\| \geq \zeta\Big\} \geq \frac{\alpha_{ij}}{4}\Big\}\Big|}{\Big|I^*_{p,q}(m)\Big|\Big|I^*_{r,s}(n)\Big|} < \frac{\beta}{3}$$

and

$$\frac{\Big|\Big\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \text{ and } \operatorname{Cr}\Big\{\|\varpi^{de}_{kl} - y_{de}\| \geq \zeta\Big\} \geq \frac{\alpha_{de}}{4}\Big\}\Big|}{\Big|I^*_{p,q}(m)\Big|\Big|I^*_{r,s}(n)\Big|} < \frac{\beta}{3}.$$

Also, we obtain

$$\tfrac{1}{|I^*_{p,q}(m)||I^*_{r,s}(n)|} \Big|\Big\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \text{ and} \\ \operatorname{Cr}\Big\{\Big\|\varpi^{ij}_{kl} - y_{ij}\Big\| \geq \zeta\Big\} \geq \tfrac{\alpha_{ij}}{4} \vee \operatorname{Cr}\Big\{\|\varpi^{de}_{kl} - y_{de}\| \geq \zeta\Big\} \geq \tfrac{\alpha_{de}}{4}\Big\}\Big| < \beta < 1.$$

So, there exists a pair of naturals of the form (k,l) such that

$$\operatorname{Cr}\Big\{\Big\|\varpi^{ij}_{kl} - y_{ij}\Big\| \geq \zeta\Big\} < \frac{\alpha_{ij}}{4}$$

and

$$\operatorname{Cr}\Big\{\|\varpi^{de}_{kl} - y_{de}\| \geq \zeta\Big\} < \frac{\alpha_{de}}{4}.$$

Hence, we get

$$\operatorname{Cr}\Big(\Big\|y_{ij} - y_{de}\Big\| \geq \zeta\Big) \leq \operatorname{Cr}\Big\{\Big\|\varpi^{ij}_{kl} - y_{ij}\Big\| \geq \zeta'\Big\} + \operatorname{Cr}\Big\{\Big\|\varpi^{ij}_{kl} - \varpi^{de}_{kl}\Big\| \geq \zeta'\Big\}$$

$$+\operatorname{Cr}\Big\{\Big\|\varpi^{de}_{kl} - y_{de}\Big\| \geq \zeta'\Big\}, \text{ for some } \zeta' \leq \tfrac{\zeta}{3}$$

$$\leq \operatorname{Cr}\Big\{\Big\|\varpi^{ij}_{kl} - y_{ij}\Big\| \geq \zeta'\Big\} + \operatorname{Cr}\Big\{\Big\|\varpi^{ij}_{kl} - \varpi\Big\| \geq \zeta''\Big\}$$

$$+\operatorname{Cr}\Big\{\Big\|\varpi^{de}_{kl} - \varpi\Big\| \geq \zeta''\Big\} + \operatorname{Cr}\Big\{\Big\|\varpi^{de}_{kl} - y_{de}\Big\| \geq \zeta'\Big\}, \text{ for some } \zeta'' \leq \tfrac{\zeta'}{2}$$

$$\leq \tfrac{\alpha_{ij}}{4} + \tfrac{\alpha_{de}}{4} + \tfrac{\alpha_{ij}}{4} + \tfrac{\alpha_{de}}{4} = \alpha_{ij}.$$

Hence, $\left\{y_{ij}\right\}$ is a Cauchy double sequence in credibility.
Therefore there is a y so that $\left\{y_{ij}\right\} \to y$ in credibility, as $i, j \to \infty$.
Now we demonstrate that $\{\varpi_{kl}\} \in {}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}})$ and ${}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}}) - \lim\{\varpi_{kl}\} = y$.
For a given $\alpha, \zeta > 0$, select $i, j \in \mathbb{N}$ such that $\alpha_{ij} < \frac{\alpha}{4}$, $\mathrm{Cr}\left\{\left\|\varpi^{ij}_{kl} - \varpi\right\| \geq \zeta\right\} < \frac{\alpha}{4}$, $\mathrm{Cr}\left\{\left\|y_{ij} - y\right\| \geq \zeta\right\} < \frac{\alpha}{4}$.
Then

$$\frac{\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \ \&\ \mathrm{Cr}\left\{\left\|\varpi_{kl} - y\right\| \geq \zeta\right\} \geq \alpha\right\}\right|}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|}$$

$$\leq \frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|} \mid \left\|\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \ \&\right.$$
$$\mathrm{Cr}\left\{\left\|\varpi^{ij}_{kl} - y_{ij}\right\| \geq \zeta\right\} + \mathrm{Cr}\left\{\left\|\varpi_{kl} - \varpi^{ij}_{kl}\right\| \geq \zeta\right\} + \mathrm{Cr}\left\{\left\|y_{ij} - y\right\| \geq \zeta\right\} \geq \alpha\right\} \mid$$

$$\leq \frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|} \mid \left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \ \&\right.$$
$$\left.\mathrm{Cr}\left\{\left\|\varpi^{ij}_{kl} - y_{ij}\right\| \geq \zeta\right\} + \frac{\alpha}{4} + \frac{\alpha}{4} \geq \alpha\right\} \mid$$

$$\leq \frac{\mid \left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \ \&\ \mathrm{Cr}\left\{\left\|\varpi^{ij}_{kl} - y_{ij}\right\| \geq \zeta\right\} \geq \frac{\alpha}{2}\right\} \mid}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|}.$$

So, for any given $\beta > 0$,

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|}\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \ \&\ \mathrm{Cr}\left\{\left\|\varpi_{kl} - y\right\| \geq \zeta\right\} \geq \alpha\right\}\right| < \beta\right\}$$

$$\supseteq \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|} \mid \left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n)\right.\right.$$
$$\left.\left.\&\ \mathrm{Cr}\left\{\left\|\varpi^{ij}_{kl} - y_{ij}\right\| \geq \zeta\right\} \geq \frac{\alpha}{2}\right\} \mid < \beta\right\} \in \mathcal{F}(\mathcal{I}_2).$$

As a result, we get

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|}\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \ \&\ \mathrm{Cr}\left\{\left\|\varpi_{kl} - y\right\| \geq \zeta\right\} \geq \alpha\right\}\right| < \beta\right\} \in \mathcal{F}(\mathcal{I}_2).$$

Hence, $\{\varpi_{kl}\} \in {}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}})$ and ${}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}}) - \lim\{\varpi_{kl}\} = y$. This finalizes the result. □

Definition 2.26. *Let $\{\varpi_{kl}\}$ be a double sequence of fuzzy variables. The double sequence $\{\varpi_{kl}\}$ is called bounded in mean if*

$$\sup_{k,l} E\left[\|\varpi_{kl}\|\right] \text{ is finite.}$$

We denote the set of such type of fuzzy variable double sequences by $\ell^2_\infty(\Theta_E)$.

Definition 2.27. *Let $\{\varpi_{kl}\}$ be a double sequence of fuzzy variables and φ_{kl} be the credibility distribution functions for the fuzzy variables ϖ_{kl}, for each k, l. Then, $\{\varpi_{kl}\}$ is said to be bounded in distribution if*

$$\sup_{k,l}\left[\left\|\varphi_{kl}(x)\right\|\right] \text{ is finite,}$$

where x are such points at which the distribution function is continuous. We denote the set of such type of fuzzy variable double sequences by $\ell^2_\infty\left(\Theta_\varphi\right)$.

Definition 2.28. *Let $\{\varpi_{kl}\}$ be a double sequence of fuzzy variables. Then, $\{\varpi_{kl}\}$ is called bounded in almost surely if $\forall \varepsilon > 0$, $\exists$ a sequence of events $\{F_t\} \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}(\{F_t\}) = 1$ such that*

$$\sup_{k,l} [\|\varpi_{kl}(\theta)\|] \text{ is finite, } \forall \theta \in \{F_t\}.$$

We denote the set of such type of fuzzy variable double sequences by $\ell^2_\infty(\Theta_{a.s.})$.

Definition 2.29. *Let $\{\varpi_{kl}\}$ be a double sequence of fuzzy variables. The double sequence $\{\varpi_{kl}\}$ is bounded w.r.t. uniformly almost surely if $\forall \varepsilon > 0$ $\exists$ a sequence of events $\{E_k\} \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{E_k\} \to 0$ and*

$$\sup_{k,l} [\|\varpi_{kl}(\theta)\|] \text{ is finite, } \forall \theta \in \Theta - E_k \text{ for each } k.$$

We denote the set of such type of fuzzy variable double sequences by $\ell^2_\infty(\Theta_{u.a.s.})$.

In order to convey the theorems without proof, we must show the subsequent theorem in the same manner as described earlier.

Theorem 2.30. *Sets ${}_\mu DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_E) \cap \ell^2_\infty(\Theta_E)$, ${}_\mu DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_D) \cap \ell^2_\infty(\Theta_D)$ and ${}_\mu DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.}) \cap \ell^2_\infty(\Theta_{a.s.})$ are closed sets of $\ell^2_\infty(\Theta_E)$, $\ell^2_\infty(\Theta_D)$ and $\ell^2_\infty(\Theta_{a.s.})$, respectively.*

Theorem 2.31. *If $\{\varpi_{kl}\} \in \ell^2_\infty(\Theta_{u.a.s.})$ then ${}_\mu DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.c.}) \subset DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{u.a.s.})$.*

Theorem 2.32. *The following statements supply:*
(i) ${}_\mu DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.}) \cap \ell^2_\infty(\Theta_{a.s.}) = DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.}) \cap \ell^2_\infty(\Theta_{a.s.})$.
(ii) ${}_\mu DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}}) \cap \ell^2_\infty(\Theta_{\mathrm{Cr}}) = DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}}) \cap \ell^2_\infty(\Theta_{\mathrm{Cr}})$.
(iii) ${}_\mu DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_E) \cap l_\infty(\Theta_E) = DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_E) \cap \ell^2_\infty(\Theta_E)$.
(iv) ${}_\mu DS^{\mathcal{I}_2}_{(\psi,\varkappa)}\left(\Theta_\varphi\right) \cap \ell^2_\infty\left(\Theta_\varphi\right) = DC^{\mathcal{I}_2}_{(\psi,\varkappa)}\left(\Theta_\varphi\right) \cap \ell^2_\infty\left(\Theta_\varphi\right)$.
(iv) ${}_\mu DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.c.}) \cap \ell^2_\infty(\Theta_{u.a.s.}) = DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{u.a.s.}) \cap \ell^2_\infty(\Theta_{u.a.s.})$.

Theorem 2.33. *If $\{\varpi_{kl}\} \in \ell^2_\infty(\Theta_{a.s.})$ then ${}_\mu DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.}) \subset DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.})$.*

Proof. Let $\{\varpi_{kl}\} \in \ell^2_\infty(\Theta_{\mathrm{a.s.}})$ and ${}_\mu DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{a.s.}}) - \lim(\varpi_{kl}) = \varpi$.
Hence $\exists K > 0$ such that $\|\varpi_{kl} - \varpi\| \le K$ for all $k, l \in \mathbb{N}$.
For a given $\varepsilon > 0$, we have

$$\frac{1}{\psi(m)\varkappa(n)} \sum_{k=p(m)+1}^{q(m)} \sum_{l=r(n)+1}^{s(n)} \|\varpi_{kl} - \varpi\|$$

$$= \frac{1}{\psi(m)\varkappa(n)} \left(\sum_{\substack{k=p(m)+1 \\ \|\varpi_{kl}-\varpi\| \ge \varepsilon}}^{q(m)} \sum_{\substack{l=r(n)+1 \\ \|\varpi_{kl}-\varpi\| \ge \varepsilon}}^{s(n)} + \sum_{\substack{k=p(m)+1 \\ \|\varpi_{kl}-\varpi\| < \varepsilon}}^{q(m)} \sum_{\substack{l=r(n)+1 \\ \|\varpi_{kl}-\varpi\| < \varepsilon}}^{s(n)} \right) \|\varpi_{kl} - \varpi\|$$

$$\le \frac{D}{|I^*_{p,q}(m)||I^*_{r,s}(n)|} \left| \left\{ (k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \ \& \ \|\varpi_{kl} - \varpi\| \ge \varepsilon \right\} \right| + \frac{\varepsilon}{2}.$$

Then, we obtain

$$\left\{ (m,n) \in \mathbb{N}^2 : \frac{1}{\psi(m)\varkappa(n)} \sum_{k=p(m)+1}^{q(m)} \sum_{l=r(n)+1}^{s(n)} \|\varpi_{kl} - \varpi\| \ge \varepsilon \right\}$$

$$\subseteq \left\{ (m,n) \in \mathbb{N}^2 : \frac{1}{|I^*_{p,q}(m)||I^*_{r,s}(n)|} \left| \left\{ (k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \ \& \ \|\varpi_{kl} - \varpi\| \ge \frac{\varepsilon}{2K} \right\} \right| \right\} \in \mathcal{I}_2.$$

Therefore, $\{\varpi_{kl}\} \in_\mu DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{a.s.}})$. □

In view of the standard techniques, the following three theorems can be established.

Theorem 2.34. *If* $\{\varpi_{kl}\} \in \ell^2_\infty(\Theta_{\mathrm{Cr}})$ *then* ${}_\mu DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}}) \subset DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}})$.

Theorem 2.35. *If* $\{\varpi_{kl}\} \in \ell^2_\infty(\Theta_E)$ *then* ${}_\mu DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_E) \subset DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_E)$.

Theorem 2.36. *If* $\{\varpi_{kl}\} \in \ell^2_\infty\left(\Theta_\varphi\right)$, *then* ${}_\mu DS^{\mathcal{I}_2}_{(\psi,\varkappa)}\left(\Theta_\varphi\right) \subset DC^{\mathcal{I}_2}_{(\psi,\varkappa)}\left(\Theta_\varphi\right)$.

Theorem 2.37. *If the double sequence* $\{\varpi_{kl}\}$ *of fuzzy variables is* $\mathcal{I}_2$*-deferred strongly Cesàro summable almost surely to* ϖ *then* $\{\varpi_{kl}\}$ *is* μ*-deferred* $\mathcal{I}_2$*-statistically convergent almost surely to* ϖ.

Proof. Let $\{\varpi_{kl}\}$ be fuzzy variable double sequence which is $\mathcal{I}_2$-deferred strongly Cesàro summable almost surely to the fuzzy variable ϖ. Then, by definition $\forall\varepsilon > 0$, there is a $T \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{T\} = 1$ such that $\forall\theta \in T$, the set

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\psi(m)\varkappa(n)} \sum_{k=p(m)+1}^{q(m)} \sum_{l=r(n)+1}^{s(n)} |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\} \in \mathcal{I}_2.$$

Now, for any $\varepsilon > 0$, we have

$$\begin{aligned}
&\sum_{k=p(m)+1}^{q(m)} \sum_{l=r(n)+1}^{s(n)} |\varpi_{kl}(\theta) - \varpi(\theta)| \\
&= \left(\sum_{\substack{k=p(m)+1 \\ |\varpi_{kl}(\theta)-\varpi(\theta)|\geq\varepsilon}}^{q(m)} \sum_{\substack{l=r(n)+1 \\ |\varpi_{kl}(\theta)-\varpi(\theta)|\geq\varepsilon}}^{s(n)} + \sum_{\substack{k=p(m)+1 \\ |\varpi_{kl}(\theta)-\varpi(\theta)|<\varepsilon}}^{q(m)} \sum_{\substack{l=r(n)+1 \\ |\varpi_{kl}(\theta)-\varpi(\theta)|<\varepsilon}}^{s(n)} \right) |\varpi_{kl}(\theta) - \varpi(\theta)| \\
&\geq \sum_{\substack{k=p(m)+1 \\ |\varpi_{kl}(\theta)-\varpi(\theta)|\geq\varepsilon}}^{q(m)} \sum_{\substack{l=r(n)+1 \\ |\varpi_{kl}(\theta)-\varpi(\theta)|\geq\varepsilon}}^{s(n)} |\varpi_{kl}(\theta) - \varpi(\theta)| \\
&\geq \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n)\ \&\ |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right| \varepsilon,
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{1}{\varepsilon\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|} \sum_{k=p(m)+1}^{q(m)} \sum_{l=r(n)+1}^{s(n)} |\varpi_{kl}(\theta) - \varpi(\theta)| \\
&\geq \frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n)\ \&\ |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right|.
\end{aligned}$$

Thus, for any $\delta > 0$ we have,

$$\begin{aligned}
&\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n), \|\varpi_{kl}(\theta) - \varpi(\theta)\| \geq \varepsilon\right\}\right| \geq \delta\right\} \\
&\qquad \subseteq \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\psi(m)\varkappa(n)} \sum_{k=p(m)+1}^{q(m)} \sum_{l=r(n)+1}^{s(n)} |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\delta\right\} \in \mathcal{I}_2.
\end{aligned}$$

Hence, $\{\varpi_{kl}\}$ is μ-deferred $\mathcal{I}_2$-statistically convergent almost surely to the fuzzy variable ϖ. □

Remark 2.38. *Converse of Theorem 2.37 is not true in general. We illustrate the same in the following example.*

Example 2.39. *Assume the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to be $\theta_1, \theta_2, \theta_3, \ldots$ with*

$$\mathrm{Cr}(T) = \begin{cases} \sup_{\theta_{k+l} \in T} \frac{1}{k+l+1}, & \text{if } \sup_{\theta_{k+l} \in T} \frac{1}{k+l+1} < \frac{1}{2}; \\ 1 - \sup_{\theta_{k+l} \in T^c} \frac{1}{k+l+1}, & \text{if } \sup_{\theta_{k+l} \in T^c} \frac{1}{k+l+1} < \frac{1}{2}; \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Let $q(m)$ and $s(n)$ be strictly increasing sequences of positive integers. Let us consider the double sequence $\{\varpi_{kl}\}$ of fuzzy variables as

$$\varpi_{kl}(\theta \in \Theta) = \begin{cases} kl, & \text{if } q(m) - \sqrt{q(m)} < k \leq q(m), s(n) - \sqrt{s(n)} < l \leq s(n) \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

and $\xi \equiv 0$. Then, for any small $\varepsilon > 0$ and any $\theta = \theta_{k+l}$,

$$\frac{\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \ \& \ |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right|}{\left|I^*_{p,q}(m)\right| \left|I^*_{r,s}(n)\right|} \leq \frac{\sqrt{q(m)}\sqrt{s(n)}}{\left|I^*_{p,q}(m)\right| \left|I^*_{r,s}(n)\right|}.$$

Thus, for any $\delta > 0$ we obtain

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \ \& \ |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right|}{\left|I^*_{p,q}(m)\right| \left|I^*_{r,s}(n)\right|} \geq \delta\right\}$$
$$\subseteq \left\{(m,n) \in \mathbb{N}^2 : \frac{\sqrt{q(m)}\sqrt{s(n)}}{\left|I^*_{p,q}(m)\right| \left|I^*_{r,s}(n)\right|} \geq \delta\right\}.$$

Since the set $\left\{(m,n) \in \mathbb{N}^2 : \frac{\sqrt{q(m)}\sqrt{s(n)}}{\left|I^*_{p,q}(m)\right| \left|I^*_{r,s}(n)\right|} \geq \delta\right\}$ is finite, so it belongs to $\mathcal{I}_2$, therefore $\{\varpi_{kl}\} \in {}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.})$. However, for any $\theta = \theta_{k+l}$

$$\frac{1}{\psi(m)\varkappa(n)} \sum_{k=p(m)+1}^{q(m)} \sum_{l=r(n)+1}^{s(n)} |\varpi_{kl}(\theta) - \varpi(\theta)| = \frac{kl}{\psi(m)\varkappa(n)} \sum_{k=q(m)-\sqrt{q(m)}}^{q(m)} \sum_{l=s(n)-\sqrt{s(n)}}^{s(n)} |kl|$$
$$= \frac{kl}{\psi(m)\varkappa(n)} \frac{\left(2q_m^{\frac{3}{2}} - \left(q(m) - \sqrt{q(m)}\right)\right)\left(2s_n^{\frac{3}{2}}\left(s(n) - \sqrt{s(n)}\right)\right)}{2} \leq \frac{kl}{\psi(m)\varkappa(n)} \frac{\left(2q_m^{\frac{3}{2}} - p(m)\right)\left(2s_n^{\frac{3}{2}} - r(n)\right)}{2}.$$

So, we have

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\psi(m)\varkappa(n)} \sum_{k=p(m)+1}^{q(m)} \sum_{l=r(n)+1}^{s(n)} |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \frac{1}{4}\right\}$$
$$\subseteq \left\{(m,n) \in \mathbb{N}^2 : \frac{\left(2q^{\frac{3}{2}} - p(m)\right)\left(2s^{\frac{3}{2}} - r(n)\right)}{\psi(m)\varkappa(n)} \geq \frac{1}{2kl}\right\}$$
$$= \{u, u+1, u+2, \ldots\}$$

for some $u \in \mathbb{N}$ and and so the set belongs to $\mathcal{F}(\mathcal{I}_2)$ as $\mathcal{I}_2$ is an admissible ideal. Thus $\{\varpi_{kl}\} \notin DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.})$.

Theorem 2.40. *If the double sequence $\{\varpi_{kl}\}$ of fuzzy variables is $\mathcal{I}_2$-deferred strongly Cesàro summable in credibility to ϖ, then $\{\varpi_{kl}\}$ is μ-deferred $\mathcal{I}_2$-statistically convergent in credibility to ϖ.*

Proof. Let $\{\varpi_{kl}\}$ be a double sequence of fuzzy variable which is $\mathcal{I}_2$-deferred strongly Cesàro summable in credibility to ϖ. Then, by the Definition 2.4 for all $\varepsilon, \delta > 0$, we obtain

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\psi(m)\varkappa(n)} \sum_{k=p(m)+1}^{q(m)} \sum_{l=r(n)+1}^{s(n)} \mathrm{Cr}\left(\|\varpi_{kl} - \varpi\| \geq \varepsilon\right) \geq \delta\right\} \in \mathcal{I}_2.$$

Now, for any $\varepsilon, \delta > 0$, we obtain

$$\sum_{k=p(m)+1}^{q(m)} \sum_{l=r(n)+1}^{s(n)} \mathrm{Cr}\left(\|\varpi_{kl} - \varpi\| \geq \varepsilon\right) \geq \sum_{\substack{k=p(m)+1 \\ \mathrm{Cr}(\|\varpi_{kl}-\varpi\|\geq\varepsilon)\geq\delta}}^{q(m)} \mathrm{Cr}\left(\|\varpi_{kl} - \varpi\| \geq \varepsilon\right)$$
$$\geq \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \;\&\; \mathrm{Cr}\left(\|\varpi_{kl} - \varpi\| \geq \varepsilon\right) \geq \delta\right\}\right| \delta.$$

which implies

$$\frac{1}{\beta\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|} \sum_{k=p(m)+1}^{q(m)} \sum_{l=r(n)+1}^{s(n)} \mathrm{Cr}\left(\|\varpi_{kl} - \varpi\| \geq \varepsilon\right)$$
$$\geq \frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \;\&\; \mathrm{Cr}\left(\|\varpi_{kl} - \varpi\| \geq \varepsilon\right) \geq \beta\right\}\right|.$$

Then, for any $\xi > 0$ we have

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n), \mathrm{Cr}\left(\|\varpi_{kl} - \varpi\| \geq \varepsilon\right) \geq \beta\right\}\right| \geq \xi\right\}$$
$$\subseteq \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\psi(m)\varkappa(n)} \sum_{i=p(m)+1}^{q(m)} \sum_{j=r(n)+1}^{s(n)} \mathrm{Cr}\left(\|\varpi_{kl} - \varpi\| \geq \varepsilon\right) \geq \delta\xi\right\} \in \mathcal{I}_2.$$

Hence, $\{\varpi_{kl}\}$ is μ-deferred $\mathcal{I}_2$-statistically convergent in credibility to the fuzzy variable ϖ. □

We state the following 3 results without proof, those can be established using standard techniques.

Theorem 2.41. *If the double sequence $\{\varpi_{kl}\}$ of fuzzy variables is $\mathcal{I}_2$-deferred strongly Cesàro summable in mean to ϖ, then $\{\varpi_{kl}\}$ is μ-deferred $\mathcal{I}_2$-statistically convergent in mean to ϖ.*

Theorem 2.42. *If the double sequence $\{\varpi_{kl}\}$ of fuzzy variables is $\mathcal{I}_2$-deferred strongly Cesàro summable in distribution to ϖ, then $\{\varpi_{kl}\}$ is μ-deferred $\mathcal{I}_2$-statistically convergent in distribution to ϖ.*

Theorem 2.43. *If the double sequence $\{\varpi_{kl}\}$ of fuzzy variables is $\mathcal{I}_2$-deferred strongly Cesàro summable uniformly almost surely to ϖ, then $\{\varpi_{kl}\}$ is μ-deferred $\mathcal{I}_2$-statistically convergent uniformly almost surely to ϖ.*

Definition 2.44. *Any two fuzzy variable double sequences $\{\varpi_{kl}\}$ and $\{q_{kl}\}$ are said to be equivalent w.r.t. (p,q) and (r,s) if for any $\varepsilon > 0$, the following set*

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{\left|M \cap I^*_{p,q}(m) \cap I^*_{r,s}(n)\right|}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|} \geq \varepsilon\right\} \in \mathcal{I}_2,$$

where $M := \left\{(k,l) \in \mathbb{N}^2 : \varpi_{kl}(\theta) \neq q_{kl}(\theta)\right\}$.

Theorem 2.45. *Suppose $\{\varpi_{kl}\}$ and $\{q_{kl}\}$ are two equivalent fuzzy variable sequences w.r.t. (p,q) and (r,s). Then, the double sequence $\{\varpi_{kl}\}$ is μ-deferred $\mathcal{I}_2$-statistically convergent almost surely implies that double sequence $\{q_{kl}\}$ is μ-deferred $\mathcal{I}_2$-statistically convergent almost surely.*

Proof. Let the double sequence $\{\varpi_{kl}\}$ of fuzzy variables be μ-deferred $\mathcal{I}_2$-statistically convergent almost surely. Then, for all $\varepsilon, \delta > 0$, there is a $T \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{T\} = 1$ and $\forall \theta \in T$, the set

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I^*_{p,q}(m)\right| \left|I^*_{r,s}(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \;\&\; |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

Consider the set

$$M := \left\{(k,l) \in \mathbb{N}^2 : \varpi_{kl}(\theta) \neq q_{kl}(\theta)\right\}.$$

So, for any preassigned $\varepsilon > 0$ and $\theta \in T$, we get

$$\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \;\&\; |q_{kl}(\theta) - q(\theta)| \geq \varepsilon\right\}$$

$$= \left((k,l) \in \mathbb{N}^2 : k \in M_{p,q}, l \in M_{r,s} \;\&\; |q_{kl}(\theta) - q(\theta)| \geq \varepsilon\right)$$

$$\cup \left((k,l) \in \mathbb{N}^2 : k \in M^c_{p,q}, l \in M^c_{r,s} \;\&\; |q_{kl}(\theta) - q(\theta)| \geq \varepsilon\right)$$

where $M_{p,q} := I^*_{p,q}(m) \cap M$, $M^c_{p,q} := I^*_{p,q}(m) \cap M^c$, $M_{r,s} := I^*_{r,s}(n) \cap M$ and $M^c_{r,s} := I^*_{r,s}(n) \cap M^c$.
Thus, we get

$$\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \;\&\; |q_{kl}(\theta) - q(\theta)| \geq \varepsilon\right\}$$

$$\subseteq \left((k,l) \in \mathbb{N}^2 : k \in M_{p,q}, l \in M_{r,s} \;\&\; |q_{kl}(\theta) - q(\theta)| \geq \varepsilon\right)$$

$$\cup \left((k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \;\&\; |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right)$$

Therefore,

$$\frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \;\&\; |q_{kl}(\theta) - q(\theta)| \geq \varepsilon\right\}\right|$$

$$\leq \frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in M_{p,q}, l \in M_{r,s} \;\&\; |q_{kl}(\theta) - q(\theta)| \geq \varepsilon\right\}\right|$$

$$+ \frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \;\&\; |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right|.$$

Therefore, for any $\delta > 0$ we obtain

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \;\&\; |q_{kl}(\theta) - q(\theta)| \geq \varepsilon\right\}\right| \geq \delta\right\}$$

$$\subseteq \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in M_{p,q}, l \in M_{r,s} \;\&\; |q_{kl}(\theta) - q(\theta)| \geq \varepsilon\right\}\right| \geq \delta\right\}$$

$$\cup \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \;\&\; |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right| \geq \delta\right\}.$$

As the right hand side set belongs to $\mathcal{I}_2$ and $\theta \in T$ was arbitrary, hence

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I^*_{p,q}(m)\right| \left|I^*_{r,s}(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \;\&\; |q_{kl}(\theta) - q(\theta)| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

This implies that $\{q_{kl}\}$ is μ-deferred $\mathcal{I}_2$-statistically convergent almost surely. □

Theorem 2.46. *Suppose $\{\varpi_{kl}\}$ and $\{q_{kl}\}$ be two equivalent fuzzy variable sequences w.r.t. (p,q) and (r,s). Then, the double sequence $\{\varpi_{kl}\}$ is μ-deferred $\mathcal{I}_2$-statistically convergent uniformly almost surely implies that double sequence $\{q_{kl}\}$ is μ-deferred $\mathcal{I}_2$-statistically convergent uniformly almost surely.*

Proof. Consider $\theta \in \Theta - \left(E_j\right)$, where $\left(E_j\right)$ is the sequence of event such that $DC^{\mathcal{I}_2}_{(\psi,\varkappa)} - \lim \mathrm{Cr}\left(E_j\right) = 0$ for each j and adapt the method which is followed in the Theorem 2.45. In this way, the proof can be obtained easily. □

Definition 2.47. *The double sequence $\{\varpi_{kl}\}$ of fuzzy variables is said to be μ-$\mathcal{I}_2$-statistically convergent in credibility to the fuzzy variable ϖ if for all $\varepsilon, \delta > 0$, and $\zeta > 0$, the set*

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{|I^*(m)|\,|I^*(n)|}\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*(m), l \in I^*(n) \;\&\; \mathrm{Cr}\left\{\|\varpi_{kl} - \varpi\| \geq \varepsilon\right\} \geq \delta\right\}\right| \geq \zeta\right\}$$

belongs to $\mathcal{I}_2$.
We denote it as ${}_{\mu}S^{\mathcal{I}_2}(\Theta_{\mathrm{Cr}}) - \lim\{\varpi_{kl}\} = \varpi$. The space of all μ-$\mathcal{I}_2$-statistically convergent fuzzy variable sequence in credibility is demonstrated by ${}_{\mu}S^{\mathcal{I}_2}(\Theta_{\mathrm{Cr}})$.

Theorem 2.48. *If $\liminf_m \frac{p_m}{q_m} \neq 1$ and $\liminf_n \frac{r_n}{s_n} \neq 1$ then ${}_{\mu}S^{\mathcal{I}_2}(\Theta_{\mathrm{Cr}}) \subset {}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}})$.*

Proof. Let $\liminf_m \frac{p_m}{q_m} = a(\neq 1)$ and $\liminf_n \frac{r_n}{s_n} = c(\neq 1)$. So, there exist $b, d > 0$ such that $\frac{p_m}{q_m} \geq a + b$ and $\frac{r_n}{s_n} \geq c + d$ for sufficiently large m, n. So, we get

$$\frac{q_m - p_m}{q_m} \geq \frac{b}{a+b}, \frac{s_n - r_n}{s_n} \geq \frac{d}{c+d}.$$

Suppose $(\varpi_{kl}) \in {}_{\mu}S^{\mathcal{I}_2}(\Theta_{\mathrm{Cr}})$. Then, for all $\varepsilon, \delta > 0$, and $\zeta > 0$, the set

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{|I^*(m)|\,|I^*(n)|}\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*(m), l \in I^*(n) \;\&\; \mathrm{Cr}\left\{\|\varpi_{kl} - \varpi\| \geq \varepsilon\right\} \geq \delta\right\}\right| \geq \zeta\right\}$$

belongs to $\mathcal{I}_2$.
As $\lim_{m\to\infty} q(m) = \infty$ and $\lim_{n\to\infty} s(n) = \infty$, we also have for any given $\varepsilon, \delta > 0$, and $\zeta > 0$, the set

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I_q^*(m)\right|\left|I_s^*(n)\right|}\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I_q^*(m), l \in I_s^*(n) \;\&\; \mathrm{Cr}\left\{\|\varpi_{kl} - \varpi\| \geq \varepsilon\right\} \geq \delta\right\}\right| \geq \zeta\right\} \in \mathcal{I}_2.$$

For any pair of sequences (p,q) and (r,s) which satisfy (1), $I_q^*(m) \supset I_{p,q}^*(m)$ and $I_s^*(n) \supset I_{r,s}^*(n)$ supply. So, we have

$$\frac{1}{\left|I_q^*(m)\right|\left|I_s^*(n)\right|}\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I_q^*(m), l \in I_s^*(n) \;\&\; \mathrm{Cr}\left\{\|\varpi_{kl} - \varpi\| \geq \varepsilon\right\} \geq \delta\right\}\right|$$

$$\geq \frac{1}{\left|I_q^*(m)\right|\left|I_s^*(n)\right|}\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I_{p,q}^*(m), l \in I_{r,s}^*(n) \;\&\; \mathrm{Cr}\left\{\|\varpi_{kl} - \varpi\| \geq \varepsilon\right\} \geq \delta\right\}\right|$$

$$= \frac{\left|I_{p,q}^*(m)\right|\left|I_{r,s}^*(n)\right|}{\left|I_q^*(m)\right|\left|I_s^*(n)\right|}\frac{1}{\left|I_{p,q}^*(m)\right|\left|I_{r,s}^*(n)\right|}\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I_{p,q}^*(m), l \in I_{r,s}^*(n)\right\} \;\&\; \mathrm{Cr}\left\{\|\varpi_{kl} - \varpi\| \geq \varepsilon\right\} \geq \delta\right|$$

$$\geq \frac{bd}{(a+b)(c+d)}\frac{1}{\left|I_{p,q}^*(m)\right|\left|I_{r,s}^*(n)\right|}\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I_{p,q}^*(m), l \in I_{r,s}^*(n)\right\} \;\&\; \mathrm{Cr}\left\{\|\varpi_{kl} - \varpi\| \geq \varepsilon\right\} \geq \delta\right|$$

and so

$$\frac{(a+b)(c+d)}{bd}\frac{1}{\left|I_q^*(m)\right|\left|I_s^*(n)\right|}\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I_q^*(m), l \in I_s^*(n) \;\&\; \mathrm{Cr}\left\{\|\varpi_{kl} - \varpi\| \geq \varepsilon\right\} \geq \delta\right\}\right|$$

$$\geq \frac{1}{\left|I_{p,q}^*(m)\right|\left|I_{r,s}^*(n)\right|}\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I_{p,q}^*(m), l \in I_{r,s}^*(n) \;\&\; \mathrm{Cr}\left\{\|\varpi_{kl} - \varpi\| \geq \varepsilon\right\} \geq \delta\right\}\right|.$$

Hence, for any $\zeta > 0$ we obtain

$$\Big\{(m,n) \in \mathbb{N}^2 : \tfrac{1}{\left|I^*_{p,q}(m)\right|\left|I^*_{r,s}(n)\right|}\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{p,q}(m), l \in I^*_{r,s}(n) \right.\right.$$
$$\left.\& \operatorname{Cr}\{\|\varpi_{kl} - \varpi\| \geq \varepsilon\} \geq \delta\}\right| \geq \zeta\}$$
$$\subseteq \Big\{(m,n) \in \mathbb{N}^2 : \tfrac{1}{\left|I^*_{q}(m)\right|\left|I^*_{s}(n)\right|}\left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*_{q}(m), l \in I^*_{s}(n) \right.\right.$$
$$\left.\& \operatorname{Cr}\{\|\varpi_{kl} - \varpi\| \geq \varepsilon\} \geq \delta\}\right| \geq \tfrac{\zeta bd}{(a+b)(c+d)}\Big\} \in \mathcal{I}_2.$$

Therefore, we obtain $(\varpi_{kl}) \in {}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}})$. □

Remark 2.49. *If* $\liminf_m \frac{p_m}{q_m} \neq 1$ *and* $\liminf_n \frac{r_n}{s_n} \neq 1$, *then* ${}_{\mu}S^{\mathcal{I}_2}(\Theta_{\mathrm{Cr}}) \subset DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}})$.

Definition 2.50. *The double sequence* (ϖ_{kl}) *of fuzzy variables is said to be* $\mathcal{I}_2$ *strongly Cesàro summable in credibility to* ϖ *if for all* $\varepsilon > 0$ *and* $\delta > 0$, *the following set*

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n}\operatorname{Cr}(\|\varpi_{kl} - \varpi\| \geq \varepsilon) \geq \delta\right\} \in \mathcal{I}_2.$$

We denote it as $C^{\mathcal{I}_2}(\Theta_{\mathrm{Cr}}) - \lim \varpi_{kl} = \varpi$. *Space of all* $\mathcal{I}_2$ *strongly Cesàro summable fuzzy variable double sequences in credibility is denoted by* $DC^{\mathcal{I}_2}(\Theta_{\mathrm{Cr}})$.

Theorem 2.51. *If* $\liminf_m \frac{p_m}{q_m} \neq 1$ *and* $\liminf_n \frac{r_n}{s_n} \neq 1$, *then* $C^{\mathcal{I}_2}(\Theta_{\mathrm{Cr}}) \subset DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\mathrm{Cr}})$.

Proof. Let $\liminf_m \frac{p_m}{q_m} = a(\neq 1)$ and $\liminf_n \frac{r_n}{s_n} = c(\neq 1)$. So there exist $b, d > 0$ such that $\frac{p_m}{q_m} \geq a + b$ and $\frac{r_n}{s_n} \geq c + d$ for sufficiently large m, n. So, we get

$$\frac{q_m - p_m}{q_m} \geq \frac{b}{a+b}, \frac{s_n - r_n}{s_n} \geq \frac{d}{c+d}.$$

Suppose $(\varpi_{kl}) \in C^{\mathcal{I}_2}(\Theta_{\mathrm{Cr}})$. Then for all $\varepsilon, \delta > 0$, the set

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n}\operatorname{Cr}(\|\varpi_{kl} - \varpi\| \geq \varepsilon) \geq \delta\right\}$$

belongs to $\mathcal{I}_2$.
As $\lim_{m\to\infty} q(m) = \infty$ and $\lim_{n\to\infty} s(n) = \infty$, we also have for any given $\varepsilon, \delta > 0$, the set

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{q_m s_n}\sum_{k=1}^{q(m)}\sum_{l=1}^{s(n)}\operatorname{Cr}(\|\varpi_{kl} - \varpi\| \geq \varepsilon) \geq \delta\right\} \in \mathcal{I}_2.$$

For any pair of sequences (p, q) and (r, s) which satisfy (1),

$$\sum_{i=1}^{q(m)}\sum_{j=1}^{s(n)}\operatorname{Cr}(\|\varpi_{kl} - \varpi\| \geq \varepsilon) \geq \sum_{k=p(m)+1}^{q(m)}\sum_{l=r(n)+1}^{s(n)}\operatorname{Cr}(\|\varpi_{kl} - \varpi\| \geq \varepsilon)$$

holds and also implies that

$$\sum_{k=1}^{q(m)}\sum_{l=1}^{s(n)}\operatorname{Cr}(\|\varpi_{kl} - \varpi\| \geq \varepsilon)$$
$$\geq \tfrac{q_m - p_m}{q_m}\tfrac{s_n - r_n}{s_n}\tfrac{1}{q_m - p_m}\tfrac{1}{s_n - r_n}\sum_{k=p(m)+1}^{q(m)}\sum_{l=r(n)+1}^{s(n)}\operatorname{Cr}(\|\varpi_{kl} - \varpi\| \geq \varepsilon)$$
$$\geq \tfrac{b}{a+b}\tfrac{d}{c+d}\tfrac{1}{\psi(m)\varkappa(n)}\sum_{k=p(m)+1}^{q(m)}\sum_{l=r(n)+1}^{s(n)}\operatorname{Cr}(\|\varpi_{kl} - \varpi\| \geq \varepsilon)$$

and so

$$\frac{(a+b)(c+d)}{bd}\sum_{k=1}^{q(m)}\sum_{l=1}^{s(n)}\text{Cr}\left(\|\varpi_{kl}-\varpi\|\geq\varepsilon\right)\geq\frac{1}{\psi(m)\varkappa(n)}\sum_{k=p(m)+1}^{q(m)}\sum_{l=r(n)+1}^{s(n)}\text{Cr}\left(\|\varpi_{kl}-\varpi\|\geq\varepsilon\right).$$

Hence, for any $\delta > 0$ we obtain

$$\left\{(m,n)\in\mathbb{N}^2:\tfrac{1}{\psi(m)\varkappa(n)}\sum_{k=p(m)+1}^{q(m)}\sum_{l=r(n)+1}^{s(n)}\text{Cr}\left(\|\varpi_{kl}-\varpi\|\geq\varepsilon\right)\geq\delta\right\}$$
$$\subseteq\left\{(m,n)\in\mathbb{N}^2:\tfrac{1}{q(m)s(n)}\sum_{k=1}^{q(m)}\sum_{l=1}^{s(n)}\text{Cr}\left(\|\varpi_{kl}-\varpi\|\geq\varepsilon\right)\geq\tfrac{\delta bd}{(a+b)(c+d)}\right\}\in\mathcal{I}_2.$$

Therefore $(\varpi_{kl})\in DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\text{Cr}})$. □

Remark 2.52. *Since each convergent double sequence $\{\varpi_{kl}\}$ of fuzzy variables in credibility belongs to ${}_{\mu}S^{\mathcal{I}_2}(\Theta_{\text{Cr}})$ as well as $C^{\mathcal{I}_2}(\Theta_{\text{Cr}})$. By Theorem 2.48 and 2.51 we have the following result, presented as a theorem.*

Theorem 2.53. *If a double sequence $\{\varpi_{kl}\}$ of fuzzy variables is convergent in credibility then $\{\varpi_{kl}\}\in {}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\text{Cr}})\cap DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{\text{Cr}})$.*

Theorem 2.54. *If $\liminf_m \frac{p_m}{q_m}\neq 1$ and $\liminf_n \frac{r_n}{s_n}\neq 1$, then ${}_{\mu}S^{\mathcal{I}_2}(\Theta_E)\subset {}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_E)$, where ${}_{\mu}S^{\mathcal{I}_2}(\Theta_E)$ demonstrates space of all $\mathcal{I}_2$-${}_{\mu}$statistically convergent fuzzy variable double sequences in mean.*

Proof. The proof follows from the Theorem 2.48, considering uncertain expected value operator. □

Theorem 2.55. *If $\liminf_m \frac{p_m}{q_m}\neq 1$ and $\liminf_n \frac{r_n}{s_n}\neq 1$, then $C^{\mathcal{I}_2}(\Theta_E)\subset DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_E)$, where $C^{\mathcal{I}_2}(\Theta_E)$ demonstrates space of all $\mathcal{I}_2$-Cesàro summable fuzzy variable sequences in mean.*

Proof. Take uncertain expected value operator in the proof of Theorem 2.51. □

Theorem 2.56. *If $\liminf_m \frac{p_m}{q_m}\neq 1$ and $\liminf_n \frac{r_n}{s_n}\neq 1$, then ${}_{\mu}S^{\mathcal{I}_2}\left(\Theta_{\varphi}\right)\subset {}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}\left(\Theta_{\varphi}\right)$, where ${}_{\mu}S^{\mathcal{I}_2}\left(\Theta_{\varphi}\right)$ demonstrates space of all $\mathcal{I}_2$-${}_{\mu}$statistically convergent fuzzy variable sequences in distribution.*

Proof. In the proof of 2.48. By taking fuzzy variable function, the proof can be established similarly. □

Theorem 2.57. *If $\liminf_m \frac{p_m}{q_m}\neq 1$ and $\liminf_n \frac{r_n}{s_n}\neq 1$, then $C^{\mathcal{I}_2}\left(\Theta_{\varphi}\right)\subset DC^{\mathcal{I}_2}_{(\psi,\varkappa)}\left(\Theta_{\varphi}\right)$, where $C^{\mathcal{I}_2}\left(\Theta_{\varphi}\right)$ demonstrates space of all $\mathcal{I}_2$-Cesàro summable fuzzy variable sequences in distribution.*

Proof. In the proof of Theorem 2.51. By taking fuzzy variable function, the proof can be established. □

Definition 2.58. *Suppose $(\Theta,\mathcal{P}(\Theta),\text{Cr})$ be a credibility space. Let $\{\varpi_{kl}\}$ be a double sequence of fuzzy variables. If $\forall\varepsilon,\delta>0$, there exists a $T\in\mathcal{P}(\Theta)$ with $\text{Cr}\{T\}=1$ and $\forall\theta\in T$, the set*

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{|I^*(m)|\,|I^*(n)|}\left|\left\{(k,l)\in\mathbb{N}^2:k\in I^*(m),l\in I^*(n)\ \&\ |\varpi_{kl}(\theta)-\varpi(\theta)|\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}_2.$$

In this case, we say $\{\varpi_{kl}\}$ is $\mathcal{I}_2$-${}_{\mu}$statistically convergent in almost surely to the fuzzy variable ϖ and ${}_{\mu}S^{\mathcal{I}_2}(\Theta_{a.s.})-\lim\{\varpi_{kl}\}=\varpi$. The space of all $\mathcal{I}_2$-${}_{\mu}$statistically convergent fuzzy variable double sequences w.r.t. almost surely is denoted by ${}_{\mu}S^{\mathcal{I}_2}(\Theta_{a.s.})$.

Theorem 2.59. *If* $\liminf_m \frac{p_m}{q_m} \neq 1$ *and* $\liminf_n \frac{r_n}{s_n} \neq 1$, *then* ${}_{\mu}S^{\mathcal{I}_2}(\Theta_{a.s.}) \subset {}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.})$.

Proof. Let $\liminf_m \frac{p_m}{q_m} = a(\neq 1)$ and $\liminf_n \frac{r_n}{s_n} = c(\neq 1)$ so there exist $b, d > 0$ such that $\frac{p_m}{q_m} \geq a+b$ and $\frac{r_n}{s_n} \geq c+d$ for sufficiently large m, n. So, we get

$$\frac{q_m - p_m}{q_m} \geq \frac{b}{a+b}, \frac{s_n - r_n}{s_n} \geq \frac{d}{c+d}.$$

Suppose $\{\varpi_{kl}\} \in {}_{\mu}S^{\mathcal{I}_2}(\Theta_{a.s.})$. Then, for all $\varepsilon, \delta > 0$, there is a $T \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{T\} = 1$ and $\forall \theta \in T$, the set

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{|I^*(m)|\,|I^*(n)|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I^*(m), l \in I^*(n) \ \&\ |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

As $\lim_{m\to\infty} q(m) = \infty$ and $\lim_{n\to\infty} s(n) = \infty$, we also have for any given $\varepsilon, \delta > 0$, the set

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I_q^*(m)\right|\left|I_s^*(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I_q^*(m), l \in I_s^*(n) \ \&\ |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}_2,$$

for all $\theta \in T$.

For any pair of sequences (p,q) and (r,s) which satisfy (1), $I_q^*(m) \supset I_{p,q}^*(m)$ and $I_s^*(n) \supset I_{r,s}^*(n)$ supply. So, we have

$$\frac{1}{\left|I_q^*(m)\right|\left|I_s^*(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I_q^*(m), l \in I_s^*(n) \ \&\ |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right|$$

$$\geq \frac{1}{\left|I_q^*(m)\right|\left|I_s^*(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I_{p,q}^*(m), l \in I_{r,s}^*(n) \ \&\ |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right|$$

$$= \frac{\left|I_{p,q}^*(m)\right|\left|I_{r,s}^*(n)\right|}{\left|I_q^*(m)\right|\left|I_s^*(n)\right|} \frac{1}{\left|I_{p,q}^*(m)\right|\left|I_{r,s}^*(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I_{p,q}^*(m), l \in I_{r,s}^*(n) \ \&\ |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right|$$

$$\geq \frac{bd}{(a+b)(c+d)} \frac{1}{\left|I_{p,q}^*(m)\right|\left|I_{r,s}^*(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I_{p,q}^*(m), l \in I_{r,s}^*(n) \ \&\ |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right|$$

and so

$$\frac{(a+b)(c+d)}{bd} \frac{1}{\left|I_q^*(m)\right|\left|I_s^*(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I_q^*(m), l \in I_s^*(n) \ \&\ |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right|$$

$$\geq \frac{1}{\left|I_{p,q}^*(m)\right|\left|I_{r,s}^*(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I_{p,q}^*(m), l \in I_{r,s}^*(n) \ \&\ |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right|.$$

Hence, for any $\delta > 0$ we obtain

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I_{p,q}^*(m)\right|\left|I_{r,s}^*(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I_{p,q}^*(m), l \in I_{r,s}^*(n) \ \&\ |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right| \geq \delta\right\}$$

$$\subseteq \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|I_q^*(m)\right|\left|I_s^*(n)\right|} \left|\left\{(k,l) \in \mathbb{N}^2 : k \in I_q^*(m), l \in I_s^*(n) \right.\right.\right.$$
$$\left.\left.\left. \&\ |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \varepsilon\right\}\right| \geq \frac{\delta bd}{(a+b)(c+d)}\right\} \in \mathcal{I}_2, \quad \forall \theta \in T.$$

Therefore, we obtain $\{\varpi_{kl}\} \in_{\mu} DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.})$. □

Remark 2.60. *If* $\liminf_m \frac{p_m}{q_m} \neq 1$ *and* $\liminf_n \frac{r_n}{s_n} \neq 1$, *then* ${}_{\mu}S^{\mathcal{I}_2}(\Theta_{a.s.}) \subset DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.})$.

Definition 2.61. *Let* $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ *be a credibility space. Let* $\{\varpi_{kl}\}$ *be a double sequence of fuzzy variables. If* $\forall \varepsilon > 0$*, there exists a* $T \in \mathcal{P}(\Theta)$ *with* $\mathrm{Cr}\{T\} = 1$ *and* $\forall \theta \in T$*, the set*

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n}|\varpi_{kl}(\theta) - \varpi(\theta)| \geq \delta\right\} \in \mathcal{I}_2.$$

In this case, we say $\{\varpi_{kl}\}$ *is* $\mathcal{I}_2$ *strongly Cesàro summable with respect to almost surely in credibility space to the fuzzy variable* ϖ *and* $C^{\mathcal{I}_2}(\Theta_{a.s.}) - \lim(\varpi_{kl}) = \varpi$*. The space of all* $\mathcal{I}_2$ *strongly Cesàro summable fuzzy variable double sequences w.r.t. almost surely is demonstrated by* $C^{\mathcal{I}_2}(\Theta_{a.s.})$*.*

Theorem 2.62. *If* $\liminf_m \frac{p_m}{q_m} \neq 1$ *and* $\liminf_n \frac{r_n}{s_n} \neq 1$*, then* $C^{\mathcal{I}_2}(\Theta_{a.s.}) \subset DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.})$*, where* $C^{\mathcal{I}_2}(\Theta_{a.s.})$ *demonstrates space of all* $\mathcal{I}_2$*-Cesàro summable fuzzy variable double sequences w.r.t. almost surely.*

Proof. Let $\liminf_m \frac{p_m}{q_m} = a(\neq 1)$ and $\liminf_n \frac{r_n}{s_n} = c(\neq 1)$ so there exist $b, d > 0$ such that $\frac{p_m}{q_m} \geq a + b$ and $\frac{r_n}{s_n} \geq c + d$ for sufficiently large m, n. So, we get

$$\frac{q_m - p_m}{q_m} \geq \frac{b}{a+b}, \frac{s_n - r_n}{s_n} \geq \frac{d}{c+d}.$$

Suppose $\{\varpi_{kl}\} \in C^{\mathcal{I}_2}(\Theta_{a.s.})$. Then, for all $\delta > 0$, there exists $T \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{T\} = 1$ and $\forall \theta \in T$, the set

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n}|\varpi_{kl}(\theta) - \varpi(\theta)| \geq \delta\right\}$$

belongs to $\mathcal{I}_2$.
As $\lim_{m\to\infty} q(m) = \infty$ and $\lim_{n\to\infty} s(n) = \infty$, we also have for any given $\varepsilon, \delta > 0$, the set

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{q_m s_n}\sum_{k=1}^{q(m)}\sum_{l=1}^{s(n)}|\varpi_{kl}(\theta) - \varpi(\theta)| \geq \delta\right\} \in \mathcal{I}_2.$$

For any pair of sequences (p, q) and (r, s) which satisfy (1),

$$\sum_{k=1}^{q(m)}\sum_{l=1}^{s(n)}\|\varpi_{kl} - \varpi\| \geq \sum_{k=p(m)+1}^{q(m)}\sum_{l=r(n)+1}^{s(n)}|\varpi_{kl}(\theta) - \varpi(\theta)|$$

holds and also implies that

$$\begin{aligned}
&\sum_{k=1}^{q(m)}\sum_{l=1}^{s(n)}|\varpi_{kl}(\theta) - \varpi(\theta)| \\
&\geq \tfrac{q_m - p_m}{q_m}\tfrac{s_n - r_n}{s_n}\tfrac{1}{q_m - p_m}\tfrac{1}{s_n - r_n}\sum_{k=p(m)+1}^{q(m)}\sum_{l=r(n)+1}^{s(n)}|\varpi_{kl}(\theta) - \varpi(\theta)| \\
&\geq \tfrac{b}{a+b}\tfrac{d}{c+d}\tfrac{1}{\psi(m)\varkappa(n)}\sum_{k=p(m)+1}^{q(m)}\sum_{l=r(n)+1}^{s(n)}|\varpi_{kl}(\theta) - \varpi(\theta)|
\end{aligned}$$

and so

$$\frac{(a+b)(c+d)}{bd}\sum_{k=1}^{q(m)}\sum_{l=1}^{s(n)}|\varpi_{kl}(\theta) - \varpi(\theta)| \geq \frac{1}{\psi(m)\varkappa(n)}\sum_{k=p(m)+1}^{q(m)}\sum_{l=r(n)+1}^{s(n)}|\varpi_{kl}(\theta) - \varpi(\theta)|$$

Hence, for any $\delta > 0$ we obtain

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\psi(m)\varkappa(n)} \sum_{k=p(m)+1}^{q(m)} \sum_{l=r(n)+1}^{s(n)} |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \delta\right\}$$
$$\subseteq \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{q(m)s(n)} \sum_{k=1}^{q(m)} \sum_{l=1}^{s(n)} |\varpi_{kl}(\theta) - \varpi(\theta)| \geq \frac{\delta bd}{(a+b)(c+d)}\right\} \in \mathcal{I}_2.$$

Therefore $\{\varpi_{kl}\} \in DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.})$. □

As every double sequence $\{\varpi_{kl}\}$ of fuzzy variables belongs to ${}_{\mu}S^{\mathcal{I}_2}(\Theta_{\mathrm{Cr}})$ as well as $C^{\mathcal{I}_2}(\Theta_{\mathrm{Cr}})$ so we have following remark which can be proved by using Theorem 2.59 and Theorem 2.62.

Remark 2.63. *If a double sequence $\{\varpi_{kl}\}$ of fuzzy variables is convergent almost surely then $\{\varpi_{kl}\} \in {}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.}) \cap DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{a.s.})$.*

Theorem 2.64. *If $\liminf_m \frac{p_m}{q_m} \neq 1$ and $\liminf_n \frac{r_n}{s_n} \neq 1$, then ${}_{\mu}S^{\mathcal{I}_2}(\Theta_{u.a.s.}) \subset {}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{u.a.s.})$, where ${}_{\mu}S^{\mathcal{I}_2}(\Theta_{u.a.s.})$ denotes space of all $\mu - \mathcal{I}_2$-statistically convergent fuzzy variable double sequences w.r.t. uniformly almost surely.*

Proof. Consider the events $\theta \in \Theta - \{F_t\}$, where sequence $\{F_t\}$ be such that ${}_{\mu}DS^{\mathcal{I}_2}_{(\psi,\varkappa)} - \lim \mathrm{Cr}\{F_t\} = 0$ and adapt the method which is used in the Theorem 2.59. □

Theorem 2.65. *If $\liminf_m \frac{p_m}{q_m} \neq 1$ and $\liminf_n \frac{r_n}{s_n} \neq 1$, then $C^{\mathcal{I}_2}(\Theta_{u.a.s.}) \subset DC^{\mathcal{I}_2}_{(\psi,\varkappa)}(\Theta_{u.a.s.})$, where $C^{I}(\Theta_{u.a.a.})$ denotes space of all $\mathcal{I}_2$-Cesàro summable fuzzy variable double sequences w.r.t. uniformly almost surely.*

Proof. Once again, considering the events from $\Theta - \{F_t\}$, one can prove the theorem by following the technique of Theorem 2.62. □

3. Conclusion

The findings of this research work adds to the domains of summability theory using the credibility theory. We have introduced various types of deferred strongly Cesaro summable double sequence of fuzzy variables using credibility theory via ideals. Existence of such sequences are shown for the purpose and characteristics of these notions are established to some extent. We also initiated the notion of deferred statistically convergent double sequences in a credibility space in different aspects using credibility measure operator, expected value operator (mean), credibility distribution function and also by reducing the domain of definition of fuzzy variables in respect certain conditions (almost surely and uniformly almost surely). For each cases the interconnection between deferred statistically convergent double sequences and deferred strongly Cesaro summable double sequence are studied. This study's findings are more generic and a natural extension of the traditional convergence of fuzzy variable double sequences.

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