Filomat 38:14 (2024), 4933–4944
https://doi.org/10.2298/FIL2414933L

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Some new characterizations of normal elements in rings with involution

**Jiahui Liang[a,*], Long Wang[a], Junchao Wei[a]**

[a] *School of Mathematical Science, Yangzhou University, Yangzhou, Jiangsu 225002, P. R. China*

**Abstract.** In this paper, we firstly illuminate some properties of normal elements by the consistency of certain equation and construct the group inverses, *MP* inverses and invertible elements. Then we gives some conditions involving $a$, $a^{\dagger}$, $a^{\#}$, $a^{*}$, $(a^{\#})^{*}$ and $(a^{\dagger})^{*}$ to guarantee that $a$ is normal. Finally, the solution of related equations to depict normality have been studied in a certain set.

## 1. Introduction

Throughout this paper, let $R$ be an associative ring with 1. An involution $* : a \mapsto a^*$ in $R$ is an anti-isomorphism of degree 2 (see [1]), that is,

$$(a^*)^* = a,\ (ab)^* = b^*a^*,\ (a+b)^* = a^* + b^*.$$

In what follows, $R$ is called a $*$-ring if $R$ is a ring with involution.

An element $a \in R$ is said to be Moore-Penrose invertible if the following equations:

$$axa = a,\ xax = x,\ (ax)^* = ax,\ (xa)^* = xa$$

have a common solution [2]. Such solution is unique if it exists we call it the Moore-Penrose inverse of $a$ and denoted it by $a^{\dagger}$. The set of all Moore-Penrose invertible elements of $R$ is denoted by $R^{\dagger}$.

An element $a \in R$ is said to be group invertible [3] if there exists $x \in R$ satisfying the following equations:

$$axa = a,\ xax = x,\ ax = xa.$$

Such solution is unique if it exists, we call it the group inverse of $a$ and denoted by $a^{\#}$. The set of all group invertible elements of $R$ is denoted by $R^{\#}$.

The core (*resp*. dual core) inverse [12] of $a \in R$ is the element $x \in R$ which satisfies

$$axa = a,\ xR = aR(resp.\ Rx = Ra),\ Rx = Ra^*(resp.\ xR = a^*R).$$

2020 *Mathematics Subject Classification*. 15A09, 16U99, 16W10.
*Keywords*. *EP* element, normal element, group inverse, Moore-Penrose inverse, solutions of equations.
Received: 27 April 2023; Revised: 20 January 2024; Accepted: 15 February 2024
Communicated by Dijana Mosić
Research supported by the Foundation of Natural Science of China(Grant No.11901510) optionally.
* Corresponding author: Jiahui Liang
*Email addresses:* 506390843@qq.com (Jiahui Liang), lwangmath@yzu.edu.cn (Long Wang), jcweiyz@126.com (Junchao Wei)

The element $x$ above is unique if it exists and is denoted by $a^{\circledast}$(*resp*. $a_{\circledast}$). The set of all core (*resp*. dual core) invertible elements of $R$ will be denoted by $R^{\circledast}$(*resp*. $R_{\circledast}$).

An element $a \in R^{\#} \cap R^{\dagger}$ satisfying $a^{\#} = a^{\dagger}$ is said to be EP [4]. We denote the set of all EP elements of $R$ by $R^{EP}$. An element $a \in R$ satisfying $a^* = a$ is called symmetric, and $a$ is called normal if $a^*a = aa^*$. We denote the set of all the normal elements of $R$ by $R^{Nor}$.

In [5] and [10], D. Mosić and D. S. Djordjević demonstrated some equivalent conditions of normal elements by a combination of $a^{\dagger}$, $a^{\#}$, $a^*$ and $a^n (n \in N^+)$. They also characterized the normality by generalizing the result holds for linear bounded operators on Hilbert spaces. In [9], W.X.Chen presented several counterexamples to further improve the characterization of normality for the results in [5]. J. C.Wei et al. in [7], [8] utilized the one-sided ideal inclusion properties and the solutions of certain equations to investigate normal elements.

Motivated by the aforementioned above, some new sufficient and necessary conditions are given to characterize the normal elements.

The following lemmas are frequently used in this paper, for the convenience of the reader, the characterization of *EP* and normal elements are given again.

**Lemma 1.1.** *[1, 5, 10] Let $a \in R^{\#} \cap R^{\dagger}$. Then $a$ is EP if and only if one of the following conditions holds:*
(1) $a = a^{\dagger}a^2$;
(2) $a = a^2a^{\dagger}$;
(3) $a^{\dagger}a = aa^{\#}$;
(4) $a^{\dagger} = aa^{\dagger}a^{\dagger}$;
(5) $a^{\dagger} = a^{\dagger}a^{\dagger}a$;
(6) $a^{\dagger} = a^{\#}aa^{\dagger}$;
(7) $a^{\dagger} = a^{\dagger}aa^{\#}$.

**Lemma 1.2.** *[5, Theorem 2.2] Let $a \in R^{\#} \cap R^{\dagger}$. Then $a$ is normal if and only if one of the following conditions holds:*
(1) $aa^{\#}a^* = a^{\#}a^*a$;
(2) $a^*a^{\#} = a^{\#}a^*$;
(3) $a^{\#}a^* = a^*a^{\dagger}$;
(4) $a^*a^{\#} = a^{\dagger}a^*$;
(5) $a^*a^{\#}a^{\#} = a^{\#}a^*a^{\#}$;
(6) $a^{\#}a^*a^{\#} = a^{\#}a^{\#}a^*$;
(7) $a^*a^{\#}a^* = a^{\#}a^*a^*$;
(8) $a^{\dagger}a^*a^{\#} = a^{\#}a^{\dagger}a^*$.

## 2. Some property characterizations of normality

Let $a \in R^{\#} \cap R^{\dagger}$. Then by the Lemma 1.2 and [8, Lemma 2.7], we know that $a \in R^{Nor}$ if and only if $a^*a^{\#} = a^{\#}a^*$ if and only if $a(a^{\#})^* = (a^{\#})^*a$ if and only if $a \in R^{EP}$ and $a^*a^{\dagger} = a^{\dagger}a^*$.

Hence we have the following characterization of normal elements.

**Theorem 2.1.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Nor}$ if and only if $a^2(a^{\dagger})^* = a^{\dagger}a^2(a^{\#})^*a$.*

*Proof.* $\Rightarrow$ Assume that $a \in R^{Nor}$. Then $a \in R^{EP}$ and $a(a^{\#})^* = (a^{\#})^*a$. This gives

$$a^2(a^{\dagger})^* = a^{\#}a^3(a^{\dagger})^* = a^{\dagger}a^3(a^{\#})^* = a^{\dagger}a^2(a^{\#})^*a.$$

$\Leftarrow$ Suppose that $a^2(a^{\dagger})^* = a^{\dagger}a^2(a^{\#})^*a$. Then multiply the equality on the left by $a^{\dagger}a$, one has $a^2(a^{\dagger})^* = a^{\dagger}a^3(a^{\dagger})^*$. Multiplying the last equality on the right by $a^*a^{\#}$, we get $a = a^{\dagger}a^2$. Hence $a \in R^{EP}$ and $a^{\dagger} = a^{\#}$ by Lemma 1.1, this implies $a(a^{\#})^*a = a^{\dagger}a^2(a^{\#})^*a = a^2(a^{\dagger})^*$ and

$$(a^{\#})^*a = a^{\dagger}a(a^{\#})^*a = a^{\dagger}a^2(a^{\dagger})^* = a(a^{\dagger})^* = a(a^{\#})^*.$$

Thus $a \in R^{Nor}$ by Lemma 1.2. □

Since $a \in R^{Nor}$, then $a \in R^{EP}$ and $a^{\dagger} = a^{\#}$. Inspired by Theorem 2.1, we have the following corollary.

**Corollary 2.2.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Nor}$ if and only if one of the following conditions are satisfied:*
(1) $a^2(a^{\#})^* = a^{\dagger}a^2(a^{\#})^*a$;
(2) $a^2(a^{\dagger})^* = a^{\dagger}a^2(a^{\dagger})^*a$;
(3) $a^2(a^{\#})^* = a^{\dagger}a^2(a^{\dagger})^*a$;
(4) $a^2(a^{\dagger})^* = a^{\dagger}a^3a^{\dagger}(a^{\#})^*a$.

*Proof.* If $a$ is normal, then it satisfies $a^{\#} = a^{\dagger}$. It is not difficult to testify that condition $(1) - (4)$ hold by Theorem 2.1.

On the contrary, we suppose that $a \in R^{\#} \cap R^{\dagger}$. To prove that $a$ is normal, we can conclude that $a \in R^{EP}$. Then according to Theorem 2.1, our conclusions are satisfied.

(1) Assume that $a^2(a^{\#})^* = a^{\dagger}a^2(a^{\#})^*a$, then we have

$$a^2(a^{\#})^* = a^{\dagger}a^2(a^{\#})^*a = a^{\dagger}aa^{\dagger}a^2(a^{\#})^*a = a^{\dagger}aa^2(a^{\#})^*.$$

Post-multiply the above equality by $a^*a^{\dagger}a^{\#}a$, one obtains $a = a^{\dagger}a^2$. Thus $a \in R^{EP}$.

(2) From the equality $a^2(a^{\dagger})^* = a^{\dagger}a^2(a^{\dagger})^*a$, one gets

$$a^2(a^{\dagger})^* = a^{\dagger}a^2(a^{\dagger})^*a = a^{\dagger}aa^{\dagger}a^2(a^{\dagger})^*a = a^{\dagger}aa^2(a^{\dagger})^*.$$

Multiplying the above equality on the right by $a^*a^{\#}$, we get $a = a^{\dagger}a^2$. Hence $a \in R^{EP}$.

The proof of (3), (4) is similar to (1), (2). □

Noting that if $a \in R^{\#} \cap R^{\dagger}$, then $(a^{\#})^* = (a^{\#})^*aa^{\dagger}$ and $(a^{\dagger})^*aa^{\#} = (a^{\dagger})^* = (a^{\dagger})^*a^{\dagger}a$. Thus Theorem 2.1 and Corollary 2.2 induces the following corollary.

**Corollary 2.3.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Nor}$ if and only if one of the following conditions are satisfied:*
(1) $a^2(a^{\dagger})^*a^{\dagger} = a^{\dagger}a^2(a^{\#})^*$;
(2) $a^2(a^{\#})^*a^{\dagger} = a^{\dagger}a^2(a^{\#})^*$;
(3) $a^2(a^{\dagger})^*a^{\#} = a^{\dagger}a^2(a^{\dagger})^*$;
(4) $a^2(a^{\#})^*a^{\#} = a^{\dagger}a^2(a^{\dagger})^*$;
(5) $a^2(a^{\dagger})^*a^{\dagger} = a^{\dagger}a^3a^{\dagger}(a^{\#})^*$.

*Proof.* If $a$ is normal, it is not difficult to verify that conditions $(1) - (5)$ hold by Theorem 2.1 and Corollary2.2.

In contrast, suppose that $a \in R^{\#} \cap R^{\dagger}$. We show that the element obeys one of the conditions established earlier in this corollary.

(1) From the equality $a^2(a^{\dagger})^*a^{\dagger} = a^{\dagger}a^2(a^{\#})^*$, one has

$$a^2(a^{\dagger})^* = (a^2(a^{\dagger})^*a^{\dagger})a = a^{\dagger}a^2(a^{\#})^*a.$$

Thus $a$ is normal by Theorem 2.1.

(2) Applying the equality $a^2(a^{\#})^*a^{\dagger} = a^{\dagger}a^2(a^{\#})^*$, this gives

$$a^2(a^{\#})^*a^{\dagger} = a^{\dagger}a^2(a^{\#})^* = a^{\dagger}a(a^{\dagger}a^2(a^{\#})^*) = a^{\dagger}aa^2(a^{\#})^*a^{\dagger}.$$

Post-multiply the above equality by $aa^*a^{\dagger}a^{\#}a$, one obtains $a = a^{\dagger}a^2$. This infers $a \in R^{EP}$ and $a^{\#} = a^{\dagger}$. Therefore $a^2(a^{\dagger})^*a^{\dagger} = a^2(a^{\#})^*a^{\dagger} = a^{\dagger}a^2(a^{\#})^*$ and the condition (1) is satisfied.

(3) Suppose that $a^2(a^{\dagger})^*a^{\#} = a^{\dagger}a^2(a^{\dagger})^*$, we get

$$a^2(a^{\dagger})^* = (a^2(a^{\dagger})^*a^{\#})a = a^{\dagger}a^2(a^{\dagger})^*a.$$

Then $a$ is normal by Corollary 2.2.

(4) Assume that $a^2(a^{\#})^*a^{\#} = a^{\dagger}a^2(a^{\dagger})^*$, it gives

$$a^2(a^{\#})^*a^{\#} = a^{\dagger}a^2(a^{\dagger})^* = a^{\dagger}a(a^{\dagger}a^2(a^{\dagger})^*) = a^{\dagger}aa^2(a^{\#})^*a^{\#}.$$

Multiplying the above equality on the right by $a^2a^{\dagger}a^*a^{\dagger}a^{\#}a$, one has $a = a^{\dagger}a^2$. Thus $a \in R^{EP}$ and $a^2(a^{\dagger})^*a^{\#} = a^2(a^{\#})^*a^{\#} = a^{\dagger}a^2(a^{\dagger})^*$. Therefore the condition (3) holds.

(5) The equality $a^2(a^\dagger)^*a^\dagger = a^\dagger a^3 a^\dagger (a^\#)^*$ gives

$$a^2(a^\dagger)^* = (a^2(a^\dagger)^*a^\dagger)a = a^\dagger a^3 a^\dagger (a^\#)^* a.$$

We conclude that $a$ is normal by Corollary 2.2. □

Considering that when $a \in R^\# \cap R^\dagger$, then

$$a^{\textcircled{\#}} = a^\# a a^\dagger;$$

$$a_{\textcircled{\#}} = a^\dagger a a^\#;$$

$$(a^\#)^\dagger = a^\dagger a^3 a^\dagger;$$

$$(a_{\textcircled{\#}})^\dagger = a^\dagger a^2;$$

$$(a^{\textcircled{\#}})^\dagger = a^2 a^\dagger.$$

Combining Theorem 2.1, Corollary 2.2 and Corollary 2.3, we have the following theorems.

**Theorem 2.4.** *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{Nor}$ if and only if one of the following conditions are satisfied:*
(1) $a^2(a^\dagger)^* = (a_{\textcircled{\#}})^\dagger (a^\#)^* a$;
(2) $a^2(a^\#)^* = (a_{\textcircled{\#}})^\dagger (a^\#)^* a$;
(3) $a^2(a^\dagger)^* = (a_{\textcircled{\#}})^\dagger (a^\dagger)^* a$;
(4) $a^2(a^\#)^* = (a_{\textcircled{\#}})^\dagger (a^\dagger)^* a$;
(5) $a^2(a^\dagger)^* = (a^\#)^\dagger (a^\#)^* a$;
(6) $a^2(a^\dagger)^* = a_{\textcircled{\#}} a^3 a^\dagger (a^\#)^* a$;
(7) $a^2(a^\dagger)^* = a^\dagger a^3 a^{\textcircled{\#}} (a^\#)^* a$.

*Proof.* Assume that $a \in R^{Nor}$, it is not difficult to prove that conditions (1) – (7) are satisfied.

Conversely, if $a \in R^\# \cap R^\dagger$, we only to verify that the element obeys one of the preceding already established conditions of this theorem.

(1) The equality satisfies

$$a^2(a^\dagger)^* = (a_{\textcircled{\#}})^\dagger (a^\#)^* a = a^\dagger a^2 (a^\#)^* a.$$

This indicates $a \in R^{Nor}$ by Theorem 2.1.

(2) Suppose that

$$a^2(a^\#)^* = (a_{\textcircled{\#}})^\dagger (a^\#)^* a = a^\dagger a^2 (a^\#)^* a.$$

Then according to Corollary 2.2, we have $a \in R^{Nor}$.

(3) Applying the equality

$$a^2(a^\dagger)^* = (a_{\textcircled{\#}})^\dagger (a^\dagger)^* a = a^\dagger a^2 (a^\dagger)^* a,$$

we obtain $a \in R^{Nor}$ by Corollary 2.2.

(4) From the equality

$$a^2(a^\#)^* = (a_{\textcircled{\#}})^\dagger (a^\dagger)^* a = a^\dagger a^2 (a^\dagger)^* a,$$

one gets $a \in R^{Nor}$ by Corollary 2.2.

(5) Noting that

$$a^2(a^\dagger)^* = (a^\#)^\dagger (a^\#)^* a = a^\dagger a^3 a^\dagger (a^\#)^* a.$$

Therefore $a \in R^{Nor}$ by Corollary 2.2.

(6), (7) Provided that $a \in R^\# \cap R^\dagger$, then $a^\dagger a^3 a^\dagger = a^\dagger a a^\# a^3 a^\dagger = a_{\textcircled{\#}} a^3 a^\dagger$ and $a^\dagger a^3 a^\dagger = a^\dagger a^3 a^\# a a^\dagger = a^\dagger a^3 a^{\textcircled{\#}}$. Thus we have

$$a^2(a^\dagger)^* = a_{\textcircled{\#}} a^3 a^\dagger (a^\#)^* a = a^\dagger a^3 a^{\textcircled{\#}} (a^\#)^* a = a^\dagger a^3 a^\dagger (a^\#)^* a.$$

Therefore the condition (5) is satisfied. □

**Theorem 2.5.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Nor}$ if and only if one of the following conditions are satisfied:*
(1) $a^2(a^{\dagger})^* a^{\dagger} = (a_{\textcircled{\#}})^{\dagger}(a^{\#})^*$;
(2) $a^2(a^{\#})^* a^{\dagger} = (a_{\textcircled{\#}})^{\dagger}(a^{\#})^*$;
(3) $a^2(a^{\dagger})^* a^{\#} = (a_{\textcircled{\#}})^{\dagger}(a^{\dagger})^*$;
(4) $a^2(a^{\#})^* a^{\#} = (a_{\textcircled{\#}})^{\dagger}(a^{\dagger})^*$;
(5) $a^2(a^{\dagger})^* a^{\dagger} = (a^{\#})^{\dagger}(a^{\#})^*$;
(6) $a^2(a^{\dagger})^* a^{\dagger} = a_{\textcircled{\#}} a^3 a^{\dagger}(a^{\#})^*$;
(7) $a^2(a^{\dagger})^* a^{\dagger} = a^{\dagger} a^3 a^{\textcircled{\#}}(a^{\#})^*$.

## 3. Constructing the group invertible elements and MP invertible elements to characterize normality

The following theorem constructs the group inverses and MP inverses of product of several elements, which proof is routine.

**Theorem 3.1.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then*
(1) $(a^2(a^{\dagger})^*)^{\dagger} = a^* a^{\#} a^{\dagger}$;
(2) $(a^2(a^{\dagger})^*)^{\#} = aa^{\#} a^* a^{\#} a^{\#}$;
(3) $(a^{\dagger} a^2(a^{\#})^* a)^{\dagger} = (a^{\dagger} a^2(a^{\#})^* a)^{\#} = a^{\dagger} a^* a^{\dagger} aa^{\#}$.

Using Theorem 3.1, we can give some characterizations of normal elements as follows.

**Theorem 3.2.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then the following conditions are equivalent:*
(1) $a \in R^{Nor}$;
(2) $(a^2(a^{\dagger})^*)^{\dagger} = a^{\#} a^* a^{\dagger}$;
(3) $(a^2(a^{\dagger})^*)^{\#} = a^{\dagger} a^* a^{\#}$;
(4) $(a^{\dagger} a^2(a^{\#})^* a)^{\dagger} = a^{\dagger} a^{\#} a^*$;
(5) $a^{\dagger} a^* a^{\#} = a^{\dagger} a^{\#} a^*$.

*Proof.* (1)⇒(2),(3),(4),(5) These implications can be easily proved by Theorem 3.1.

(2)⇒(1) Suppose that $(a^2(a^{\dagger})^*)^{\dagger} = a^* a^{\#} a^{\dagger} = a^{\#} a^* a^{\dagger}$ by Theorem 3.1. This infers $a^* a^{\#} a^{\dagger} = a^{\#} a^* a^{\dagger} = aa^{\dagger} a^{\#} a^* a^{\dagger} = aa^{\dagger} a^* a^{\#} a^{\dagger}$. Post-multiply the equality $a^* a^{\#} a^{\dagger} = aa^{\dagger} a^* a^{\#} a^{\dagger}$ by $a^2(a^{\dagger})^* a^{\dagger}$, we have $aa^{\dagger} a^{\dagger} = a^{\dagger}$. Hence $a \in R^{EP}$. Then $a^* a^{\#} = a^* a^{\#} a^{\dagger} a = a^{\#} a^* a^{\dagger} a = a^{\#} a^*$. Therefore $a \in R^{Nor}$ by Lemma 1.2.

(3)⇒(1) Assume that $(a^2(a^{\dagger})^*)^{\#} = aa^{\#} a^* a^{\#} a^{\#} = a^{\dagger} a^* a^{\#}$ by Theorem 3.1. It implies that $a^{\dagger} a^* a^{\#} = aa^{\#} a^* a^{\#} a^{\#} = aa^{\#} aa^{\#} a^* a^{\#} a^{\#} = aa^{\#} a^{\dagger} a^* a^{\#}$. Multiplying the equality $a^{\dagger} a^* a^{\#} = aa^{\#} a^{\dagger} a^* a^{\#}$ on the right by $a(a^{\dagger})^*(a^{\#})^* a^*$, one obtains $aa^{\#} a^{\dagger} = a^{\dagger}$. This infers $a \in R^{EP}$ and $a^{\#} = a^{\dagger}$. Applying $aa^{\#} a^* a^{\#} a^{\#} = a^{\dagger} a^* a^{\#}$, we can reduce to $a^* a^{\#} a^{\#} = a^{\#} a^* a^{\#}$. Accordingly, $a \in R^{Nor}$ by Lemma 1.2.

(4)⇒(1) The assumption $(a^{\dagger} a^2(a^{\#})^* a)^{\dagger} = a^{\dagger} a^{\#} a^*$ implies $a^{\dagger} a^* a^{\dagger} aa^{\#} = a^{\dagger} a^{\#} a^*$ by Theorem 3.1. Then we can deduce from the following equality $a^{\dagger} a^* a^{\dagger} aa^{\#} = a^{\dagger} a^{\#} a^* aa^{\dagger} = a^{\dagger} a^* a^{\dagger} aa^{\#} aa^{\dagger} = a^{\dagger} a^* a^{\dagger}$. It implies $a^{\dagger} = (a^{\#})^* aa^{\dagger} a^* a^{\dagger} = (a^{\#})^* aa^{\dagger} a^* a^{\dagger} aa^{\#} = a^{\dagger} aa^{\#}$ and $a \in R^{EP}$. Therefore $a^{\dagger} a^* a^{\#} = a^{\dagger} a^* a^{\dagger} aa^{\#} = a^{\dagger} a^{\#} a^* = a^{\#} a^{\dagger} a^*$. Hence $a \in R^{Nor}$ by Lemma 1.2.

(5)⇒(1) The equality can deduce $a^{\dagger} a^{\#} a^* = a^{\dagger} a^* a^{\#} = a^{\dagger} a^* a^{\#} a^{\dagger} a = a^{\dagger} a^{\#} a^* a^{\dagger} a$. Then pre-multiply the equality $a^{\dagger} a^{\#} a^* = a^{\dagger} a^{\#} a^* a^{\dagger} a$ by $a^{\dagger}(a^{\dagger})^* a^2$, one gets $a^{\dagger} = a^{\dagger} a^{\dagger} a$. This indicates $a \in R^{EP}$ and $a^{\dagger} a^* a^{\#} = a^{\dagger} a^{\#} a^* = a^{\#} a^{\dagger} a^*$. Thus $a \in R^{Nor}$ by Lemma 1.2. □

**Theorem 3.3.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then the following conditions are equivalent:*
(1) $a \in R^{Nor}$;
(2) $a^2(a^{\dagger})^* = a(a^{\#})^* a^2 a^{\dagger}$;
(3) $a^2(a^{\dagger})^* = a(a^{\dagger})^* a^2 a^{\dagger}$;
(4) $a^2(a^{\dagger})^* = a(a^{\#})^* a^{\dagger} a^3 a^{\dagger}$;
(5) $a(a^{\dagger})^* = aa^{\dagger}(a^{\#})^* a(aa^{\#})^*$;
(6) $(a^{\dagger})^* = a^{\textcircled{\#}}(a^{\#})^* a(aa^{\#})^*$.

*Proof.* (1)⇒(2) Since $a \in R^{Nor}$, thus $a \in R^{EP}$ and $a(a^{\#})^* = (a^{\#})^* a$. We can conclude that

$$a(a^{\#})^* a^2 a^{\dagger} = a^2 (a^{\#})^* a a^{\dagger} = a^2 (a^{\#})^* = a^2 (a^{\dagger})^*.$$

(2)⇒(3) Assume that

$$a^2 (a^{\dagger})^* = a(a^{\#})^* a^2 a^{\dagger} = a(a^{\#})^* a^2 a^{\dagger} a a^{\dagger} = a^2 (a^{\dagger})^* a a^{\dagger}.$$

This gives

$$(a^{\dagger})^* = a^{\#} a^{\#} a^2 (a^{\dagger})^* = a^{\#} a^{\#} a^2 (a^{\dagger})^* a a^{\dagger} = (a^{\dagger})^* a a^{\dagger}.$$

Then take the involution on both sides of the equality, one obtains $a^{\dagger} = a a^{\dagger} a^{\dagger}$. Hence $a \in R^{EP}$. We get

$$a^2 (a^{\dagger})^* = a(a^{\#})^* a^2 a^{\dagger} = a(a^{\dagger})^* a^2 a^{\dagger}.$$

(3)⇒(4) Using the assumption

$$a^2 (a^{\dagger})^* = a(a^{\dagger})^* a^2 a^{\dagger} = a(a^{\dagger})^* a^2 a^{\dagger} a a^{\dagger} = a^2 (a^{\dagger})^* a a^{\dagger}.$$

The following proof is similar to (2)⇒(3). Therefore we have $a \in R^{EP}$ and then

$$a^2 (a^{\dagger})^* = a(a^{\dagger})^* a^2 a^{\dagger} = a(a^{\dagger})^* a^{\dagger} a^3 a^{\dagger} = a(a^{\#})^* a^{\dagger} a^3 a^{\dagger}.$$

(4)⇒(5) From the equality

$$a^2 (a^{\dagger})^* = a(a^{\#})^* a^{\dagger} a^3 a^{\dagger} = a(a^{\#})^* a^{\dagger} a^3 a^{\dagger} a a^{\dagger} = a^2 (a^{\dagger})^* a a^{\dagger}.$$

The following proof is similar to (2)⇒(3). Thus we have $a \in R^{EP}$ and

$$a(a^{\dagger})^* = a^{\dagger} a^2 (a^{\dagger})^* = a^{\dagger} a (a^{\#})^* a^{\dagger} a^3 a^{\dagger} = a a^{\dagger} (a^{\#})^* a a a^{\dagger} = a a^{\dagger} (a^{\#})^* a (a a^{\#})^*.$$

(5)⇒(6) By the equality $a(a^{\dagger})^* = a a^{\dagger} (a^{\#})^* a (a a^{\#})^*$. Pre-multiply the above equality by $a^{\#}$, we get

$$(a^{\dagger})^* = a^{\text{\#}\!\!\!\bigcirc} (a^{\#})^* a (a a^{\#})^*.$$

(6)⇒(1) Since we have

$$(a^{\dagger})^* = a^{\text{\#}\!\!\!\bigcirc} (a^{\#})^* a (a a^{\#})^* = a^{\text{\#}\!\!\!\bigcirc} (a^{\#})^* a (a a^{\#})^* a a^{\dagger} = (a^{\dagger})^* a a^{\dagger}.$$

Then take the involution on both sides of the equality, one obtains $a^{\dagger} = a a^{\dagger} a^{\dagger}$. This indicates $a \in R^{EP}$. Hence we have

$$a(a^{\#})^* = a(a^{\dagger})^* = a a^{\text{\#}\!\!\!\bigcirc} (a^{\#})^* a (a a^{\#})^* = a a^{\dagger} (a^{\#})^* a (a a^{\dagger})^* = (a^{\#})^* a a a^{\dagger} = (a^{\#})^* a.$$

Thus $a \in R^{Nor}$ by Lemma 1.2. □

Since $a \in R^{EP}$, $a(a^{\dagger})^* a^{\dagger} = a^{\dagger} a^2 (a^{\dagger})^* a^{\dagger}$ and $a^{\dagger} a (a^{\#})^* a^2 a^{\dagger} a^{\dagger} = (a^{\#})^* a a^{\dagger} = (a^{\#})^*$. Hence Theorem 3.3 leads to the following corollary.

**Corollary 3.4.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Nor}$ if and only if one of the following conditions are satisfied:*
(1) $a(a^{\dagger})^* a^{\dagger} = (a^{\#})^*$;
(2) $a(a^{\dagger})^* a^{\dagger} = (a^{\dagger})^*$;
(3) $a(a^{\#})^* a^{\dagger} = (a^{\#})^*$;
(4) $a(a^{\#})^* a^{\dagger} = (a^{\dagger})^*$.

*Proof.* If $a$ is normal, then $a \in R^{EP}$ and $a(a^{\#})^* = (a^{\#})^* a$. This gives

$$a(a^{\dagger})^* a^{\dagger} = a(a^{\#})^* a^{\dagger} = (a^{\#})^* a a^{\dagger} = (a^{\#})^* = (a^{\dagger})^*.$$

On the contrary, we suppose that $a \in R^{\#} \cap R^{\dagger}$. Then we can obtain the element is subject to one of the preceding already established conditions of this corollary to prove that $a$ is normal.

(1) Assume the equality $a(a^\dagger)^*a^\dagger = (a^\#)^*$. Multiplying the last equality on the right by $a$, we have

$$a(a^\dagger)^* = (a^\#)^*a.$$

Then take the involution, one gets $a^\dagger a^* = a^*a^\#$. Consequently $a \in R^{Nor}$ by Lemma 1.2.

(2) From the equality $a(a^\dagger)^*a^\dagger = (a^\dagger)^*$, we get

$$(a^\dagger)^* = a(a^\dagger)^*a^\dagger = (a(a^\dagger)^*a^\dagger)aa^\dagger = (a^\dagger)^*aa^\dagger.$$

Applying the involution to the above equality, one has $a^\dagger = aa^\dagger a^\dagger$. This infers $a \in R^{EP}$ and then

$$a(a^\dagger)^*a^\dagger = (a^\dagger)^* = (a^\#)^*.$$

Therefore the condition (1) holds.

(3) Applying the equality $a(a^\#)^*a^\dagger = (a^\#)^*$, one obtains

$$(a^\#)^* = a(a^\#)^*a^\dagger = aa^\dagger(a(a^\#)^*a^\dagger) = aa^\dagger(a^\#)^*.$$

Taking the involution to the above equality, we have $a^\# = a^\# aa^\dagger$. Therefore $a \in R^{EP}$ and

$$a(a^\dagger)^*a^\dagger = a(a^\#)^*a^\dagger = (a^\#)^*.$$

Thus the condition (1) is satisfied.

(4) Suppose that $a(a^\#)^*a^\dagger = (a^\dagger)^*$, we have

$$(a^\dagger)^* = a(a^\#)^*a^\dagger = (a(a^\#)^*a^\dagger)aa^\dagger = (a^\dagger)^*aa^\dagger.$$

Then we take the invotution to the above equality, one gets $a^\dagger = aa^\dagger a^\dagger$. Hence $a \in R^{EP}$ and

$$a(a^\#)^* = (a(a^\#)^*a^\dagger)a = (a^\dagger)^*a = (a^\#)^*a.$$

Consequently $a \in R^{Nor}$. □

Taking the involution to the equality of Corollary 3.4, we have the following corollary.

**Corollary 3.5.** *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{Nor}$ if and only if one of the following conditions are satisfied:*
(1) $(a^\dagger)^*a^\dagger a^* = a^\#$;
(2) $(a^\dagger)^*a^\dagger a^* = a^\dagger$;
(3) $(a^\dagger)^*a^\# a^* = a^\#$;
(4) $(a^\dagger)^*a^\# a^* = a^\dagger$.

Certainly, using the symmetricity of Corollary 3.5, we can give the following corollary.

**Corollary 3.6.** *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{Nor}$ if and only if one of the following conditions are satisfied:*
(1) $a^*a^\dagger(a^\dagger)^* = a^\#$;
(2) $a^*a^\dagger(a^\dagger)^* = a^\dagger$;
(3) $a^*a^\#(a^\dagger)^* = a^\#$;
(4) $a^*a^\#(a^\dagger)^* = a^\dagger$.

## 4. Constructing invertible elements to characterize normality

It is well known that if $a \in R^\#$, then $a + 1 - aa^\# \in R^{-1}$ and $(a + 1 - aa^\#)^{-1} = a^\# + 1 - aa^\#$. By Theorem 3.1, we have $(a^2(a^\dagger)^*)^\# = aa^\# a^*a^\# a^\#$ and $a^2(a^\dagger)^*(a^2(a^\dagger)^*)^\# = aa^\#$, hence we have the following lemma.

**Lemma 4.1.** *Let $a \in R^\# \cap R^\dagger$. Then $a^2(a^\dagger)^* + 1 - aa^\# \in R^{-1}$ and $(a^2(a^\dagger)^* + 1 - aa^\#)^{-1} = aa^\# a^*a^\# a^\# + 1 - aa^\#$.*

Using the representation form of invertible elements, the following theorem gives a new characterization of normality.

**Theorem 4.2.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Nor}$ if and only if $(a^2(a^{\dagger})^* + 1 - aa^{\#})^{-1} = a^{\dagger}a^*a^{\#} + 1 - aa^{\#}$.*

*Proof.* $\Rightarrow$ It can be easily proved by Theorem 3.2 and Lemma 4.1.

$\Leftarrow$ According to Lemma 4.1, we can get $(a^2(a^{\dagger})^* + 1 - aa^{\#})^{-1} = aa^{\#}a^*a^{\#}a^{\#} + 1 - aa^{\#}$. By assumption, $(a^2(a^{\dagger})^* + 1 - aa^{\#})^{-1} = a^{\dagger}a^*a^{\#} + 1 - aa^{\#}$. Then $aa^{\#}a^*a^{\#}a^{\#} + 1 - aa^{\#} = a^{\dagger}a^*a^{\#} + 1 - aa^{\#}$. This indicates $aa^{\#}a^*a^{\#}a^{\#} = a^{\dagger}a^*a^{\#}$. Hence $a \in R^{Nor}$ by Theorem 3.1 and Theorem 3.2. □

Also, by Theorem 3.1, we can easily obtain the following lemma.

**Lemma 4.3.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a(a^{\dagger})^*a + 1 - aa^{\#} \in R^{-1}$ and $(a(a^{\dagger})^*a + 1 - aa^{\#})^{-1} = a^{\#}a^*a^{\#} + 1 - a^{\#}a$.*

**Theorem 4.4.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Nor}$ if and only if $(a(a^{\dagger})^*a + 1 - aa^{\#})^{-1} = a^{\#}a^{\dagger}a^* + 1 - a^{\#}a$.*

*Proof.* $\Rightarrow$ Suppose that $a \in R^{Nor}$. Then $a^{\#}a^{\dagger}a^* = a^{\#}a^*a^{\dagger} = a^{\#}a^*a^{\#}$. Hence, it can be proved by Lemma 4.3.

$\Leftarrow$ Provided that $(a(a^{\dagger})^*a + 1 - aa^{\#})^{-1} = a^{\#}a^{\dagger}a^* + 1 - a^{\#}a$, and by Lemma 4.3, $(a(a^{\dagger})^*a + 1 - aa^{\#})^{-1} = a^{\#}a^*a^{\#} + 1 - a^{\#}a$. We can illuminate that

$$a^{\#}a^*a^{\#} = a^{\#}a^{\dagger}a^* = a^{\#}a^{\dagger}a^*aa^{\dagger} = a^{\#}a^*a^{\#}aa^{\dagger}.$$

Then $a^{\#} = aa^{\dagger}a^{\#} = (a^{\dagger})^*aa^{\#}a^*a^{\#} = (a^{\dagger})^*aa^{\#}a^*a^{\#}aa^{\dagger} = a^{\#}aa^{\dagger}$. This indicates $a \in R^{EP}$ and $a^{\#}a^*a^{\#} = a^{\#}a^{\dagger}a^* = a^{\#}a^{\#}a^*$. By Lemma 1.2, $a \in R^{Nor}$. □

**Corollary 4.5.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Nor}$ if and only if $(a(a^{\dagger})^*a + 1 - aa^{\#})^{-1} = a^{\#}a^{\dagger}a^*a^{\#}a + 1 - a^{\dagger}a$.*

*Proof.* $\Rightarrow$ Assume that $a \in R^{Nor}$. Then $a \in R^{EP}$ and by Theorem 4.4, we have

$$(a(a^{\dagger})^*a + 1 - aa^{\#})^{-1} = a^{\#}a^{\dagger}a^* + 1 - a^{\#}a.$$

Noting that $a \in R^{EP}$. Then $a^{\#}a = a^{\dagger}a$ and $a^{\#}a^{\dagger}a^* = a^{\#}a^{\dagger}a^*a^{\dagger}a = a^{\#}a^{\dagger}a^*a^{\#}a$. Thus $(a(a^{\dagger})^*a + 1 - aa^{\#})^{-1} = a^{\#}a^{\dagger}a^*a^{\#}a + 1 - a^{\dagger}a$.

$\Leftarrow$ From the assumption, one gets

$$\begin{aligned} 1 &= (a^{\#}a^{\dagger}a^*a^{\#}a + 1 - a^{\dagger}a)(a(a^{\dagger})^*a + 1 - aa^{\#}) \\ &= a^{\#}a^{\dagger}a^*a(a^{\dagger})^*a + a(a^{\dagger})^*a - a^{\dagger}a^2(a^{\dagger})^*a + 1 - aa^{\#} - a^{\dagger}a + a^{\dagger}a^2a^{\#}. \end{aligned}$$

This gives

$$a^{\#}a^{\dagger}a^*a(a^{\dagger})^*a + a(a^{\dagger})^*a - a^{\dagger}a^2(a^{\dagger})^*a = aa^{\#}. \tag{4.1}$$

Pre-multiply the equality by $aa^{\#}$, one yields

$$a^{\dagger}a^2(a^{\dagger})^*a = aa^{\#}a^{\dagger}a^2(a^{\dagger})^*a = a(a^{\dagger})^*a.$$

Post-multiply the equality $a^{\dagger}a^2(a^{\dagger})^*a = a(a^{\dagger})^*a$ by $a^{\#}a^*a^{\#}$, we have $a^{\dagger}a = aa^{\#}$. Hence $a \in R^{EP}$ and the Eq.(4.1) changes to $a^{\#}a^{\dagger}a^*a(a^{\dagger})^*a = aa^{\#}$. This leads to $a^*a(a^{\dagger})^*a = a^2a^{\#}a^{\dagger}a^*a(a^{\dagger})^*a = a^3a^{\#} = a^2$ and $a^*a(a^{\dagger})^* = a^*a(a^{\dagger})^*aa^{\#} = a^2a^{\#} = a$. Therefore $a^*a = a^*a^2a^{\dagger} = a^*a(a^{\dagger})^*a^* = aa^*$. Thus $a \in R^{Nor}$. □

The proof of the following lemma is also routine.

**Lemma 4.6.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then $(a^{\dagger})^*a^2 + 1 - aa^{\#} \in R^{-1}$ and $((a^{\dagger})^*a^2 + 1 - aa^{\#})^{-1} = a^{\#}a^{\#}a^*a^{\#}a + 1 - a^{\#}a$.*

**Theorem 4.7.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Nor}$ if and only if $((a^{\dagger})^*a^2 + 1 - aa^{\#})^{-1} = a^{\#}a^{\#}a^{\dagger}a^*a + 1 - a^{\dagger}a$.*

*Proof.* $\Rightarrow$ It is clear.

$\Leftarrow$ From the assumption, one obtains $1 = (a^{\#}a^{\#}a^{\dagger}a^*a + 1 - a^{\dagger}a)((a^{\dagger})^*a^2 + 1 - aa^{\#})$. Then simplify the above equality, we get

$$a^{\#}a^{\#}a^{\dagger}a^*a(a^{\dagger})^*a^2 + (a^{\dagger})^*a^2 = a^{\dagger}a(a^{\dagger})^*a^2 + aa^{\#}. \tag{4.2}$$

Pre-multiply the equality by $aa^{\#}$, one has $a^{\dagger}a(a^{\dagger})^*a^2 = (a^{\dagger})^*a^2$. Post-multiply the last equality by $a^{\#}a^{\#}a^*a$, we obtain $a^{\dagger}a^2 = a$. Therefore $a \in R^{EP}$ and the Eq.(4.2) can be simplified to $a^{\#}a^{\#}a^{\dagger}a^*a(a^{\dagger})^*a^2 = aa^{\#}$. Pre-multiply the last equality by $a^3$ and according to the equality $a^* = aa^{\#}a^*$, one yields $a^*a(a^{\dagger})^*a^2 = a^3$. Thus $a^*a(a^{\dagger})^* = a^*a(a^{\dagger})^*a^2a^{\#}a^{\#} = a^3a^{\#}a^{\#} = a$. Then $a^*a = a^*a^2a^{\dagger} = a^*a(a^{\dagger})^*a^* = aa^*$, we conclude $a \in R^{Nor}$. □

## 5. Characterizing normality by solutions in a definite set of equations

Observing Theorem 3.2, we establish the following equation:

$$a^{\#}a^{*}x = xa^{\#}a^{*}. \tag{5.1}$$

**Theorem 5.1.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Nor}$ if and only if Eq.*(5.1) *has at least one solution in $\rho_a = \{a, a^{\#}, a^{\dagger}, a^{*}, (a^{\dagger})^{*}, (a^{\#})^{*}, (a^{\dagger})^{\#}, (a^{\#})^{\dagger}\}$.*

*Proof.* $\Rightarrow$ Since $a \in R^{Nor}$, it follows from Lemma 1.2 that $x = a$ is a solution.

$\Leftarrow$ We know that $x = a, a^{\#}, a^{\dagger}, a^{*}$ as solutions have been proved by Lemma 1.2. Thus we only prove the rest solutions.

(1) If $x = (a^{\dagger})^{*}$ is a solution, then $a^{\#} = a^{\#}a^{*}(a^{\dagger})^{*} = (a^{\dagger})^{*}a^{\#}a^{*}$. This implies $a^{\#} = (a^{\dagger})^{*}a^{\#}a^{*} = (a^{\dagger})^{*}a^{\#}a^{*}aa^{\dagger} = a^{\#}aa^{\dagger}$ and then $a \in R^{EP}$. Therefore $(a^{\#})^{*} = ((a^{\dagger})^{*}a^{\#}a^{*})^{*} = a(a^{\#})^{*}a^{\dagger}$ and $(a^{\#})^{*}a = a(a^{\#})^{*}a^{\dagger}a = a(a^{\#})^{*}$. Thus $a \in R^{Nor}$ by Lemma 1.2.

(2) If $x = (a^{\#})^{*}$ is a solution, then we have $(a^{\#})^{*}a^{\#}a^{*} = a^{\#}a^{*}(a^{\#})^{*} = aa^{\dagger}a^{\#}a^{*}(a^{\#})^{*} = aa^{\dagger}(a^{\#})^{*}a^{\#}a^{*}$. Post-multiply the equality $(a^{\#})^{*}a^{\#}a^{*} = aa^{\dagger}(a^{\#})^{*}a^{\#}a^{*}$ by $(a^{\dagger})^{*}a^{2}a^{\dagger}$ and then take the involution, it obtains $a^{\#} = a^{\#}aa^{\dagger}$. This infers $a \in R^{EP}$. Thus $x = (a^{\#})^{*} = (a^{\dagger})^{*}$. By (1), we get $a \in R^{Nor}$.

(3) If $x = (a^{\dagger})^{\#} = (aa^{\#})^{*}a(aa^{\#})^{*}$ is a solution, then $a^{\#}a^{*}(a^{\dagger})^{\#} = (a^{\dagger})^{\#}a^{\#}a^{*}$. Simplify the equality to $a^{\#}a^{*}a(aa^{\#})^{*} = (aa^{\#})^{*}a(aa^{\#})^{*}a^{\#}a^{*}$. Pre-multiply the equality by $a^{\dagger}a$, we have

$$a^{\#}a^{*}a(a^{\#})^{*}a^{*} = a^{\dagger}aa^{\#}a^{*}a(a^{\#})^{*}a^{*}.$$

And then $a^{\#} = a^{\#}a^{*}a(a^{\#})^{*}a^{*}a^{\dagger}(a^{\dagger})^{*} = a^{\dagger}aa^{\#}a^{*}a(a^{\#})^{*}a^{*}a^{\dagger}(a^{\dagger})^{*} = a^{\dagger}aa^{\#}$. Hence $a \in R^{EP}$. Therefore $x = (a^{\dagger})^{\#} = (a^{\#})^{\#} = a$. By Lemma 1.2, we get $a \in R^{Nor}$.

(4) If $x = (a^{\#})^{\dagger} = a^{\dagger}a^{3}a^{\dagger}$ is a solution, then $a^{\#}a^{*}(a^{\#})^{\dagger} = (a^{\#})^{\dagger}a^{\#}a^{*}$. Simplify the equality to $a^{\#}a^{*}a^{\dagger}a^{3}a^{\dagger} = a^{*}$. It follows from $a^{\#}a^{*}a^{*} = a^{\#}a^{*}a^{\dagger}aa^{*} = a^{\#}a^{*}a^{\dagger}a^{3}a^{\dagger}a^{\#}a^{*} = a^{*}a^{\#}a^{*}$. Thus $a \in R^{Nor}$ by Lemma 1.2. □

Similarly, Eq.(5.1) can modified as follows

$$a^{\#}a^{*}x = xa^{\dagger}a^{*}. \tag{5.2}$$

**Theorem 5.2.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Nor}$ if and only if Eq.*(5.2) *has at least one solution in $\rho_a$.*

*Proof.* $\Rightarrow$ Since $a \in R^{Nor}$, then $a \in R^{EP}$ and $a^{*}a^{\#} = a^{\dagger}a^{*}$ by Lemma 1.2. Hence $a^{\#}a^{*}a^{\#} = a^{\#}a^{\dagger}a^{*}$. It follows from that $x = a^{\#}$ is a solution.

$\Leftarrow$ (1) If $x = a$ is a solution, then $a^{\#}a^{*}a = aa^{\dagger}a^{*} = aa^{\dagger}a^{*}aa^{\dagger} = a^{\#}a^{*}a^{2}a^{\dagger}$. Multiplying the equality $a^{\#}a^{*}a = a^{\#}a^{*}a^{2}a^{\dagger}$ on the left by $(a^{\dagger})^{*}a$, one gets $a = a^{2}a^{\dagger}$. Hence $a \in R^{EP}$. This indicates $a^{\#}a^{*}a = aa^{\dagger}a^{*} = aa^{\#}a^{*}$. Therefore $a \in R^{Nor}$ by Lemma 1.2.

(2) If $x = a^{\#}$ is a solution, then $a^{\#}a^{*}a^{\#} = a^{\#}a^{\dagger}a^{*} = a^{\#}a^{\dagger}a^{*}aa^{\dagger} = a^{\#}a^{*}a^{\#}aa^{\dagger}$. Thus $a^{\#} = aa^{\dagger}a^{\#} = (a^{\dagger})^{*}aa^{\#}a^{*}a^{\#} = (a^{\dagger})^{*}aa^{\#}a^{*}a^{\#}aa^{\dagger} = a^{\#}aa^{\dagger}$. It implies $a \in R^{EP}$ and $a^{\#}a^{*}a^{\#} = a^{\#}a^{\dagger}a^{*} = a^{\#}a^{\#}a^{*}$. By Lemma 1.2, $a \in R^{Nor}$.

(3) If $x = a^{\dagger}$ is a solution, then $a^{\dagger}a^{\dagger}a^{*} = a^{\#}a^{*}a^{\dagger} = aa^{\dagger}a^{\#}a^{*}a^{\dagger} = aa^{\dagger}a^{\dagger}a^{\dagger}a^{*}$. Post-multiply the equality $a^{\dagger}a^{\dagger}a^{*} = aa^{\dagger}a^{\dagger}a^{\dagger}a^{*}$ by $(a^{\#})^{*}a(aa^{\#})^{*}$, we have $a^{\dagger} = aa^{\dagger}a^{\dagger}$. This infers $a \in R^{EP}$ and $x = a^{\dagger} = a^{\#}$. Consequently, $a \in R^{Nor}$ by (2).

(4) If $x = a^{*}$ is a solution, then $a^{\#}a^{*}a^{*} = a^{*}a^{\dagger}a^{*}$. We have $a^{\#}a^{*} = a^{\#}a^{*}a^{*}(a^{\#})^{*} = a^{*}a^{\dagger}a^{*}(a^{\#})^{*} = a^{*}a^{\dagger}$. Hence $a \in R^{Nor}$ by Lemma 1.2.

(5) If $x = (a^{\dagger})^{*}$ is a solution, then

$$a^{\#} = a^{\#}a^{*}(a^{\dagger})^{*} = (a^{\dagger})^{*}a^{\dagger}a^{*} = (a^{\dagger})^{*}a^{\dagger}a^{*}aa^{\dagger} = a^{\#}aa^{\dagger}.$$

This indicates $a \in R^{EP}$. Thus $(a^{\#})^{*} = ((a^{\dagger})^{*}a^{\dagger}a^{*})^{*} = a(a^{\dagger})^{*}a^{\dagger}$ and $(a^{\#})^{*}a = a(a^{\dagger})^{*}a^{\dagger}a = a(a^{\dagger})^{*} = a(a^{\#})^{*}$. Accordingly, $a \in R^{Nor}$ by Lemma 1.2.

(6) If $x = (a^{\#})^{*}$ is a solution, then $a^{\#}a^{*}(a^{\#})^{*} = (a^{\#})^{*}a^{\dagger}a^{*} = a^{\dagger}a(a^{\#})^{*}a^{\dagger}a^{*} = a^{\dagger}aa^{\#}a^{*}(a^{\#})^{*}$. Then $a^{\#} = a^{\#}a^{*}(a^{\#})^{*}a^{\dagger}a = a^{\dagger}aa^{\#}a^{*}(a^{\#})^{*}a^{\dagger}a = a^{\dagger}aa^{\#}$. Hence $a \in R^{EP}$ and $x = (a^{\#})^{*} = (a^{\dagger})^{*}$. By (5), $a \in R^{Nor}$.

(7) If $x = (a^\dagger)^\# = (aa^\#)^* a(aa^\#)^*$ is a solution, then $a^\# a^* (a^\dagger)^\# = (a^\dagger)^\# a^\dagger a^*$. Simplify the equality to $a^\# a^* a(aa^\#)^* = a^*$. So we have

$$a^* = a^\# a^* a(aa^\#)^* = aa^\dagger a^\# a^* a(aa^\#)^* = aa^\dagger a^*.$$

This infers $a = a^2 a^\dagger$ by involution and then $a \in R^{EP}$. Thus $x = (a^\dagger)^\# = (a^\#)^\# = a$. We conclude $a \in R^{Nor}$ by (1).

(8) If $x = (a^\#)^\dagger = a^\dagger a^3 a^\dagger$ is a solution, then

$$(a^\#)^\dagger a^\dagger a^* = a^\# a^* (a^\#)^\dagger = aa^\dagger a^\# a^* (a^\#)^\dagger = aa^\dagger (a^\#)^\dagger a^\dagger a^*.$$

Post-multiply the equality $(a^\#)^\dagger a^\dagger a^* = aa^\dagger (a^\#)^\dagger a^\dagger a^*$ by $(a^\#)^* a(a^\#)^*$, we get $aa^\dagger (a^\#)^\dagger (a^\#)^* = (a^\#)^\dagger (a^\#)^*$. Post-multiply the last equality by $a^* a^\# a^\dagger$, one yields $aa^\dagger a^\dagger = a^\dagger$. This infers $a \in R^{EP}$ and $x = (a^\#)^\dagger = (a^\dagger)^\dagger = a$. Therefore $a \in R^{Nor}$ by (1). □

Noting that $a \in R^{Nor}$ if and only if $a^\dagger \in R^{Nor}$. Meanwhile, we know that $a \in R^{Nor}$ can deduce $a \in R^{EP}$ and $a^\dagger aa^\# = a^\#$ by [5, Lemma 1.3]. Thus replace $a$ in Eq.(5.2) by $a^\dagger$, we have Eq.(5.3). As consequence, Theorem 5.2 leads to the following theorem.

$$(aa^\#)^* a(a^\#)^* x = xa(a^\dagger)^*. \tag{5.3}$$

**Theorem 5.3.** *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{Nor}$ if and only if Eq.*(5.3) *has at least one solution in $\rho_a$.*

Similarly, replace $a$ in Eq.(5.2) by $a^\#$, we have Eq.(5.4). By Theorem 5.2, we can easily induce the following theorem.

$$a(a^\#)^* x = xa^\dagger a^3 a^\dagger (a^\#)^*. \tag{5.4}$$

**Theorem 5.4.** *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{Nor}$ if and only if Eq.*(5.4) *has at least one solution in $\rho_a$.*

## 6. The general solution of bivariate equations

Now we generalize Eq.(5.2) as follows

$$a^\# a^* x = ya^\dagger a^*. \tag{6.1}$$

**Theorem 6.1.** *Let $a \in R^\# \cap R^\dagger$. Then the general solution to the Eq.*(6.1) *is given by*

$$\begin{cases} x = pa^\dagger a^* + u - aa^\dagger u \\ y = a^\# a^* p + v - va^\dagger a \end{cases}, \text{where } p,\ u,\ v \in R\ . \tag{6.2}$$

*Proof.* Firstly,

$$a^\# a^* (pa^\dagger a^* + u - aa^\dagger u) = a^\# a^* pa^\dagger a^* = (a^\# a^* p + v - va^\dagger a)a^\dagger a^*.$$

Therefore the formula (6.2) is the solution of Eq.(6.1).

Then let $\begin{cases} x = x_0 \\ y = y_0 \end{cases}$ be any solution to the Eq.(6.1), then $a^\# a^* x_0 = y_0 a^\dagger a^*$.

Choose $u = x_0$, $p = aa^\dagger (a^\#)^* a^\dagger a^2 y_0$, $v = y_0 - a^\# a^* p$. Then

$$pa^\dagger a^* = aa^\dagger (a^\#)^* a^\dagger a^2 y_0 a^\dagger a^* = aa^\dagger (a^\#)^* a^\dagger a^2 a^\# a^* x_0 = aa^\dagger x_0 = aa^\dagger u.$$

Hence $x_0 = u = pa^\dagger a^* + u - aa^\dagger u$.

Since

$$\begin{aligned} va^\dagger a &= (y_0 - a^\# a^* p)a^\dagger a = y_0 a^\dagger a - a^\# a^* (aa^\dagger (a^\#)^* a^\dagger a^2 y_0)a^\dagger a \\ &= y_0 a^\dagger a - aa^\# y_0 a^\dagger a = y_0 a^\dagger a - aa^\# (y_0 a^\dagger a^*)(a^\#)^* a = y_0 a^\dagger a - aa^\# (a^\# a^* x_0)(a^\#)^* a \\ &= y_0 a^\dagger a - a^\# a^* x_0 (a^\#)^* a = y_0 a^\dagger a - y_0 a^\dagger a^* (a^\#)^* a = y_0 a^\dagger a - y_0 a^\dagger a = 0. \end{aligned}$$

This infers $y_0 = a^\# a^* p + y_0 - a^\# a^* p = a^\# a^* p + v = a^\# a^* p + v - va^\dagger a$. Thus the general solution to the Eq.(6.1) is given by formula (6.2). □

**Theorem 6.2.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Nor}$ if and only if the general solution to the Eq.*(6.1) *is given by*

$$\begin{cases} x = pa^{\dagger}a^{*} + u - aa^{\dagger}u \\ y = a^{*}a^{\#}p + v - va^{\dagger}a \end{cases}, \text{where } p,\ u,\ v \in R\ . \tag{6.3}$$

*Proof.* $\Rightarrow$ Since $a \in R^{Nor}$, then $a^{*}a^{\#} = a^{\#}a^{*}$ by Lemma 1.2.

Hence the formula (6.3) is the same as the formula (6.2). By Theorem 6.1, the general solution of Eq.(6.1) is given by the formula (6.3).

$\Leftarrow$ From the assumption, one obtains

$$a^{\#}a^{*}(pa^{\dagger}a^{*} + u - aa^{\dagger}u) = (a^{*}a^{\#}p + v - va^{\dagger}a)a^{\dagger}a^{*}.$$

That is $a^{\#}a^{*}pa^{\dagger}a^{*} = a^{*}a^{\#}pa^{\dagger}a^{*}$ for all $p \in R$. Choose $p = a$, one has

$$a^{\#}a^{*}aa^{\dagger}a^{*} = a^{*}a^{\#}aa^{\dagger}a^{*}.$$

Post-multiply the above equality by $(a^{\#})^{*}aa^{\#}$, we get $a^{*}a^{\#} = a^{\#}a^{*}$. Consequently, $a \in R^{Nor}$ by Lemma 1.2. □

Now we construct the following equation

$$a^{*}a^{\#}(aa^{\#})^{*}x = ya^{\dagger}a^{*}. \tag{6.4}$$

**Theorem 6.3.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then the general solution to the Eq.*(6.4) *is given by*

$$\begin{cases} x = pa^{\dagger}a^{*} + u - aa^{\dagger}u \\ y = a^{*}a^{\#}p + v - va^{\dagger}a \end{cases}, \text{where } p,\ u,\ v \in R \text{ with } a^{\dagger}p = a^{\dagger}a^{\dagger}ap. \tag{6.5}$$

*Proof.* The proof is similar to Theorem 6.1. □

**Theorem 6.4.** *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Nor}$ if and only if the Eq.*(6.1) *and Eq.*(6.4) *have the same solution.*

*Proof.* $\Rightarrow$ Since $a \in R^{Nor}$, then by the Theorem 6.2, the general solution to the Eq.(6.1) is given by the formula (6.3). We know that $a \in R^{EP}$, then $a^{\dagger} = a^{\dagger}a^{\dagger}a$. Thus the general solution to the Eq.(6.1) is given by the formula (6.5). According to the Theorem 6.3, the Eq.(6.1) and Eq.(6.4) have the same solution.

$\Leftarrow$ Provided that the Eq.(6.1) and Eq.(6.4) have the same solution, then the Eq.(6.1) is given by the formula (6.5) by Theorem 6.3. This indicates

$$a^{\#}a^{*}(pa^{\dagger}a^{*} + u - aa^{\dagger}u) = (a^{*}a^{\#}p + v - va^{\dagger}a)a^{\dagger}a^{*}.$$

Therefore $a^{\#}a^{*}pa^{\dagger}a^{*} = a^{*}a^{\#}pa^{\dagger}a^{*}$ for all $p \in R$ with $a^{\dagger}p = a^{\dagger}a^{\dagger}ap$.

Especially, choose $p = a^{\dagger}$, we get $a^{\#}a^{*}a^{\dagger}a^{\dagger}a^{*} = a^{*}a^{\#}a^{\dagger}a^{\dagger}a^{*}$. Post-multiply the equality by $(a^{\#})^{*}a(aa^{\#})^{*}$, one yields $a^{\#}a^{*}a^{\dagger} = a^{*}a^{\#}a^{\dagger}$. By Theorem 3.2, $a \in R^{Nor}$. □

## References


[1] D. Mosić, D. S. Djordjević, J. J. Koliha, *EP elements in rings*, Linear Algebra Appl. **431** (2009), 527-535.

[2] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. **51** (1955), 406-413.

[3] A. Ben-Israel, T. N. E Greville, *Generalized inverses: theory and applications*, Springer Science & Business Media, 2003.

[4] R. E. Hartwig, *Block generalized inverses*, Arch. Retion. Mech. Anal. **61** (1976), 197-251.

[5] D. Mosić, D. S. Djordjević, *Moore-Penrose invertible normal and Hermitian elements in rings*, Linear Algebra Appl. **431** (2009), 732–745.

[6] D. Mosić, *Generalized inverses*, Faculty of Sciences and Mathematics, University of Niš, Niš, 2018.

[7] L. Shi, J. C.Wei, *Some new characterizations of normal elements*, Filomat. **33(13)** (2019), 4115-4120.

[8] Y. C. Qu, J. C.Wei, H. Yao, *Characterizations of normal elements in rings with involution*, Acta. Math. Hungar. **156(2)** (2018), 459-464.

[9] W. X. Chen, *On EP elements, normal elements and paritial isometries in rings with involution*, Electron. J. Linear Algebra. **23** (2012), 553-561.

[10] D. Mosić, D. S. Djordjević, *New characterizations of EP, generalized normal and generalized Hermitian elements in rings*, Appl. Math. Comput. **218** (2012), 6702-6710.

[11] R. J. Zhao, H. Yao, J. C. Wei, *EP elements and the solutions of equation in rings with involution*, Filomat. **32(13)** (2018), 4537-4542.
[12] D. S. Rakić, N. Č. Dinčić, D. S. Djordjević, *Group, Moore-Penrose, core and dual core inverse in rings with involution*, Linear Algebra & Its Applications. **463** (2014), 115-133.
[13] R. E. Harte, M. Mbekhta, *On generalized inverses in $C^*$-algebras*, Studia Math. **103** (1992), 71-77.
[14] S. Karanasios, *EP elements in rings and semigroup with involution and $C^*$-algebras*, Serdica Math. J. **41** (2015), 83-116.
[15] S. Z. Xu, J. L. Chen, J. Benitez, *EP elements in rings with involution*, Bull. Malay. Math. Scien. Soc. **42(6)** (2019), 3409-3426.
[16] R. J. Zhao, H. Yao, J. C Wei, *Characterizations of partial isometries and two special kinds of EP elements*, Czecho. Math. J, **70(2)** (2020), 539-551.