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Uniqueness results related to L-functions satisfying same functional equation under sharing pre-images of range sets

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Abstract. In this article we have studied the uniqueness problem of L-function $\mathcal{L}$ in extended-Selberg class with some meromorphic function f having finitely many poles, when the pre-image of a set with respect to $\mathcal{L}$ coincides with the same of another set with respect to f. We have studied the problem corresponding to two Selberg class L-functions as well. With the new notion of sharing which is the generalization of the definition of traditional set sharing, we have significantly improved the results of [Houston. J. Math., **46**(**4**)(2020), 915-923], and some of [Complex Var. Elliptic Equ., Published online (DOI: 10.1080/17476933.2022.2069759)]. In another result, we have extended a result of [Ann. Polon. Math., **126**(2021), 265-278] which in turn provid the best possible answer to a question raised in the same paper.

1. Introduction

This paper concerns the question of how L-functions can be uniquely determined by sharing of any arbitrary finite set. The most common example of L-function is the Riemann zeta function, which was first introduced by Riemann in 1859 as a function in complex plane. It is defined in $Re(s) > 1$, as $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ (where $s = \sigma + it$), converges absolutely and admits analytic continuation as a meromorphic function throughout the complex plane, having simple pole at $s = 1$ with residue 1. Moreover the unique factorization theorem of natural numbers into primes helps us to express $\zeta(s)$ as a product over primes in $Re(s) > 1$, as $\zeta(s) = \prod_p (1 - p^{-s})$, where p is prime, known as a Euler product. The functional equation of $\zeta(s)$ that is $\zeta(s) = \zeta(1 - s)$ plays an important role in some properties of zeta function. In addition, the Riemann hypothesis, proposed by Riemann is a conjecture that all the non-trivial zeros of $\zeta(s)$ lies on $Re(s) = 1/2$, plays a very important role in analytic number theory. So in order to understand the analytic number theory it is important to study zeta function.

Riemann hypothesis can be generalized by replacing Riemann's zeta function by the formally similar, but much more general, L-function. Considering $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ as a prototype in 1989, Selberg defined a rather general class $\mathcal{S}$ of convergent Dirichlet series $L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ which is absolutely convergent for $Re(s) > 1$ In the meantime, this so-called Selberg class L-function became an important object of research

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as it plays an pivotal role in analytic number theory. An L-function in $\mathcal{S}$ needs to satisfy the following axioms:(see [15]):
(i) Ramanujan hypothesis: $a(n) \ll n^{\epsilon}$ for every $\epsilon > 0$.
(ii) Analytic continuation: There is a non-negative integer k such that $(s-1)^k\mathcal{L}(s)$ is an entire function of finite order.
(iii) Functional equation: $\mathcal{L}$ satisfies a functional equation of type

$$\Lambda_{\mathcal{L}}(s) = \omega\overline{\Lambda_{\mathcal{L}}(1-\overline{s})},$$

where

$$\Lambda_{\mathcal{L}}(s) = \mathcal{L}(s)Q^s\prod_{j=1}^{K}\Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q, λ_j and complex numbers ν_j, ω with $Re\ \nu_j \geq 0$ and $|\omega| = 1$.
(iv) Euler product hypothesis: $\mathcal{L}$ can be written over prime as

$$\mathcal{L}(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} b(p^k)/p^{ks}\right)$$

with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$ where the product is taken over all prime numbers p. The degree $d_{\mathcal{L}}$ of an L-function $\mathcal{L}$ is defined to be

$$d_{\mathcal{L}} = 2\sum_{j=1}^{K}\lambda_j,$$

where λ_j and K are respectively the positive real number and the positive integer as in axiom (iii) above.

In 1999, Kaczorowski-Perelli [6] introduced the extended Selberg class $\mathcal{S}^{\#}$, defined as the collection of not identically vanishing L-functions $\mathcal{L}$ which satisfies the axioms (i)-(iii) above but can not be expressed as a product over primes. Naturally a function is called L-function in Selberg class, if it satisfies the Euler product hypothesis but in $\mathcal{S}^{\#}$ we can have some functions with degree zero which do not have a Euler product and so it is worthwhile to study the extended Selberg class. We see that $\mathcal{S}$ is automatically included in $\mathcal{S}^{\#}$.

In this paper we are going to discuss some results under the periphery of value distribution of L-functions in $\mathcal{S}^{\#}$. Throughout this paper, by an L function we will mean an L-function of non-zero degree with the normalized condition $a(1) = 1$. On the other hand, by a meromorphic function f we mean meromorphic function in the whole complex plane $\mathbb{C}$. Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\overline{\mathbb{N}} = \mathbb{N} \cup \{0\}$, where $\mathbb{C}$ and $\mathbb{N}$ denote the set of all complex numbers and natural numbers respectively and by $\mathbb{Z}$ we denote the set of all integers.

Before entering into the detail literature, let us assume $\mathcal{M}(\mathbb{C})$ as the field of meromorphic functions over $\mathbb{C}$. The proofs of the theorems of the paper are heavily depending on Nevanlinna theory and we assume that the readers are familiar with the standard notations like the characteristic function $T(r, f)$, the proximity function $m(r, f)$, counting (reduced counting) function $N(r, f)$ ($\overline{N}(r, f)$) that are also explained in [4], [17]. By $S(r, f)$ we mean any quantity that satisfies $S(r, f) = O(\log(rT(r, f)))$ when $r \longrightarrow \infty$, except possibly on a set of finite Lebesgue measure. When f has finite order, then $S(r, f) = O(\log r)$ for all r.

Let us take $f \in \mathcal{M}(\mathbb{C})$, then the order of f is defined as

$$\rho(f) := \limsup_{r\longrightarrow\infty}\frac{\log T(r, f)}{\log r}.$$

Before proceeding further, we require the following definitions.

Definition 1.1. *Let f and g be two non-constant meromorphic functions in $\mathcal{M}(\mathbb{C})$. For some $S \subset \mathbb{C}$, we define $E_f(S) = \cup_{a\in S}\{z : f(z) - a = 0\}$, where each point is counted according to its multiplicity. If we do not count the multiplicity then the set $\cup_{a\in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ then we say f and g share the set S CM. On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$ then we say f and g share the set S IM. In particular, for some element $a \in \overline{\mathbb{C}}$, if $E_f(a) = E_g(a)$ ($\overline{E}_f(a) = \overline{E}_g(a)$) we call f, g share a CM (IM).*

In this paper we have not dealt with the usual sharing of sets, rather we have generalized it in a much broader perspective. The following definition is more generalized than *Definition 1.1* and somehow been inspired from the idea in Khoai-An-Ninh [7].

Definition 1.2. *Let S_1, $S_2 \subset \mathbb{C}$ and f, $g \in \mathcal{M}(\mathbb{C})$. If $E_f(S_1) = E_g(S_2)$ ($\overline{E}_f(S_1) = \overline{E}_g(S_2)$) holds then we say that f, g have the same inverse image with respect to the sets S_1 and S_2 respectively, counting multiplicity (ignoring multiplicity) and abbreviated it as IICM $\{(S_1)(f), (S_2)(g)\}$
(IIIM $\{(S_1)(f), (S_2)(g)\}$).*

The following examples show that the *Definition 1.2* actually exist and it extends the realm of *Definition 1.1*. In fact, the definition exists irrespective of the cases $\#(S_1) = \#(S_2)$ or $\#(S_1) \neq \#(S_2)$, where $\#(S)$ denotes the cardinality of the set S.

Example 1.3. *Set $f = e^{3z} + e^{2z} + 5$, $g = e^{3z} + e^{2z} - 1$ and $S_1 = \{1, 3\}$, $S_2 = \{-3, -5\}$ and here $E_f(S_1) = E_g(S_2)$, i.e., IICM $\{(S_1)(f), (S_2)(g)\}$.*

Example 1.4. *Set $f = e^z$ and $g = e^{2z} + 2e^z$and $S_1 = \{-2, -1, 0\}$, $S_2 = \{0, -1\}$ and here $\overline{E}_f(S_1) = \overline{E}_g(S_2)$, i.e., IIIM $\{(S_1)(f), (S_2)(g)\}$.*

Example 1.5. *Set $f = e^z$ and $g = e^{2z} - ae^z$ and $S_1 = \{0, a\}$, $S_2 = \{0\}$ and here $E_f(S_1) = E_g(S_2)$, i.e., IICM $\{(S_1)(f), (S_2)(g)\}$.*

Example 1.6. *Set $f = e^z - 1$ and $g = e^z(e^z - 2)$, and $S_1 = \{0\}$, $S_2 = \{-1\}$ and here $\overline{E}_f(S_1) = \overline{E}_g(S_2)$, i.e., IIIM $\{(S_1)(f), (S_2)(g)\}$.*

The purpose of the paper is to prove some results based on the notion introduced in *Definition 1.2*. Actually, we strive to classify those sets S_1, S_2; with some sufficient conditions such that there do not exist two different meromorphic functions or even L-functions, f, g such that IICM $\{(S_1)(f), (S_2)(g)\}$ (IIIM $\{(S_1)(f), (S_2)(g)\}$).

We are not going to explain the well known definitions of $N(r, a; f| \geq m)$ ($N(r, a; f| \leq m)$) and $N_*(r, a; f, g)$, $\overline{N}_L(r, a; f)$, $\overline{N}_L(r, a; g)$, as one can easily find the same in [8], [1]. When f and g share the value a, then by $N_E^{1)}(r, a; f)$ we mean the counting function of all the simple zeros of $f - a$ and $g - a$.

Recently the value distribution of L-function has been studied exhaustively by so many researchers ([5], [10], [11], [16] etc). The value distribution of L-function is all about the roots of $\mathcal{L}(s) = c$.

In 2007, regarding uniqueness problem of two $\mathcal{L}$ functions, Steuding (p. 152, [16]) proved that the number of shared values can be reduced significantly than that happens in case of ordinary meromorphic function. Below we invoke the result.

Theorem 1.7. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two non-constant L-functions with $a(1) = 1$ and $c \in \mathbb{C}$. If $E_{\mathcal{L}_1}(c) = E_{\mathcal{L}_2}(c)$ holds, then $\mathcal{L}_1 = \mathcal{L}_2$.*

Providing a counterexample with a zero degree L-function, Hu-Li [5] pointed out that *Theorem 1.7* is not true when c = 1. Since L-functions possess meromorphic continuations, it is quite natural to investigate up to which extent an L-function can share a set and value with an arbitrary meromorphic function.

Inspired by the question of Gross [3] for meromorphic functions, Yuan-Li-Yi [19] proposed the analogous question for a meromorphic function f and an L-function $\mathcal{L}$ sharing one or two finite sets. Yuan-Li-Yi [19] answered the question by themselves by proving the following uniqueness result.

Theorem 1.8. *[19] Let f be a meromorphic function having finitely many poles in $\mathbb{C}$ and let $\mathcal{L} \in \mathcal{S}$ be a non-constant L-function. Let us consider the set $S = \{w : w^n + aw^m + b = 0\}$, where $(n, m) = 1$, $n > 2m + 4$. If $E_f(S) = E_{\mathcal{L}}(S)$, then we will have $f = \mathcal{L}$.*

Recently there are a number of results in the direction of value distribution of L-functions under sharing of one or two sets. We have already mentioned that it was Khoai-An-Ninh [7] who explored the sharing of set in a different angle. In their paper [7] the following question was posed.

Question 1.1. *[7]*
(i) *What can be said about the relationship between L-functions $\mathcal{L}_1$ and $\mathcal{L}_2$, if $\overline{E}_{\mathcal{L}_1}(S) = \overline{E}_{\mathcal{L}_2}(T)$:*
or more generally, between a non-constant meromorphic function f and a non-constant L-function, i.e.,
(ii) *What happens if $\overline{E}_f(S) = \overline{E}_{\mathcal{L}}(T)$, where $S, T \subset \mathbb{C}$?*

In connection to deal with *Question 1.1* (ii), considering the zero set of more generalized form of Yi's polynomial [18], Khoai-An-Ninh [7] obtained the following result.

Theorem 1.9. *[7]Let n, m be positive integers, $n \geq 2m+8$, $a, b, d, u, v, t \in \mathbb{C}^*$ and let $S = \{z : P_H(z) = az^n + bz^m + d = 0\}$, $T = \{z : Q_H(z) = uz^n + vz^m + t = 0\}$ be two sets respectively. Suppose $\overline{E}_{\mathcal{L}}(S) = \overline{E}_f(T)$ for a non-constant meromorphic function f with finitely many poles in the complex plane, and a non-constant L-function $\mathcal{L}$. Then we have:*
(i) *There exists a non-zero constant h, such that $f = h\mathcal{L}$.*
(ii) *In particular, if for some L-functions $\mathcal{L}_1$, $\mathcal{L}_2$ we have $\overline{E}_{\mathcal{L}_1}(S) = \overline{E}_{\mathcal{L}_2}(T)$, then $\mathcal{L}_1 = \mathcal{L}_2$ and $P_H = Q_H$.*

Clearly in the above question if $S = T$ then the sharing is same with the usual sharing as in *Theorem 1.8* and hence the approach of Khoai-An-Ninh is unique and much more generalized than the usual sharing. So it will be quite interesting to investigate the uniqueness problems of L-functions together with the meromorphic function under sharing of set, not in usual way but following the approach resorted by Khoai-An-Ninh in [7].

Motivated by the approach of Khoai-An-Ninh in [7], in this paper we have tried to re-investigate the situation of *Theorem 1.9* considering a different gap polynomials. Here first let us introduce the following two polynomials, having no multiple zeros, of the forms

$$P(z) = z^n + az^{2m} + bz^m + c, \tag{1.1}$$

$$Q(z) = z^n + a_1z^{2m} + b_1z^m + c_1, \tag{1.2}$$

where n, m are positive integers such that $(n, m) = 1$ and $a, a_1, b, b_1, c, c_1 \in \mathbb{C}^*$.

The possible answer of *Question 1.1* for the case if $a = 0 = a_1$, were already investigated in detail in [7]. So here we will investigate the situation when $a \cdot a_1 \neq 0$. This is the prime motivation of writing this article.

Theorem 1.10. *Let $\mathcal{L}$ be a non-constant L-function and f be a non-constant meromorphic function having finitely many poles. Also let us consider two sets $S_P = \{w : P(w) = 0\}$, $S_Q = \{w : Q(w) = 0\}$; where P, Q are the polynomials defined as in (1.1), (1.2). Now suppose $\overline{E}_f(S_P) = \overline{E}_{\mathcal{L}}(S_Q)$ and $b^2 = 4ac$, $b_1^2 = 4a_1c_1$. Then for n with $n > 4m+7$, there exists a non-zero constant h such that $f = h\mathcal{L}$.*

Corollary 1.11. *Under the condition of Theorem 1.10, if we choose $f = \mathcal{L}_1$ and $\mathcal{L} = \mathcal{L}_2$, where $\mathcal{L}_1, \mathcal{L}_2$ are two non-constant L-functions, then we will get, $\mathcal{L}_1 = \mathcal{L}_2$.*

Recently considering the set sharing of L-functions Li-Wu-Yi [12] obtained the following results.

Theorem 1.12. *[12] Suppose that two L-function $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{S}^\#$ satisfy the same functional equation. If $E_{\mathcal{L}_1}(S) = E_{\mathcal{L}_2}(S)$ for a finite set $S = \{w : w^5 + \frac{5}{2}w^4 + \frac{5}{3}w^3 + c = 0\}$, where c is a distinct complex numbers, then $\mathcal{L}_1 = \mathcal{L}_2$.*

Theorem 1.13. *[12] Suppose that two L-function $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{S}^\#$ satisfy the same functional equation. If for some $S = \{c_1, c_2\}$, where c_1, c_2 distinct finite complex numbers, $E_{\mathcal{L}_1}(S) = E_{\mathcal{L}_2}(S)$ holds, then $\mathcal{L}_1 = \mathcal{L}_2$.*

Considering an arbitrary set of three elements Li-Du-Yi [13] obtained the following result.

Theorem 1.14. *[13] Suppose that two L-function $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{S}^\#$ satisfy the same functional equation. If for some $S = \{c_1, c_2, c_3\}$, where c_1, c_2 and c_3 are three distinct finite complex numbers, $E_{\mathcal{L}_1}(S) = E_{\mathcal{L}_2}(S)$ holds, then $\mathcal{L}_1 = \mathcal{L}_2$.*

Now in view of *Definition 1.2*, it will be interesting to re-investigate *Theorems 1.12, 1.13* and *1.14*. In this respect the following question is inevitable,

Question 1.2. *What can be said about the relationship between L-functions $\mathcal{L}_1$ and $\mathcal{L}_2$ if $E_{\mathcal{L}_1}(S_1) = E_{\mathcal{L}_2}(S_2)$ holds, for some $S_1,\ S_2 \subset \mathbb{C}$?*

In order to provide the answer of *Question 1.2*, here we have generalized the sharing in *Theorems 1.12, 1.13, 1.14* and obtained the following results.

Theorem 1.15. *If two non-constant L-function $\mathcal{L}_1, \mathcal{L}_2$ in $\mathcal{S}^\#$, satisfy the same functional equation, $S_1 = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, $S_2 = \{\beta_1, \beta_2, \dots, \beta_n\} \subset \mathbb{C}$ and $E_{\mathcal{L}_1}(S_1) = E_{\mathcal{L}_2}(S_2)$, then we will have $\mathcal{L}_1 = \mathcal{L}_2$, also we will get $S_1 = S_2$*

Remark 1.1. *From Theorem 1.15, it is clear there can not be two distinct L-functions with positive degree and same functional equation, sharing any arbitrary set in $\mathbb{C}$ CM.*

Remark 1.2. *If in Theorem 1.15, $S = S_1 = S_2$ a set with two and three elements then it is actually Theorems 1.13, 1.14 respectively, so it is a huge extension as well as improvement of Theorems 1.13, 1.14. Also since $S_1,\ S_2$ both are arbitrary, for the case $S = S_1 = S_2$, Theorem 1.15 is a drastic improvement of Theorem 1.12. In fact for the case $S = S_1 = S_2$ Theorem 1.15 is independent of cardinality and irrespective of choice of sets, it is a two fold improvement of Theorem 1.12.*

In the following example we will show, first leading coefficient $a(1)$ of L-function must be one, otherwise *Theorem 1.15* cease to hold.

Definition 1.16. *ζ and $c\zeta$ for some non-zero real number c satisfy same functional equation and $E_{\mathcal{L}_1}(\{0,1\}) = E_{\mathcal{L}_2}(\{0,c\})$ but still $\zeta \neq c\zeta$.*

Now considering arbitrary set we have tried to provide answer to the *Question 1.1 (i)*. Here before stating the next result, let us consider the polynomials $P_1(z),\ P_2(z)$ having the set $S_1,\ S_2$ it's simple zeros.

$$\begin{aligned} P_1(z) &= (z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_n) \\ &= z^n - (\sum \alpha_i)z^{n-1} + \dots + (-1)^{n-1}(\sum \alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_{n-1}})z - (-1)^{n+1}\alpha_1\alpha_2\dots\alpha_n, \end{aligned} \tag{1.3}$$

$$\begin{aligned} P_2(z) &= (z-\beta_1)(z-\beta_2)\dots(z-\beta_n) \\ &= z^n - (\sum \beta_i)z^{n-1} + \dots + (-1)^{n-1}(\sum \beta_{i_1}\beta_{i_2}\dots\beta_{i_{n-1}})z - (-1)^{n+1}\beta_1\beta_2\dots\beta_n. \end{aligned} \tag{1.4}$$

Let $k_i,\ i = 1,2$ denote the number of distinct zeros of $P_i'(z)$, $i = 1,2$.

Theorem 1.17. *Let $\mathcal{L}_1,\ \mathcal{L}_2$ and $S_1,\ S_2$ be defined same as in Theorem 1.15 and here $n > 2k+5$ where $k = \max\{k_1, k_2\}$. Now if $\overline{E}_{\mathcal{L}_1}(S_1) = \overline{E}_{\mathcal{L}_2}(S_2)$, then we will have $\mathcal{L}_1 = \mathcal{L}_2$. Also we will get $S_1 = S_2$*

2. Lemma

Next, we present some lemmas that will be needed in the sequel. Henceforth, we denote by H, the following function :

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

Lemma 2.1. *[2] Let F and G be non-constant meromorphic functions and let F, G share 1 IM. Then,*

$$N_E^{1)}(r,1;F) \leq N(r,\infty;H) + S(r,F) + S(r,G).$$

Lemma 2.2. *[9] Let F and G be non-constant meromorphic functions and let F, G share 1 IM (i.e., $\overline{E}_F(1) = \overline{E}_G(1)$). Then,*

$$\begin{aligned} N(r,\infty;H) &\leq \overline{N}_*(r,1;F,G) + \overline{N}(r,\infty;F \mid\geq 2) + \overline{N}(r,\infty;G \mid\geq 2) + \overline{N}(r,0;F \mid\geq 2) \\ &\quad + \overline{N}(r,0;G \mid\geq 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G), \end{aligned}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' where $F(F-1) \neq 0$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Lemma 2.3. *[1] Let F and G be non-constant meromorphic functions and let F, G share 1 IM. Then,*

$$\overline{N}(r,1;F) + \overline{N}(r,1;G) - N_E^{1)}(r,1;F) - \frac{1}{2}\overline{N}_*(r,1;F,G) \leq \frac{1}{2}[N(r,1;F) + N(r,1;G)].$$

Lemma 2.4. *[14] Let f be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum\limits_{k=0}^{n} a_k f^k}{\sum\limits_{j=0}^{m} b_j f^j},$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r,R(f)) = dT(r,f) + S(r,f),$$

where $d = \max\{n,m\}$.

Lemma 2.5. (see Theorem 1.14, [17]) Let $f(z)$, $g(z) \in M(\mathbb{C})$ and let $\rho(f)$, and $\rho(g)$ be the order of f and g, respectively. Then

$$\rho(f.g) \leq \max\{\rho(f), \rho(g)\}.$$

Lemma 2.6. *Let $F = -\frac{f^n}{af^{2m}+bf^m+c}$ and $G = -\frac{\mathcal{L}^n}{a_1\mathcal{L}^{2m}+b_1\mathcal{L}^m+c_1}$, then for $n > 4$, $FG \neq 1$.*

Proof. If not let us assume,

$$\frac{f^n}{af^{2m}+bf^m+c} \cdot \frac{\mathcal{L}^n}{a_1\mathcal{L}^{2m}+b_1\mathcal{L}^m+c_1} = 1,$$

we have from *Lemma 2.4*,

$$T(r,f) + S(r,f) = T(r,\mathcal{L}) + S(r,\mathcal{L}), \tag{2.1}$$

and clearly $S(r,f) = S(r,\mathcal{L})$.

Now we have

$$\frac{\mathcal{L}^n}{a_1\mathcal{L}^{2m}+b_1\mathcal{L}^m+c_1} = \frac{af^{2m}+bf^m+c}{f^n}.$$

From the given condition $b^2 = 4ac$, $b_1^2 = 4a_1c_1$ we have $af^{2m} + bf^m + c = a(f^m + \delta)^2 = a\prod_{i=1}^{m}(f - \delta_i)^2$ and $a_1\mathcal{L}^{2m} + b_1\mathcal{L}^m + c_1 = a_1(\mathcal{L}^m + \eta)^2 = a_1\prod_{i=1}^{m}(\mathcal{L} - \eta_i)^2$ for some δ_i, η_i are distinct roots of the equation $aw^{2m} + bw^m + c$ and $a_1w^{2m} + b_1w^m + c_1$ respectively.

Using the Second Fundamental Theorem and (2.1), we have,

$$\begin{aligned} mT(r,f) &\leq \sum_{i=1}^{m}\overline{N}(r,\delta_i;f) + \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leq \frac{2m}{n}T(r,f) + \overline{N}(r,\infty;\mathcal{L}) + \sum_{i=1}^{m}\overline{N}(r,\eta_i;\mathcal{L}) + S(r,f) \\ &\leq \frac{4m}{n}T(r,f) + S(r,f), \end{aligned}$$

for $n > 4$ we arrived at a contradiction.

□

Lemma 2.7. *[11] Let $\mathcal{L}$ be a non-constant L-function and $\alpha \in \mathbb{C}$. Then $\mathcal{L} - \alpha$ has infinitely many zeros in $\mathbb{C}$.*

Lemma 2.8. *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be two non-constant L-functions, satisfy same functional equation and α, β be two non-zero constants in $\mathbb{C}$ and $\alpha\mathcal{L}_1^n + \beta\mathcal{L}_2^n \neq 0$. There exist a large positive $k_0 > 0$ such that $\mathcal{L}_1$, $\mathcal{L}_2$, $\alpha\mathcal{L}_1^n + \beta\mathcal{L}_2^n$ and $\prod_{j=1}^{K}(\Gamma(\lambda_j s + \nu_j))^{-1}$ have the same zeros (irrespective of their multiplicities) in the region $\{s \in \mathbb{C} : Re(s) < -k_0, \text{and } |Im(s)| < k\}$. Here $k, k_0 > 0$ is a large constant such that no one of the three functions $\mathcal{L}_1$, $\mathcal{L}_2$ and $\alpha\mathcal{L}_1^n + \beta\mathcal{L}_2^n$ vanishes in the the positive half plane $Re(s) > k_0$.*

Proof.

$$\mathcal{L}_i(s) = \chi(s)\overline{\mathcal{L}_i(1-\bar{s})}, \quad i = 1, 2 \tag{2.2}$$

where $\chi(s) = \frac{\omega Q^{1-2s}\prod_{j=1}^{K}\Gamma(\lambda_j(1-s)+\bar{\nu}_j)}{\prod_{j=1}^{K}\Gamma(\lambda_j s+\nu_j)}$.

Now clearly from (2.2) we have,

$$\mathcal{L}_i^n = (\chi(s))^n(\overline{\mathcal{L}_i(1-\bar{s})})^n, \quad \text{for } i = 1, 2.$$

Now from above we have

$$\begin{aligned} \alpha\mathcal{L}_1^n(s) + \beta\mathcal{L}_2^n(s) &= (\chi(s))^n(\alpha\overline{\mathcal{L}_1(1-\bar{s})}^n + \beta\overline{\mathcal{L}_2(1-\bar{s})}^n) \\ &= \alpha\cdot(\chi(s))^n\left(\overline{\mathcal{L}_1(1-\bar{s})}^n + \frac{\beta}{\alpha}\overline{\mathcal{L}_2(1-\bar{s})}^n\right) \\ &= \alpha\cdot(\chi(s))^n\prod_{i=1}^{n}\left(\overline{\mathcal{L}_1(1-\bar{s})} + x_i\overline{\mathcal{L}_2(1-\bar{s})}\right) \\ &= \alpha\cdot\frac{\omega^n Q^{n(1-2s)}\left(\prod_{j=1}^{K}\Gamma(\lambda_j(1-s)+\bar{\nu}_j)\right)^n}{\left(\prod_{j=1}^{K}\Gamma(\lambda_j s+\nu_j)\right)^n}\prod_{i=1}^{n}\overline{(\mathcal{L}_1(1-\bar{s}) + \overline{x_i}\mathcal{L}_2(1-\bar{s}))}, \end{aligned} \tag{2.3}$$

where x_i, $1 \leq i \leq n$ be distinct roots of the equation $x^n + \beta/\alpha$.

Here each of the $\mathcal{L}_1 + \overline{x_i}\mathcal{L}_2$ is a convergent Dirichlet series and hence zero free in some positive half plane. So it is possible to find a large $k_0 > 0$ such that no one of $\mathcal{L}_1(1-\bar{s}) + \overline{x_i}\mathcal{L}_2(1-\bar{s})$, $1 \leq i \leq n$ and $\mathcal{L}_k(1-\bar{s})$, $k = 1, 2$ vanishes in the half plane $Re(s) < -k_0$. In this negative half plane $Re(s) < -k_0$ the zeros of $\alpha\mathcal{L}_1^n(s) + \beta\mathcal{L}_2^n(s)$ and $\mathcal{L}_1(s)$, $\mathcal{L}_2(s)$ actually come from the zeros of $\chi(s)$ which are actually the poles of the Gamma factors in their functional equation, i.e., in this negative half plane the zeros of $\alpha\mathcal{L}_1^n(s) + \beta\mathcal{L}_2^n(s)$ are actually zeros of $\prod_{j=1}^{K}(\Gamma(\lambda_j s + \nu_j))^{-1}$ which are also trivial zeros of $\mathcal{L}_1(s)$, $\mathcal{L}_2(s)$. Now it is possible to find a positive large k such that $\prod_{j=1}^{K}\Gamma(\lambda_j s + \nu_j)$ has no pole in $|Im(s)| > k$.

Therefore in $\{s : Re(s) < -k_0, |Im(s)| < k\}$ $\mathcal{L}_1(s)$, $\mathcal{L}_2(s)$ and $\alpha\mathcal{L}_1^n(s) + \beta\mathcal{L}_2^n(s)$ and $\chi(s)$ have same zeros (irrespective of their multiplicities) which are actually poles of $\prod_{j=1}^{K}\Gamma(\lambda_j s + \nu_j)$. □

Lemma 2.9. *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be two non-constant L-functions, satisfy same functional equation and Q_1, Q_2 be any two arbitrary polynomials over $\mathbb{C}$, of degree n, then $Q_1(\mathcal{L}_1) = Q_2(\mathcal{L}_2) \implies \mathcal{L}_1 = \mathcal{L}_2$ and $Q_1 = Q_2$.*

Proof. Let us assume the polynomials $Q_1(z) = a_0z^n + a_1z^{n-1} + a_2z^{n-2} + \ldots + a_{n-1}z + a_n$ and $Q_2(z) = b_0z^n + b_1z^{n-1} + b_2z^{n-2} + \ldots + b_{n-1}z + b_n$, where a_i, b_i $i = 0, 1, \ldots, n$ and b_i, a_i are constants in $\mathbb{C}$ and $a_0 \cdot b_0 \neq 0$. Now from the given condition we have,

$$\begin{aligned} a_0\mathcal{L}_1{}^n + a_1\mathcal{L}_1{}^{n-1} + a_2\mathcal{L}_1{}^{n-2} + \ldots + a_{n-1}\mathcal{L}_1 + a_n &= b_0\mathcal{L}_2{}^n + b_1\mathcal{L}_2{}^{n-1} + b_2\mathcal{L}_2{}^{n-2} \\ &\quad + \ldots + b_{n-1}\mathcal{L}_2 + b_n. \end{aligned} \tag{2.4}$$

$\mathcal{L}_1$ and $\mathcal{L}_2$ satisfy same functional type equation and hence they have same set of trivial zeros in the negative half plane. Now let s_0 be a trivial zero of $\mathcal{L}_1$, $\mathcal{L}_2$, then from (2.4) we have $a_n = b_n$.

Now from above we get,

$$(a_0\mathcal{L}_1^n - b_0\mathcal{L}_2^n) + (a_1\mathcal{L}_1^{n-1} - b_1\mathcal{L}_2^{n-1}) + \ldots + (a_{n-2}\mathcal{L}_1^2 - b_{n-2}\mathcal{L}_2^2) + (a_{n-1}\mathcal{L}_1 - b_{n-1}\mathcal{L}_2) = 0$$

$$(a_0\mathcal{L}_1^n - b_0\mathcal{L}_2^n) + (a_1\mathcal{L}_1^{n-1} - b_0\mathcal{L}_2^{n-1}) + \ldots + (a_{n-2}\mathcal{L}_1^2 - b_{n-2}\mathcal{L}_2^2) = -(a_{n-1}\mathcal{L}_1 - b_{n-1}\mathcal{L}_2). \tag{2.5}$$

Now let for some large k, in $Re(s) < -k$, s_1 be a trivial zero of $\mathcal{L}_1$ and $\mathcal{L}_2$ of order p, then from *Lemma 2.8*, s_1 will be also a zero of $a_{n-i}\mathcal{L}_1^i - b_{n-i}\mathcal{L}_2^i$ of order ip, $1 \leq i \leq n$. Now clearly from the above (2.5) we have s_1 is the zero of the left side of multiplicity at least $2p$, whereas the right part has multiplicity p, hence a contradiction.

Therefore we must have, $a_{n-1}\mathcal{L}_1 - b_{n-1}\mathcal{L}_2 = 0$, and letting $\sigma \longrightarrow +\infty$, we have $a_{n-1} = b_{n-1}$. Now doing similarly we will have, $a_{n-i} = b_{n-i}$, for $i = 1, 2, \ldots, n-1$ and finally we will have $a_0\mathcal{L}_1^n = b_0\mathcal{L}_2^n$ and $a_0 = b_0$ and then from $\mathcal{L}_1^n = \mathcal{L}_2^n$, we have $\mathcal{L}_1 = \mathcal{L}_2$.

□

3. Proofs of the theorems

Proof. [Proof of Theorem 1.15] As it is given to us that, $E_{\mathcal{L}_1}(S_1) = E_{\mathcal{L}_2}(S_2)$, where $S_1 = \{\alpha_i;\ 1 \leq i \leq n\}$, $S_2 = \{\beta_i;\ 1 \leq i \leq n\}$ and $\#(S_1) = \#(S_2)$. Therefore here $\prod_{i=1}^n(\mathcal{L}_1 - \alpha_i)$ and $\prod_{i=1}^n(\mathcal{L}_2 - \beta_i)$ share 0 CM.

We can write it as

$$\mathcal{G}_0 = \frac{(\mathcal{L}_1 - \alpha_1)(\mathcal{L}_1 - \alpha_2)\ldots(\mathcal{L}_1 - \alpha_n)}{(\mathcal{L}_2 - \beta_1)(\mathcal{L}_2 - \beta_2)\ldots(\mathcal{L}_2 - \beta_n)} = (s-1)^k e^p, \tag{3.1}$$

for some integer k. Since here the non-zero degree L-function is of order one then from *Lemma 2.5*, p is a polynomial of degree at most one. Let us consider $p(s) = as + b$, where a, b are some complex constants. Next we will show $(s-1)^k e^{as+b} = c$, for some constant $c \in \mathbb{C}^*$. To do this let us consider the following cases.

Case-1. First let us assume $1 \notin S = S_1 \cup S_2$.

Now taking $\sigma \longrightarrow +\infty$ we get from (3.1)

$$\lim_{\sigma \longrightarrow +\infty} \frac{(\mathcal{L}_1 - \alpha_1)(\mathcal{L}_1 - \alpha_2)\ldots(\mathcal{L}_1 - \alpha_n)}{(\mathcal{L}_2 - \beta_1)(\mathcal{L}_2 - \beta_2)\ldots(\mathcal{L}_2 - \beta_n)} = \frac{(1-\alpha_1)\ldots(1-\alpha_n)}{(1-\beta_1)\ldots(1-\beta_n)} = \lim_{\sigma \longrightarrow +\infty} (s-1)^k e^{as+b},$$

Now $\lim_{\sigma \longrightarrow +\infty}(s-1)^k e^{as+b} = \infty$ or 0 according as $Re(a) >$ or < 0, so we must have $Re(a) = 0$. Also the limit is independent of t, so it can be shown that $a = 0$. Similarly we will have $k = 0$ otherwise $\lim_{\sigma \longrightarrow +\infty}(s-1)^k e^b = \infty$ or 0 according as $k > 0$ or < 0. Hence we have,

$$\lim_{\sigma \longrightarrow +\infty} \frac{(\mathcal{L}_1 - \alpha_1)(\mathcal{L}_1 - \alpha_2)\ldots(\mathcal{L}_1 - \alpha_n)}{(\mathcal{L}_2 - \beta_1)(\mathcal{L}_2 - \beta_2)\ldots(\mathcal{L}_2 - \beta_n)} = \frac{(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_n)}{(1-\beta_1)(1-\beta_2)\ldots(1-\beta_n)} = \lim_{\sigma \longrightarrow +\infty} e^b,$$

hence we have $e^b = \frac{(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_n)}{(1-\beta_1)(1-\beta_2)\ldots(1-\beta_n)} = C$ (constant).

Therefore we have,

$$(\mathcal{L}_1 - \alpha_1)(\mathcal{L}_1 - \alpha_2)\ldots(\mathcal{L}_1 - \alpha_n) = C(\mathcal{L}_2 - \beta_1)(\mathcal{L}_2 - \beta_2)\ldots(\mathcal{L}_2 - \beta_n).$$

Now using *Lemma 2.9* we will have $\mathcal{L}_1 = \mathcal{L}_2$ and also we will get $S_1 = S_2$.

Case-2. $1 \in S = S_1 \cup S_2$ but $0 \notin S$. Therefore at least one of S_i, $i = 1, 2$ contains 1.

Case-2.1. Let us assume 1 lie in both S_i for $i = 1, 2$. Without loss of generality let us assume $\alpha_1 = \beta_1 = 1$.

Again $\mathcal{L}_i$ can be represented by a Dirichlet series, i.e., $\mathcal{L}_i(s) = \sum_{n=1}^{\infty} \frac{a_i(n)}{n^s}$, $i = 1, 2$, absolutely convergent for $\sigma > 1$ where $a_i(1) = 1$. Also let n_1, n_2 be two integers such that $n_i = \min\{n\ (\geq 2) : a_i(n) \neq 0,\ i = 1, 2\}$.

Now we clearly have,

$$\begin{aligned}\frac{\mathcal{L}_1-1}{\mathcal{L}_2-1} &= \frac{\frac{1}{n_1^s}\left(a_1(n_1)+\sum_{n>n_1}^{\infty}a_1(n)(\frac{n_1}{n})^s\right)}{\frac{1}{n_2^s}\left(a_2(n_2)+\sum_{n>n_2}^{\infty}a_2(n)(\frac{n_2}{n})^s\right)} \\ &= \left(\frac{n_2}{n_1}\right)^s G(s),\end{aligned} \tag{3.2}$$

where,

$$G(s)=\frac{a_1(n_1)+\sum_{n>n_1}^{\infty}a_1(n)(\frac{n_1}{n})^s}{a_2(n_2)+\sum_{n>n_2}^{\infty}a_2(n)(\frac{n_2}{n})^s}.$$

By the construction of $G(s)$ we note that $\lim_{\sigma\longrightarrow+\infty}G(s)=\frac{a_1(n_1)}{a_2(n_2)}$.
Then by (3.2) we have,

$$G(s)=\frac{a_1(n_1)+\sum_{n>n_1}^{\infty}a_1(n)(\frac{n_1}{n})^s}{a_2(n_2)+\sum_{n>n_b}^{\infty}a_2(n)(\frac{n_2}{n})^s}=\left(\frac{n_1}{n_2}\right)^s\cdot\frac{\mathcal{L}_1-1}{\mathcal{L}_2-1}. \tag{3.3}$$

In view of (3.1), let us consider the following function

$$\begin{aligned}\mathcal{G}_1 &= G(s)\cdot\frac{(\mathcal{L}_1-\alpha_2)\dots(\mathcal{L}_1-\alpha_n)}{(\mathcal{L}_2-\beta_2)\dots(\mathcal{L}_2-\beta_n)} \\ &= \frac{\mathcal{L}_1-1}{\mathcal{L}_2-1}\cdot q^s\cdot\frac{(\mathcal{L}_1-\alpha_2)\dots(\mathcal{L}_1-\alpha_n)}{(\mathcal{L}_2-\beta_2)\dots(\mathcal{L}_2-\beta_n)} \\ &= q^s\frac{(\mathcal{L}_1-1)(\mathcal{L}_1-\alpha_2)\dots(\mathcal{L}_1-\alpha_n)}{(\mathcal{L}_2-1)(\mathcal{L}_2-\beta_2)\dots(\mathcal{L}_2-\beta_n)} \\ &= q^s\mathcal{G}_0=q^s(s-1)^k e^{as+b},\end{aligned} \tag{3.4}$$

for some $q=\frac{n_1}{n_2}(\in\mathbb{Q}^+)$. We can write $q=e^{\log q}=e^{q'}$, then we can write it as, $q^s(s-1)^k e^{as+b}=(s-1)^k e^{(q'+a)s+\hat{b}}=(s-1)^k e^{a's+\hat{b}}$ where $a'=q'+a$. Let us consider $a'=a_1+ia_2$ and $b=b_1+ib_2$.

With respect to the first equality of (3.4), taking limit $\sigma\longrightarrow+\infty$, we have $\lim_{\sigma\longrightarrow+\infty}\mathcal{G}_1=C_1$, for some constant in $C_1\in\mathbb{C}^*$. Next from the second and last equality of (3.4), taking limit $\sigma\longrightarrow+\infty$, we have

$$\begin{aligned}\lim_{\sigma\longrightarrow+\infty}\left|q^s\frac{(\mathcal{L}_1-1)}{(\mathcal{L}_2-1)}\cdot\frac{(\mathcal{L}_1-\alpha_2)\dots(\mathcal{L}_1-\alpha_n)}{(\mathcal{L}_2-\beta_2)\dots(\mathcal{L}_2-\beta_n)}\right|=|C_1| &= \lim_{\sigma\longrightarrow+\infty}|(s-1)^k e^{a's+\hat{b}}| \\ =\text{constant} &= \lim_{\sigma\longrightarrow+\infty}|\sigma-1+it|^k e^{a_1\sigma-a_2t+b_1}.\end{aligned}$$

Therefore we must have $a_1=0=k$, other wise $\lim_{\sigma\longrightarrow+\infty}|\sigma-1+it|^k e^{a_1\sigma-a_2t+b_1}=\infty$ or 0 according as $a_1>$ or <0 and with the same argument it can be shown that $k=0$.

Also,

$$\lim_{\sigma\longrightarrow+\infty}e^{-a_2t+b_1}=\text{constant},\quad\forall t\in\mathbb{R},$$

implies $a_2=0$. Hence we have $a=a_1+ia_2=0$ and $k=0$.
Therefore, $\mathcal{G}_1=e^b$ and from the last equality of (3.4) $\mathcal{G}_0=q^{-s}e^b$, i.e.,

$$\frac{(\mathcal{L}_1-1)}{(\mathcal{L}_2-1)}\cdot\frac{(\mathcal{L}_1-\alpha_2)\dots(\mathcal{L}_1-\beta_n)}{(\mathcal{L}_2-\beta_2)\dots(\mathcal{L}_2-\beta_n)}=q^{-s}e^b. \tag{3.5}$$

From *Lemma 2.8*, we know for some sufficiently large constant $\kappa(>0)$ and large $\kappa_0(>0)$, $\mathcal{L}_1$, $\mathcal{L}_2$ and the Gamma function $\prod_{j=1}^K(\Gamma(\lambda_j s+\nu_j))^{-1}$, in their functional equation have same zeros in the region $Re\ s<-\kappa_0$ and $|Im\ s|<\kappa$ and have no zero in $Re\ s\geq\kappa_0$.

Again Gamma function $\Gamma(s)$ has simple poles at $s = -1, -2, \ldots, -n, \ldots$. Therefore $\prod_{j=1}^{K}(\Gamma(\lambda_j s + \nu_j))^{-1}$ has zeros at $s = \frac{-n-\nu_j}{\lambda_j}$ for $n = 1, 2, \ldots$ and $j = 1, 2, \ldots, K$. Since it is assumed that $d_{\mathcal{L}} > 0$, we must have $K > 0$. Here for some j or fixing some j it is possible to get a infinite sequence $\left\{s_n = \frac{-N-\nu_j}{\lambda_j}\right\}_{n=1}^{\infty}$ of common zeros of $\mathcal{L}_1$, $\mathcal{L}_2$ and $\prod_{j=1}^{K}(\Gamma(\lambda_j s + \nu_j))^{-1}$ in $Re\ s < -\kappa_0$ and $|Im\ s| < \kappa$, where $N = n +$ large constant. Since here $Re(s_n) = \frac{-N-Re(\nu_j)}{\lambda_j} < -\kappa_0 < 0$, we have $n \longrightarrow \infty \implies N \longrightarrow \infty$ and $Re(s_n) \longrightarrow -\infty$, it implies $-Re(s_n) \longrightarrow +\infty$.

From (3.5) we have,

$$\left|\frac{(\mathcal{L}_1(s_n)-1)(\mathcal{L}_1(s_n)-\alpha_2)\ldots(\mathcal{L}_1(s_n)-\alpha_n)}{(\mathcal{L}_2(s_n)-1)(\mathcal{L}_2(s_n)-\alpha_2)\ldots(\mathcal{L}_2(s_n)-\alpha_n)}\right| = \frac{|\alpha_1\alpha_2\ldots\alpha_n|}{|\beta_1\beta_2\ldots\beta_n|} = e^{Re(b)}q^{-Re(s_n)}. \tag{3.6}$$

So from (3.6) letting $n \longrightarrow +\infty$ we get

$$\lim_{n\longrightarrow+\infty} e^{Re(b)}q^{-Re(s_n)} = \text{non-zero constant},$$

but $e^{Re(b)}q^{-Re(s_n)} \longrightarrow +\infty$ or 0 as $n \longrightarrow \infty$ according as $q > 1$ or < 1 but $\{e^{Re(b)}q^{Re(-s_n)}\}_{n=1}^{\infty}$ is a constant sequence. Hence we must have $q = 1$ and then $e^b = \frac{\alpha_1\alpha_2\ldots\alpha_n}{\beta_1\beta_2\ldots\beta_n} = C_2$, and then from (3.5), with the help of *Lemma 2.9*, we will have $\mathcal{L}_1 = \mathcal{L}_2$.

Sub case-2.2. Now let exactly one of $S_i,\ i = 1, 2$ contains 1, say S_1 and $\alpha_1 = 1$. Then following the same procedure as done in (3.2) we can have, $\mathcal{L}_1 - 1 = \frac{1}{n_1^s}\left(a_1(n_1) + \sum_{n>n_1}^{\infty} a_1(n)(\frac{n_1}{n})^s\right) = \frac{1}{n_1^s}G_1(s)$, where $G_1(s) = a_1(n_1) + \sum_{n>n_1}^{\infty} a_1(n)(\frac{n_1}{n})^s$ and $\lim_{\sigma\longrightarrow+\infty} G_1 = a_1(n_1)$.

Now, $\mathcal{G}_0 = \frac{(\mathcal{L}_1-1)(\mathcal{L}_1-\alpha_2)\ldots(\mathcal{L}_1-\alpha_n)}{(\mathcal{L}_2-\beta_1)(\mathcal{L}_2-\beta_2)\ldots(\mathcal{L}_2-\beta_n)} = e^{as+b}$.

Let us set a function

$$\begin{aligned}\mathcal{G}_2 = n_1^s\mathcal{G}_0 &= n_1^s\frac{(\mathcal{L}_1-1)(\mathcal{L}_1-\alpha_2)\ldots(\mathcal{L}_1-\alpha_n)}{(\mathcal{L}_2-\beta_1)(\mathcal{L}_2-\beta_2)\ldots(\mathcal{L}_2-\beta_n)}\\ &= G_1\frac{(\mathcal{L}_1-\alpha_2)\ldots(\mathcal{L}_1-\alpha_n)}{(\mathcal{L}_2-\beta_1)(\mathcal{L}_2-\beta_2)\ldots(\mathcal{L}_2-\beta_n)} = (s-1)^k n_1^s e^{as+b},\end{aligned}$$

therefore we can write, $\mathcal{G}_2 = (s-1)^k e^{a''s}e^b$, where $a'' = a + \log n_1$.

Now $\lim_{\sigma\longrightarrow+\infty}\mathcal{G}_2 =$ constant but $\lim_{\sigma\longrightarrow+\infty} e^{a''s+b}(s-1)^k = 0$ or ∞ according as $Re(a'') <$ or > 0. Since the limit is independent of t, we will have $a'' = 0$. With similar arguments we will have $k = 0$. Therefore we will have,

$$\begin{aligned}\mathcal{G}_2 = e^b \implies \mathcal{G}_0 &= n_1^{-s}e^b\\ i.e., \frac{(\mathcal{L}_1-1)(\mathcal{L}_1-\alpha_2)\ldots(\mathcal{L}_1-\alpha_n)}{(\mathcal{L}_2-\beta_1)(\mathcal{L}_2-\beta_2)\ldots(\mathcal{L}_2-\beta_n)} &= n_1^{-s}e^b.\end{aligned}$$

Proceeding similarly as in (3.6) we will have, $n_1 = 1$ and then with the help of *Lemma 2.9* we will have $\mathcal{L}_1 = \mathcal{L}_2$.

Case-3. Let us assume $1, 0 \in S = S_1 \cup S_2$. Now since $\mathcal{L}_1$, $\mathcal{L}_2$ satisfy same functional equation then clearly $0 \in S_1 \implies 0 \in S_2$, also $0 \in S_2 \implies 0 \in S_1$. Therefore at least one of $S_i,\ i = 1, 2$ contains both $1, 0$.

Sub case-3.1. First assume $0, 1 \in S_1 \cap S_2$. Without loss of generality let us assume $\alpha_1 = 1 = \beta_1,\ \alpha_2 = 0 = \beta_2$. Now dealing exactly in the same way as done in **Case-2.1** we will have,

$$\frac{(\mathcal{L}_1-1)}{(\mathcal{L}_2-1)}\cdot\frac{\mathcal{L}_1(\mathcal{L}_1-\alpha_3)\ldots(\mathcal{L}_1-\beta_n)}{\mathcal{L}_2(\mathcal{L}_2-\beta_3)\ldots(\mathcal{L}_2-\beta_n)} = \hat{q}^{-s}e^b, \tag{3.7}$$

for some $\hat{q} \in \mathbb{Q}^+$.

On the other hand, by noting that $\mathcal{L}_1, \mathcal{L}_2$ satisfy same functional equation with degree equal to $d_{\mathcal{L}}$ ($\neq 0$), therefore we have

$$\frac{\mathcal{L}_1(s)}{\mathcal{L}_2(s)} = \frac{\overline{\mathcal{L}_1(1-\bar{s})}}{\overline{\mathcal{L}_2(1-\bar{s})}},$$

using this in (3.7) we have,

$$\frac{\overline{\mathcal{L}_1(1-\bar{s})}}{\overline{\mathcal{L}_2(1-\bar{s})}} \cdot \frac{(\mathcal{L}_1-1)(\mathcal{L}_1-\alpha_3)\dots(\mathcal{L}_1-\beta_n)}{(\mathcal{L}_2-1)(\mathcal{L}_2-\beta_3)\dots(\mathcal{L}_2-\beta_n)} = \hat{q}^{-s}e^{b}. \tag{3.8}$$

Using the sequence of trivial zeros $\{s_n\}_{n=1}^{\infty}$ in some negative half plane $Re(s) < -k_0$, putting $s = s_n$ we have from (3.8) we have,

$$\left|\frac{\mathcal{L}_1(1-\bar{s}_n)}{\mathcal{L}_2(1-\bar{s}_n)}\right| \cdot \frac{|\alpha_3\alpha_4\dots\alpha_n|}{|\beta_3\beta_4\dots\beta_n|} = \hat{q}^{-Re(s_n)}e^{Re(b)}.$$

Letting $n \longrightarrow +\infty$, we have from above, $\lim_{n\longrightarrow+\infty}\hat{q}^{-Re(s_n)}e^{Re(b)}$ = non-zero constant, but $\hat{q}^{-Re(s_n)} \longrightarrow \infty$ or 0 as $n \longrightarrow \infty$ according as $\hat{q} > 1$ or < 1, hence we must have $\hat{q} = 1$.

From (3.7) we have,

$$\mathcal{L}_1(\mathcal{L}_1-1)(\mathcal{L}_1-\alpha_3)\dots(\mathcal{L}_1-\beta_n) = e^{b}\mathcal{L}_2(\mathcal{L}_2-1)(\mathcal{L}_2-\beta_3)\dots(\mathcal{L}_2-\beta_n),$$

and then from *Lemma 2.9* we have, $\mathcal{L}_1 = \mathcal{L}_2$.

Case-3.2. Let us assume exactly one of S_i, $i = 1, 2$ contain 1, say S_1. So, here $0 \in S$ and $1 \in S_1$. Now proceeding in same manner as done in **Sub case-2.2.**, **Sub case-3.1.** we will have $\mathcal{L}_1 = \mathcal{L}_2$.

□

Proof. [Proof of Theorem 1.17] Here, it is given that $\overline{E}_{\mathcal{L}_1}(S_1) = \overline{E}_{\mathcal{L}_2}(S_2)$. First let us consider $\mathcal{L}_1 \neq \mathcal{L}_2$ and two non-constant meromorphic functions as follows,

$$F_1 = P_1(\mathcal{L}_1) \quad and \quad G_1 = P_2(\mathcal{L}_2),$$

P_1, P_2 are given same as in (1.3), (1.4). Clearly here F_1 and G_1 share 0 IM.

Now we know L-function can have only one pole at $z = 1$, hence $\overline{N}(r,\infty;\mathcal{L}_1) = \overline{N}(r,\infty;\mathcal{L}_2) = O(\log r)$. Again, $F_1' = (P_1(\mathcal{L}_1))' = P_1'(\mathcal{L}_1)\mathcal{L}_1'$, $G_1' = P_2'(\mathcal{L}_2)\mathcal{L}_2'$. Also $T(r,F_1) = nT(r,\mathcal{L}_1) + O(\log r)$ and $T(r,G_1) = nT(r,\mathcal{L}_2) + O(\log r)$.

Next let α_{i_1}, $i = 1, 2, \dots, k_1$ be the distinct zeros of $P_1'(z)$ and α_{i_2}, $i = 1, 2, \dots, k_2$ be the distinct zeros of $P_2'(z)$.

First let us take $F - 1 = F_1$ and $G - 1 = G_1$ in H and then consider $H \neq 0$. Now using the same method as adopted in from *Lemma 2.2* we can have,

$$\begin{aligned} N(r,\infty;H) \quad &\leq \quad \overline{N}_*(r,0;F_1,G_1) + \sum_{i=1}^{k_1}\overline{N}(r,\alpha_{i_1};\mathcal{L}_1) + \sum_{i=1}^{k_2}\overline{N}(r,\alpha_{i_2};\mathcal{L}_2) + \overline{N}(r,\infty;\mathcal{L}_1) \\ &\quad + \overline{N}(r,\infty;\mathcal{L}_2) + \overline{N}_{o_1}(r,0;\mathcal{L}_1') + \overline{N}_{o_2}(r,0;\mathcal{L}_2') + O(\log r), \end{aligned} \tag{3.9}$$

where $\overline{N}_{o_1}(r,0;\mathcal{L}_1')$ is the reduced counting function of those zeros of $\mathcal{L}_1'$ which are not zeros of $P_1(\mathcal{L}_1)\prod_{i=1}^{k_1}(\mathcal{L}-\alpha_{i_1})$, similarly $\overline{N}_{o_2}(r,0;\mathcal{L}_2')$ can be defined.

Applying the Second Fundamental Theorem to $\mathcal{L}_1$ and $\mathcal{L}_2$ we have,

$$\begin{aligned} &(n-1)(T(r,\mathcal{L}_1) + T(r,\mathcal{L}_2)) + k_1T(r,\mathcal{L}_1) + k_2T(r,\mathcal{L}_2) \\ \leq \quad &\overline{N}(r,\infty;\mathcal{L}_1) + \overline{N}(r,\infty;\mathcal{L}_2) + \overline{N}(r,0;F_1) + \overline{N}(r,0;G_1) + \sum_{i=1}^{k_1}\overline{N}(r,\alpha_{i_1};\mathcal{L}_1) + \sum_{i=1}^{k_2}\overline{N}(r,\alpha_{i_2};\mathcal{L}_2) \\ &-N_{o_1}(r,0;\mathcal{L}_1') - N_{o_2}(r,0;\mathcal{L}_2') + S(r,\mathcal{L}_1) + S(r,\mathcal{L}_2). \end{aligned}$$

i.e.,

$$\begin{aligned}(n-1)T(r) &\leq \overline{N}(r,0;F_1)+\overline{N}(r,0;G_1)-N_{o_1}(r,0;\mathcal{L}_1') \\ &\quad -N_{o_2}(r,0;\mathcal{L}_2')+S(r),\end{aligned} \tag{3.10}$$

where $T(r)=T(r,\mathcal{L}_1)+T(r,\mathcal{L}_2)$ and $S(r)=S(r,\mathcal{L}_1)+S(r,\mathcal{L}_2)=O(\log r)$.

Applying *Lemmas 2.3, 2.4* from (3.10) we have,

$$\begin{aligned}(n-1)T(r) &\leq N_E^{1)}(r,0;F_1)+\frac{1}{2}\overline{N}_*(r,0;F_1,G_1)+\frac{1}{2}[N(r,0;F_1)+N(r,0;G_1)] \\ &\quad -N_{o_1}(r,0;\mathcal{L}_1')-N_{o_2}(r,0;\mathcal{L}_2')+S(r) \\ &\leq N(r,\infty;H)+\frac{1}{2}\overline{N}_*(r,0;F_1,G_1)+\frac{1}{2}(T(r,F_1)+T(r,G_1))-N_{o_1}(r,0;\mathcal{L}_1') \\ &\quad -N_{o_2}(r,0;\mathcal{L}_2')+O(\log r) \\ &\leq (n/2+k)T(r)+\frac{3}{2}\overline{N}_*(r,0;F_1,G_1)+O(\log r) \\ &\leq (n/2+k)T(r)+\frac{3}{2}(\overline{N}_L(r,0;F_1)+\overline{N}_L(r,0;G_1))+O(\log r) \\ &\leq (n/2+k)T(r)+\frac{3}{2}(\overline{N}(r,0;F_1\mid\geq 2)+\overline{N}(r,0;G_1\mid\geq 2))+O(\log r) \\ &\leq (n/2+k)T(r)+\frac{3}{2}(\overline{N}(r,0;\mathcal{L}_1'\mid F_1=1)+\overline{N}(r,0;\mathcal{L}_2'\mid G_1=1))+O(\log r) \\ &\leq (n/2+k)T(r)+\frac{3}{2}(\overline{N}(r,0;\mathcal{L}_1)+\overline{N}(r,0;\mathcal{L}_2))+O(\log r),\end{aligned}$$

for $n>2k+5$, where, $k=max\{k_1,k_2\}$, hence we arrive at a contradiction.

Therefore, $H=0$. Then integrating we have,

$$\frac{1}{P_1(\mathcal{L}_1)}=\frac{A}{P_2(\mathcal{L}_2)}+B, \tag{3.11}$$

where $A(\neq 0), B$ are two constants.

Next we will show that $B=0$. If $B\neq 0$ then we have from (3.11),

$$\frac{P_2(\mathcal{L}_2)}{P_1(\mathcal{L}_1)}=A+BP_2(\mathcal{L}_2),$$

and here now $P_1(\mathcal{L}_1)$, $P_2(\mathcal{L}_2)$ share 0 CM, then the zeros of $P_2(\mathcal{L}_2)+A/B$ can not be zeros of $P_2(\mathcal{L}_2)$. Hence the zeros of $P_2(\mathcal{L}_2)+A/B$ will be poles of $P_1(\mathcal{L}_1)$ i.e., $\overline{N}(r,0;A/B+P_2(\mathcal{L}_2))=O(\log r)$. This implies $\mathcal{L}_2$ has some generalized Picard exceptional value, which contradicts *Lemma 2.7*.

Therefore $B=0$ and then from (3.11), using *Lemma 2.9* we will have $\mathcal{L}_1=\mathcal{L}_2$ and $S_1=S_2$.

□

Proof. [Proof of Theorem 1.10] Let f be a non-constant meromorphic function and $\mathcal{L}$ be a non-constant L-function and $\overline{E}_f(S_P)=\overline{E}_{\mathcal{L}}(S_Q)$ and assume

$$F=-\frac{f^n}{af^{2m}+bf^m+c} \quad \text{and} \quad G=-\frac{\mathcal{L}^n}{a_1\mathcal{L}^{2m}+b_1\mathcal{L}^m+c_1}.$$

Clearly here F and G share 1 IM. First assume that $H \neq 0$. Now using the Second Fundamental Theorem and *Lemmas 2.3, 2.1, 2.2* and then *Lemma 2.4* we have,

$$\begin{aligned}
nT_1(r) + S_1(r) &= T(r,F) + T(r,G) + S(r,F) + S(r,G) \qquad (3.12)\\
&\leq \overline{N}(r,1;F) + \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;G) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G)\\
&\quad -N_0(r,0;F') - N_0(r,0;G') + S(r,F) + S(r,G)\\
&\leq \frac{n}{2}T_1(r) + \frac{1}{2}\overline{N}_*(r,1;F,G) + N_E^{1)}(r,1;F) + \overline{N}(r,0;F) + \overline{N}(r,\infty;F)\\
&\quad +\overline{N}(r,0;G) + \overline{N}(r,\infty;G) - N_0(r,0;F') - N_0(r,0;G') + S_1(r)\\
\frac{n}{2}T_1(r) &\leq N_2(r,0;F) + \overline{N}_2(r,\infty;F) + N_2(r,0;G) + \overline{N}_2(r,\infty;G) + \frac{3}{2}\overline{N}_*(r,1;F,G) + S_1(r)\\
&\leq 2\overline{N}(r,0;f) + 2\overline{N}(r,0;\mathcal{L}) + N_2(r,0;af^{2m} + bf^m + c) + N_2(r,0;a_1\mathcal{L}^{2m} + b_1\mathcal{L}^m + c_1)\\
&\quad +\frac{3}{2}(\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G)) + S_1(r)\\
&\leq (2m+2)T_1(r) + \frac{3}{2}(\overline{N}(r,0;G-1 \mid\geq 2) + \overline{N}(r,0;F-1 \mid\geq 2)) + S_1(r)\\
&\leq (2m + \frac{7}{2})T_1(r) + S_1(r),
\end{aligned}$$

where $T_1(r) = T(r,f) + T(r,\mathcal{L})$ and $S_1(r) = S(r,\mathcal{L}) + S(r,f)$. Clearly for $n > 4m+7$ we have a contradiction.

Therefore $H = 0$ and so integrating both sides we get,

$$\frac{1}{G-1} = \frac{A}{F-1} + B, \qquad (3.13)$$

where $A,\ B$ are two constants, $A \neq 0$.
Now from *Lemma 2.4* we have,

$$T(r,\mathcal{L}) = T(r,f) + O(1). \qquad (3.14)$$

Case-1. At first let $B \neq 0$.

Then

$$G - 1 = \frac{F-1}{A + B(F-1)}. \qquad (3.15)$$

Sub case-1.1. If $A - B \neq 0$ then zeros of $F + (A-B)/B$ are poles of $G-1$.

Now using the Second Fundamental Theorem we have,

$$\begin{aligned}
nT(r,f) + S(r,f) = T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;F + (A-B)/B) + S(r,F)\\
&\leq (1+m+m)T(r) + S(r),
\end{aligned}$$

a contradiction.
Sub case-1.2. When $A - B = 0$. From (3.15) we have, $G = \frac{(B+1)F-1}{BF}$ and let $B+1 \neq 0$, hence $\overline{N}(r,0;F - \frac{1}{B+1}) = \overline{N}(r,0;G)$. Using the Second Fundamental Theorem and *Lemma 2.4*, (3.14) we have,

$$\begin{aligned}
nT(r,f) + S(r,f) = T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;F - 1/(B+1))\\
&\leq (1+m+1)T(r,f) + S(r,f),
\end{aligned}$$

a contradiction. Therefore we have $B + 1 = 0$.

Therefore we have $FG = 1$. But from *Lemma 2.6* we arrive at a contradiction.
Case-2. Suppose, $B = 0$. Now we have from (3.13),

$$\begin{aligned} G-1 &= \frac{1}{A}(F-1) \\ G &= \frac{1}{A}(F+A-1), \end{aligned} \tag{3.16}$$

Sub case-2.1. Let $A-1 \neq 0$, then using the Second Fundamental Theorem we have,

$$\begin{aligned} nT(r,f)+S(r,f) = T(r,F) &\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;F+A-1) \\ &\leq (1+m+1)T(r,f)+S(r,f), \end{aligned}$$

a contradiction.
Sub case-2.2. Now for $A = 1$, we have from (3.16) we must have, $F = G$, i.e.,

$$\begin{aligned} \frac{f^n}{af^{2m}+bf^m+c} &= \frac{\mathcal{L}^n}{a_1\mathcal{L}^{2m}+b_1\mathcal{L}^m+c_1} \\ a_1f^n\mathcal{L}^{2m}+b_1f^n\mathcal{L}^m+c_1f^n &= a\mathcal{L}^nf^{2m}+b\mathcal{L}^nf^m+c\mathcal{L}^n \end{aligned} \tag{3.17}$$

i.e.,

$$\begin{aligned} (a_1f^n\mathcal{L}^{2m}-a\mathcal{L}^nf^{2m})+(b_1f^n\mathcal{L}^m-b\mathcal{L}^nf^m)+(c_1f^n-c\mathcal{L}^n) &= 0 \\ \mathcal{L}^nf^{2m}(a_1h^{n-2m}-a)+\mathcal{L}^nf^m(b_1h^{n-m}-b)+\mathcal{L}^n(c_1h^n-c) &= 0 \end{aligned}$$

$$f^{2m}(a_1h^{n-2m}-a)+f^m(b_1h^{n-m}-b)+(c_1h^n-c) = 0, \tag{3.18}$$

where $h = \frac{f}{\mathcal{L}}$.

Let us assume h is a non-constant meromorphic function, from above we get

$$f^{2m}+f^m\frac{b_1h^{n-m}-b}{a_1h^{n-2m}-a}+\frac{c_1h^n-c}{a_1h^{n-2m}-a} = 0$$

i.e.,

$$\begin{aligned} \left(f^m+\frac{b_1h^{n-m}-b}{2(a_1h^{n-2m}-a)}\right)^2 &= \frac{(b_1h^{n-m}-b)^2-4(a_1h^{n-2m}-a)(c_1h^n-c)}{4(a_1h^{n-2m}-a)^2} \\ &= \frac{ac_1h^{n-2m}(h^m-\frac{bb_1}{4ac_1})^2}{(a_1h^{n-2m}-a)^2}. \end{aligned} \tag{3.19}$$

Sub case-2.2.1.

Now if n is even, let $n = 2p$, then from above we have,

$$\begin{aligned} \left(f^m+\frac{b_1h^{n-m}-b}{2(a_1h^{n-2m}-a)}\right) &= \pm(c_1a)^{\frac{1}{2}}\frac{h^{p-m}\left(h^m-\frac{bb_1}{4ac_1}\right)}{a_1h^{n-2m}-a} \\ f^m &= -\frac{b_1h^{n-m}-b}{2(a_1h^{n-2m}-a)}\pm(c_1a)^{\frac{1}{2}}\frac{h^{p-m}\left(h^m-\frac{bb_1}{4ac_1}\right)}{a_1h^{n-2m}-a} \\ &= \frac{-(b_1h^{n-m}-b)\pm2(c_1a)^{\frac{1}{2}}h^{p-m}\left(h^m-\frac{bb_1}{4ac_1}\right)}{2(a_1h^{n-2m}-a)}. \end{aligned} \tag{3.20}$$

Clearly from above we have, $S(r,h) = S(r,f)$.

Now for each cases in (3.20), if the numerator $-(b_1h^{n-m}-b)\pm2(c_1a)^{\frac{1}{2}}h^{p-m}\left(h^m-\frac{bb_1}{4ac_1}\right)$ and the denominator $a_1h^{n-2m}-a$ have any common factor $h-x$ then x will be a zero of $-(\frac{b_1a}{a_1}w^m-b)\pm2(c_1a)^{\frac{1}{2}}w^{p-m}\left(w^m-\frac{bb_1}{4ac_1}\right)$.

Hence the numerator and denominator can have at most $p = \frac{n}{2}$ common factors. Hence $a_1h^{n-2m} - a$ has at least $n - 2m - \frac{n}{2} = \frac{n}{2} - 2m$ factors $h - \mu_i$, say $(i = 1, 2, \ldots, \frac{n}{2} - 2m)$ which are not factors of the numerator.

Again,

$$\left(\frac{n}{2} - 2m - 2\right)T(r,h) \leq \sum_{i=1}^{n/2-2m} \overline{N}(r,\mu_i;h) \leq \overline{N}(r,\infty;f) = O(\log r) = S(r,h),$$

a contradiction for $n > 4m + 4$.

Therefore h must be a constant satisfying (3.18).

Sub case-2.2.2.

Let n is odd and $n = 2p + 1$ (say). Then from (3.17) we have,

$$\begin{aligned} h^n &= \frac{af^{2m} + bf^m + c}{a_1\mathcal{L}^{2m} + b_1\mathcal{L}^m + c_1} \\ &= \frac{a\left(f^m + \frac{b}{2a}\right)^2}{a_1\left(\mathcal{L}^m + \frac{b_1}{2a_1}\right)^2} \\ h &= \frac{a_*\left(f^m + \frac{b}{2a}\right)^2}{h^{2p}\left(\mathcal{L}^m + \frac{b_1}{2a_1}\right)^2} = \kappa^2, \end{aligned} \tag{3.21}$$

where, $a_* = \frac{a}{a_1}$ and $\kappa = \frac{(a_*)^{\frac{1}{2}}(f^m + \frac{b}{2a})}{h^p(\mathcal{L}^m + \frac{b_1}{2a})}$, a meromorphic function.

Putting $h = \kappa^2$ we have from (3.19),

$$\begin{aligned} \left(f^m + \frac{b_1\kappa^{2n-2m} - b}{2(a_1\kappa^{2n-4m} - a)}\right) &= \pm (c_1a)^{\frac{1}{2}} \frac{\kappa^{n-2m}\left(\kappa^{2m} - \frac{bb_1}{4ac_1}\right)}{a_1\kappa^{2n-4m} - a} \\ \text{i.e.,}\quad f^m &= -\frac{b_1\kappa^{2n-2m} - b}{2(a_1\kappa^{2n-4m} - a)} \pm (c_1a)^{\frac{1}{2}} \frac{\kappa^{n-2m}\left(\kappa^{2m} - \frac{bb_1}{4ac_1}\right)}{a_1\kappa^{2n-4m} - a} \\ &= \frac{-(b_1\kappa^{2n-2m} - b) \pm 2(c_1a)^{\frac{1}{2}}\kappa^{n-2m}\left(\kappa^{2m} - \frac{bb_1}{4ac_1}\right)}{2(a_1\kappa^{2n-4m} - a)}. \end{aligned} \tag{3.22}$$

Now for each cases in (3.22), if the numerator $-(b_1\kappa^{2n-4m} - b) \pm 2(c_1a)^{\frac{1}{2}}a^2\kappa^{n-2m}\left(\kappa^{2m} - \frac{bb_1}{4ac_1}\right)$ and the denominator $a_1\kappa^{2n-4m} - a$ have any common factor $\kappa - y$ then y will be a zero of $-(\frac{b_1a}{a_1}z^{2m} - b) \pm 2(c_1a)^{\frac{1}{2}}a^2z^{n-2m}\left(z^{2m} - \frac{bb_1}{4ac_1}\right)$. Hence the numerator and denominator can have at most n common factors. Hence $a_1\kappa^{2n-4m} - a$ has at least $2n - 4m - n = n - 4m$ factors $\kappa - \nu_i$, $(i = 1, 2, \ldots, n - 4m)$ which are not factors of numerator.

Again,

$$(n - 4m - 2)T(r,\kappa) \leq \sum_{i=1}^{n-4m} \overline{N}(r,\nu_i;\kappa) \leq \overline{N}(r,\infty;f) = O(\log r) = S(r,\kappa),$$

a contradiction for $n > 4m + 2$.

Therefore h must be a constant satisfying (3.17). Hence from the above cases finally we have $h^{n-2m} = \frac{a}{a_1}$, $h^{n-m} = \frac{b}{b_1}$ and $h^n = \frac{c}{c_1}$.

In particular, if (a, a_1), (b, b_1) or (b, b_1), (c, c_1) are identical then we will automatically have $f = \mathcal{L}$ also $P = Q \implies S_P = S_Q$.

Now if $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1} = k$ then we have, $f = \omega\mathcal{L}$ where ω is mth root of unity.

□

Proof. [Proof of Corollary 1.1] Adopting the same procedure as done in *Theorem 1.10*, we will have $\mathcal{L}_1 = h\mathcal{L}_2$. Then taking $\sigma \longrightarrow +\infty$ we will have, $h = 1$ and hence we get $\mathcal{L}_1 = \mathcal{L}_2$. $\square$

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