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Cohomology and deformation of oriented Hom-associative algebra

Jingru Ren[a]

[a]*Department of Public Courses, Shenzhen Institute of Information Technology, Shenzhen 518172, P.R.China*

Abstract. A Hom version oriented associative algebra is introduced in this paper. In particular, a cohomology theory is introduced for these algebras, and it is shown that the formal deformation theory is controlled by the cohomology, as in the classical case.

1. Introduction

The de-Rham cohomology of a closed oriented manifold carries the structure of a cyclic A_∞-algebra and if the manifold is equipped with an involution, the de-Rham cohomology is an A_∞-algebra equipped with an involution. If the involution is orientation preserving, it is in addition a cyclic involutive A_∞-algebra [4]. Involutive algebras are usually found in geometric contexts in which the underlying geometric objects come equipped with an involution. For example, it was shown in [5] that cyclic involutive A_∞-algebras appear as algebras over the modular operad of chains on certain moduli spaces of Klein surfaces.

In order to develop an equivariant version of Hochschild cohomology for associative algebras, Koam and Pirashvili have recently introduced a notion of oriented algebra in [15]. These algebras are simultaneous generalizations of involutive associative algebras and G-algebras. In the same paper, a theory of cohomology has been developed for oriented associative algebras, and the authors have shown how deformations are related to the cohomology. Oriented version of many other algebras were considered thereafter, including oriented dialgebras [16] and oriented dendriform algebras [12].

Many authors have recently been studying Hom-type algebras. These are the algebras where the original identities defining the algebra structures are twisted by an endomorphism of the underlying space. For example, a Hom-associaitve algebra is a pair $(A, \cdot)$, where A is a vector space and a binary operation $\cdot\ : A \otimes A \to A$, together with a linear map $\alpha : A \to A$ such that the 'Hom-associativity'

$$\alpha(a) \cdot (b \cdot c) = (a \cdot b) \cdot \alpha(c)$$

holds for any $a, b, c \in A$. Many other type of Hom-algebras were studied and the first such was the study of Hom-Lie algebra, which first appeared in q-deformations of Heisenberg algebras, Witt and Virasoro algebras [1] [6] [8] [14]. Hom-type algebras share many of the properties of the original algebras, such as Gerstenhaber algebra structure and the relationship between cohomology and deformation theory [2] [10] [23].

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Email address: jingrur124@outlook.com (Jingru Ren)

In this paper, we introduce the oriented Hom-associative algebras and their cohomology theory. We also discuss formal deformation theory for such algebras and show how cohomology controls the deformation of oriented Hom-associative algebras. This is consistent with other algebraic structures, such as cohomology and deformation for (Hom-)associative algebras [2] [9] [10] [13] [23], (Hom-)Lie algebras [2], (Hom-)Loday-algebras (dialgebra, dendriform, triassociative, ...) [21] [23], (Hom-)algebras/operads with (higher) derivations [11] [19] [22], (Hom-)algebras with (relative) Rota-Baxter/$\mathcal{O}$-operators [3] [7] [17] [18] [20], to mention a few.

The paper is organized as follows: In Section 2, we discuss involutive Hom-associative algebras and their cohomology, as a preparation of the oriented Hom-associative algebra. In Section 3, the oriented Hom-associative algebra and their cohomology is introduced. Finally, in Section 4, we study the formal deformation of oriented Hom-associative algebras, and show how the cohomology defined in Section 3 controls the deformation.

Throughout the paper, all vector spaces are defined on a field k with characteristic zero.

2. Involutive (Hom-)associative algebra and cohomology

2.1. Usual context

Definition 2.1. *An* ***involutive associative algebra*** *is an associative algebra A together with an involution $(-)^*: A \to A$ (that is, a linear map of the underlying vector space whose composition with itself is the identity) such that $(a \cdot b)^* = b^* \cdot a^*$ for all $a, b \in A$. We will call an involutive associative algebra an involutive algebra for simplicity.*

An ***involutive A-bimodule*** *is an ordinary A-bimodule M together with an involution $*: M \to M$ such that $(am)^* = m^*a^*$ and $(ma)^* = a^*m^*$ for all $a \in A$ and $m \in M$. Similarly, we simply call to an involutive A-bimodule by an involutive bimodule.*

Let A be an involutive algebra and M an involutive bimodule, for each $n \geqslant 0$, define $C^n(A, M) := Hom(A^{\otimes n}, M)$ and a map

$$d : C^n(A, M) \to C^{n+1}(A, M)$$

by

$$(df)(a_0, \ldots, a_n) := a_0 f(a_1, \ldots, a_n) + \sum_{k=1}^{n} (-1)^k f(a_0, \ldots, a_{k-1}a_k, \ldots, a_n) + (-1)^{n+1} f(a_0, \ldots, a_{n-1})a_n$$

for any $a_0, \ldots, a_n \in A$. This defines a complex $(C^\bullet(A, M), d)$ whose homology is called the Hochschild cohomology of the associative algebra A with coefficients in usual A-bimodule M.

Define

$$iC^0(A, M) := \{m \in C^0(A, M) : m^* = -m\}$$

$$iC^n(A, M) := \{f \in C^n(A, M) : f(a_1, \ldots, a_n)^* = (-1)^{\frac{(n-1)(n-2)}{2}} f(a_n^*, \ldots, a_1^*)\}, \quad \text{for } n \geqslant 1$$

One can directly verify that $d(iC^n(A, M)) \subseteq iC^{n+1}(A, M)$ so $(iC^\bullet(A, M), d)$ is also a complex. The homology of this complex is the **Hochschild cohomology of the involutive algebra** A **with coefficients in the involutive bimodule** M.

2.2. Hom-context

We now recall the definitions of Hom-associative algebra and module, which serve as the foundation for the Hom-version of involutive algebra and module.

Definition 2.2. *A* ***Hom-associative algebra*** *is a vector space A together with a bilinear map $\cdot : A \times A \to A$ and a linear map $\alpha : A \to A$ such that $\alpha(a) \cdot (b \cdot c) = (a \cdot b) \cdot \alpha(c)$ for all $a, b, c \in A$.*

A Hom-associative algebra is ***multiplicative*** *if $\alpha(a \cdot b) = \alpha(a) \cdot \alpha(b)$, for all $a, b \in A$.*

Definition 2.3. *Let A be a Hom-associative algebra, a* ***left Hom A-module*** *is a vector space M together with a bilinear map $\cdot_l : A \times M \to M$ and a linear map $\alpha_M : M \to M$ such that $\alpha_M(a \cdot_l m) = \alpha_A(a) \cdot_l \alpha_M(m)$ and $\alpha_A(a) \cdot_l (b \cdot_l m) = (a \cdot b) \cdot_l \alpha_M(m)$, for all $a, b \in A$ and $m \in M$. A right Hom-module is defined in a similar way.*

A ***Hom-bimodule*** *is a vector space that's a left and right A-module at the same time while also satisfies the compatibility condition $\alpha_A(a) \cdot_l (m \cdot_r b) = (a \cdot_l m) \cdot_r \alpha_A(b)$.*

We will drop all the unnecessary subscripts in the notation starting from now, unless there is a possibility of confusion.

Definition 2.4. *An* ***involutive Hom-associative algebra*** *is a Hom-associative algebra (A, α) together with an involution $(-)^* : A \to A$ such that $(a \cdot b)^* = b^* \cdot a^*$ and $\alpha(a^*) = \alpha(a)^*$, for all $a, b \in A$. An involutive Hom-associative algebra is* ***multiplicative*** *if the underlying Hom-associative algebra (A, α) is multiplicative, that is, $\alpha(a \cdot b) = \alpha(a) \cdot \alpha(b)$.*

Similarly, an ***involutive Hom-A-bimodule*** *is a Hom A-bimodule M together with an involution $(-)^* : M \to M$ such that $(am)^* = m^*a^*$, $(ma)^* = a^*m^*$ and $\alpha(m^*) = \alpha(m)^*$, for all $a \in A$ and $m \in M$.*

We call involutive Hom-associative algebras and bimodules simply by involutive Hom-algebra and Hom-bimodules.

Lemma 2.5. *Let $(A, \cdot, *)$ be an involutive algebra and $\alpha : A \to A$ is an algebra homomorphism such that it commutes with the involution, i.e., $\alpha(a^*) = \alpha(a)^*$ for all $a \in A$. Define $a \cdot_\alpha b := \alpha(a \cdot b)$, then $(A, \cdot_\alpha, \alpha, *)$ is an involutive Hom-algebra. This is called the* ***Yau twist*** *of the involutive algebra A.*

Proof. One checks directly that $(A, \cdot_\alpha, \alpha)$ is a Hom-associative algebra and $(a \cdot_\alpha b)^* = (\alpha(a \cdot b))^* = \alpha((a \cdot b)^*) = \alpha(b^* \cdot a^*) = b^* \cdot_\alpha a^*$. □

The above Lemma gives us examples of involutive Hom-algebras.

Example 2.6. *Consider $A = \mathbb{C}$ with the usual complex number multiplication and addition, this gives A an associative algebra structure. The map $(-)^* : \mathbb{C} \to \mathbb{C}$ that sends a complex number $c \mapsto \bar{c}$ to its complex conjugate is obviously an involution of A. Now, take $\alpha : A \to A$ to be any algebra homomorphism that commutes with the involution, for example, $\alpha(c) := c \cdot c$, and gives rise to an involutive Hom-algebra $(\mathbb{C}, \cdot_\alpha, \alpha, (-)^*)$.*

Similarly, consider the algebra $C(S)$ of continuous complex-valued functions on a compact set S of $\mathbb{C}$ with function addition and multiplication. The map

$$(-)^* : C(S) \to C(S)$$
$$f \mapsto (f^* : S \to \mathbb{C})$$

where $f^*(c) := \overline{f(c)}$ is the complex conjugate of $f(c)$ for any $c \in S$. One checks directly that this defines an involution on $C(S)$. Now, once again, any algebra homomorphism $\alpha : C(S) \to C(S)$ that commutes with the involution gives rise to an involutive Hom-algebra $(C(S), \cdot_\alpha, \alpha, (-)^*)$. For example, one can take $\alpha := (-)^*$.

Let A be a multiplicative involutive Hom-algebra and M an involutive Hom-bimodule, define

$$C^n_\alpha(A, M) := \{f \in C^n(A, M) : \alpha(f(a_1, \ldots, a_n)) = f(\alpha(a_1), \ldots, \alpha(a_n))\}$$

$$iC^n_\alpha(A, M) := \{f \in C^n_\alpha(A, M) : f(a_1, \ldots, a_n)^* = (-1)^{\frac{(n-1)(n-2)}{2}} f(a_n^*, \ldots, a_1^*)\}$$

and

$$d_\alpha : C^n_\alpha(A, M) \to C^{n+1}_\alpha(A, M)$$

by

$$(d_\alpha f)(a_0, \ldots, a_n) := \alpha^{n-1}(a_0) f(a_1, \ldots, a_n) + \sum_{k=1}^{n} (-1)^k f(\alpha(a_0), \ldots, a_{k-1}a_k, \ldots, \alpha(a_n)) + (-1)^{n+1} f(a_0, \ldots, a_{n-1}) \alpha^{n-1}(a_n)$$

for any $n \geqslant 1$ and $a_1, \ldots, a_n \in A$. Note that $(C^\bullet_\alpha(A, M), d_\alpha)$ is a complex and its homologies are the cohomologies of the Hom-associative algebra A with coefficients in the Hom-bimodule M.

Lemma 2.7. *$d_\alpha(iC^n_\alpha(A,M)) \subseteq iC^{n+1}_\alpha(A,M)$ so $(iC^\bullet_\alpha(A,M), d_\alpha)$ is a complex.*

Proof. Pick $f \in iC^n_\alpha(A,M)$, i.e., $f \in C^n_\alpha(A,M)$ and

$$f(a_1,\dots,a_n)^* = (-1)^{\frac{(n-1)(n-2)}{2}} f(a_n^*,\dots,a_1^*)$$

we need to show

1. $d_\alpha f \in C^{n+1}_\alpha(A,M)$.
2. $((d_\alpha f)(a_0,\dots,a_n))^* = (-1)^{\frac{n(n-1)}{2}} (d_\alpha f)(a_n^*,\dots,a_0^*)$.

First, we have

$$\alpha((d_\alpha f)(a_0,\dots,a_n)) = \alpha\Big(\alpha^{n-1}(a_0)f(a_1,\dots,a_n)+\sum_{k=1}^{n}(-1)^k f(\alpha(a_0),\dots,a_{k-1}a_k,\dots,\alpha(a_n))+(-1)^{n+1}f(a_0,\dots,a_{n-1})\alpha^{n-1}(a_n)\Big)$$

$$= \alpha^n(a_0)\cdot\alpha(f(a_1,\dots,a_n)) + \sum_{k=1}^{n}(-1)^k\alpha(f(\alpha(a_0),\dots,a_{k-1}a_k,\dots,\alpha(a_n))) + (-1)^{n+1}\alpha(f(a_0,\dots,a_{n-1}))\cdot\alpha^n(a_n)$$

$$= \alpha^n(a_0)\cdot f(\alpha(a_1),\dots,\alpha(a_n)) + \sum_{k=1}^{n}(-1)^k\alpha(f(\alpha(a_0),\dots,a_{k-1}a_k,\dots,\alpha(a_n))) + (-1)^{n+1}f(\alpha(a_0),\dots,\alpha(a_{n-1}))\cdot\alpha^n(a_n)$$

and

$$(d_\alpha f)(\alpha(a_0),\dots,\alpha(a_n)) = \alpha^n(a_0)f(\alpha(a_1),\dots,\alpha(a_n)) + \sum_{k=1}^{n}(-1)^k f(\alpha^2(a_0),\dots,\alpha(a_{k-1})\alpha(a_k),\dots,\alpha^2(a_n))$$
$$+(-1)^{n+1}f(\alpha(a_0),\dots,\alpha(a_{n-1}))\alpha^n(a_n)$$

this proves (1) as A is multiplicative. Next, the right hand side of (2) is

$$(-1)^{\frac{n(n-1)}{2}}\Big(\alpha^{n-1}(a_n^*)f(a_{n-1}^*,\dots,a_0^*) + \sum_{k=1}^{n}(-1)^k f(\alpha(a_n^*),\dots,a_{n-(k-1)}^*a_{n-k}^*,\dots,\alpha(a_0^*)) + (-1)^{n+1}f(a_n^*,\dots,a_1^*)\alpha^{n-1}(a_0^*)\Big)$$

$$= (-1)^{\frac{n(n-1)}{2}}\Big((-1)^{\frac{(n-1)(n-2)}{2}}(\alpha^{n-1}(a_n))^*\cdot f(a_0,\dots,a_{n-1})^* + \sum_{k=1}^{n}(-1)^k f(\alpha(a_n)^*,\dots,a_{n-(k-1)}^*a_{n-k}^*,\dots,\alpha(a_0)^*)$$
$$+(-1)^{n+1}(-1)^{\frac{(n-1)(n-2)}{2}}f(a_1,\dots,a_n)^*\cdot(\alpha^{n-1}(a_0))^*\Big)$$

$$= (-1)^{\frac{n(n-1)}{2}}\cdot(-1)^{\frac{(n-1)(n-2)}{2}}\Big((f(a_0,\dots,a_{n-1})\cdot\alpha^{n-1}(a_n))^* + \sum_{k=1}^{n}(-1)^k f(\alpha(a_0),\dots,a_{n-k}\cdot a_{n-(k-1)},\dots,\alpha(a_n))^*$$
$$+(-1)^{n+1}(\alpha^{n-1}(a_0)\cdot f(a_1,\dots,a_n))^*\Big)$$

$$= (-1)^{(n-1)^2}\Big(f(a_0,\dots,a_{n-1})\cdot\alpha^{n-1}(a_n) + \sum_{k=1}^{n}(-1)^k f(\alpha(a_0),\dots,a_{n-k}\cdot a_{n-(k-1)},\dots,\alpha(a_n))$$
$$+(-1)^{n+1}\alpha^{n-1}(a_0)\cdot f(a_1,\dots,a_n)\Big)^*$$

$$= (-1)^{(n-1)^2}(-1)^{n+1}\Big((-1)^{n+1}f(a_0,\dots,a_{n-1})\cdot\alpha^{n-1}(a_n) + \sum_{k=1}^{n}(-1)^k(-1)^{n+1}f(\alpha(a_0),\dots,a_{n-k}\cdot a_{n-(k-1)},\dots,\alpha(a_n))$$

$$+\alpha^{n-1}(a_0)\cdot f(a_1,\dots,a_n)\Big)^*$$

$$=\Big((-1)^{n+1}f(a_0,\dots,a_{n-1})\cdot\alpha^{n-1}(a_n)+\sum_{k=1}^{n}(-1)^{n-k+1}f(\alpha(a_0),\dots,a_{n-k}\cdot a_{n-(k-1)},\dots,\alpha(a_n))$$

$$+\alpha^{n-1}(a_0)\cdot f(a_1,\dots,a_n)\Big)^*$$

since $(-1)^{(n-1)^2+(n+1)}=(-1)^{n(n-1)+2}=1$ and $(-1)^{k+(n+1)}=(-1)^{n-k+1}$ as $k+n+1-(n-k+1)=2k$ is even. Therefore, by letting $i=n-k+1$, the above further equals to

$$\Big((-1)^{n+1}f(a_0,\dots,a_{n-1})\cdot\alpha^{n-1}(a_n)+\sum_{i=n}^{1}(-1)^{i}f(\alpha(a_0),\dots,a_{i-1}\cdot a_i,\dots,\alpha(a_n))+\alpha^{n-1}(a_0)\cdot f(a_1,\dots,a_n)\Big)^*$$

$$=\Big(\alpha^{n-1}(a_0)\cdot f(a_1,\dots,a_n)+\sum_{i=1}^{n}(-1)^{i}f(\alpha(a_0),\dots,a_{i-1}\cdot a_i,\dots,\alpha(a_n))+(-1)^{n+1}f(a_0,\dots,a_{n-1})\cdot\alpha^{n-1}(a_n)\Big)^*$$

$$=((d_\alpha f)(a_0,\dots,a_n))^* \quad \square$$

Definition 2.8. *Let A be a multiplicative involutive Hom-algebra and M an involutive Hom-bimodule, the* ***cohomology of the involutive Hom-algebra** A **with coefficients in the involutive Hom-bimodule*** *M is defined to be the homology of the complex $(iC^\bullet_\alpha(A,M),d_\alpha)$. We denote the nth cohomology group by $iH^n_\alpha(A,M)$.*

Recall from [9] that there is a Gerstenhaber bracket for Hom-associative algebra cochain

$$[-,-]_\alpha : C^m_\alpha(A,A)\times C^n_\alpha(A,A)\to C^{m+n-1}_\alpha(A,A)$$

for $m,n\geqslant 1$, given by

$$[f,g]_\alpha := f\circ g-(-1)^{(m-1)(n-1)}g\circ f,$$

for $f\in C^m_\alpha(A,A)$ and $g\in C^n_\alpha(A,A)$, where

$$(f\circ g)(a_1,\dots,a_{m+n-1}):=\sum_{i=1}^{m}(-1)^{(n-1)(i-1)}(f\circ_i g)(a_1,\dots,a_{m+n-1})=\sum_{i=1}^{m}(-1)^{(n-1)(i-1)}f(\alpha^{n-1}a_1,\dots,g(a_i,\dots,a_{i+n-1}),\dots,\alpha^{n-1}a_{m+n-1}).$$

In particular, the bracket is a degree -1 graded Lie bracket on $C^\bullet_\alpha(A,A)$.

Suppose $f\in iC^m_\alpha(A,A)$ and $g\in iC^n_\alpha(A,A)$ for some involutive Hom-algebra A, we have

$$f(\alpha^{n-1}a_1,\dots,g(a_i,\dots,a_{i+n-1}),\dots,a^{n-1}a_{m+n-1})^*=(-1)^{\frac{(m-1)(m-2)}{2}}f\Big((\alpha^{n-1}a_{m+n-1})^*,\dots,g(a_i,\dots,a_{i+n-1})^*,\dots,(\alpha^{n-1}a_1)^*\Big)$$

$$=(-1)^{\frac{(m-1)(m-2)}{2}}f\Big(\alpha^{n-1}(a^*_{m+n-1}),\dots,(-1)^{\frac{(n-1)(n-2)}{2}}g(a^*_{i+n-1},\dots,a^*_i),\dots,\alpha^{n-1}(a^*_1)\Big)$$

$$=(-1)^{\frac{(m-1)(m-2)}{2}+\frac{(n-1)(n-2)}{2}}f\Big(\alpha^{n-1}(a^*_{m+n-1}),\dots,g(a^*_{i+n-1},\dots,a^*_i),\dots,\alpha^{n-1}(a^*_1)\Big),$$

i.e.,

$$(f\circ_i g)(a_1,\dots,a_{m+n-1})^*=(-1)^{\frac{(m-1)(m-2)+(n-1)(n-2)}{2}}(f\circ_{m-i+1}g)(a^*_{m+n-1},\dots,a^*_1).$$

Proposition 2.9. *If $f\in iC^m_\alpha(A,A)$ and $g\in iC^n_\alpha(A,A)$, then $[f,g]_\alpha\in iC^{m+n-1}_\alpha(A,A)$. Therefore the bracket induces a shifted graded Lie algebra structure on $iC^\bullet_\alpha(A,A)$.*

Proof. Note that we have

$$\Big(\sum_{i=1}^{m}(-1)^{(i-1)(n-1)}f\circ_i g\Big)(a_1,\dots,a_{m+n-1})^* = (-1)^{\frac{(m-1)(m-2)+(n-1)(n-2)}{2}}\sum_{i=1}^{m}(-1)^{(i-1)(n-1)}(f\circ_{m-i+1}g)(a^*_{m+n-1},\dots,a^*_1)$$

$$= (-1)^{\frac{(m-1)(m-2)+(n-1)(n-2)}{2}+(m-1)(n-1)}\sum_{i=1}^{m}(-1)^{(m-i)(n-1)}(f\circ_{m-i+1}g)(a^*_{m+n-1},\dots,a^*_1)$$

because

$$(-1)^{(m-1)(n-1)+(m-i)(n-1)} = (-1)^{(2m-i-1)(n-1)} = (-1)^{(-i-1)(n-1)} = (-1)^{(-i+1)(n-1)} = (-1)^{(i-1)(n-1)}.$$

Therefore

$$[f,g]_\alpha(a_1,\dots,a_{m+n-1})^* = (-1)^{\frac{(m-1)(m-2)+(n-1)(n-2)}{2}+(m-1)(n-1)}[f,g]_\alpha(a^*_{m+n-1},\dots,a^*_1) = (-1)^{\frac{(m+n-2)(m+n-3)}{2}}[f,g]_\alpha(a^*_{m+n-1},\dots,a^*_1),$$

that is, $[f,g]_\alpha \in iC^{m+n-1}_\alpha(A,A)$. □

Definition 2.10. *An element $f \in iC^2_\alpha(A,A)$ is a **Maurer-Cartan element** of the shifted graded Lie algebra $(iC^\bullet_\alpha(A,A),[-,-]_\alpha)$ if $[f,f]_\alpha = 0$.*

Proposition 2.11. *Let $(A,\cdot,\alpha,*)$ be a multiplicative involutive Hom-associative algebra, then the Maurer-Cartan elements of $(iC^\bullet_\alpha(A,A),[-,-]_\alpha)$ are precisely the multiplicative involutive Hom-associative algebra structures on A.*

Proof. $0 = [f,f]_\alpha = 2(f\circ f)$ means for any element $a,b,c \in A$,

$$(f\circ f)(a,b,c) = \sum_{i=1}^{2}(-1)^{i-1}(f\circ_i f)(a,b,c) = (f\circ_1 f)(a,b,c) - (f\circ_2 f)(a,b,c)$$

$$= f(f(a,b),\alpha(c)) - f(\alpha(a),f(b,c)) = 0$$

Also, f is an element of $iC^2_\alpha(A,A)$ so $f(a,b)^* = f(b^*,a^*)$. Other conditions of a multiplicative involutive Hom-associative algebra only concern the involution $(-)^*$ and α so they hold for free. □

3. Oriented (Hom-)associative algebra and cohomology

Definition 3.1. *Let G be a group and $\epsilon : G \to \{\pm1\}$ be a group homomorphism. A (G,ϵ)-**oriented associative algebra** is an associative algebra A together with a G-action $(g,a)\mapsto ga$ such that*

$$g(ab) = \begin{cases} g(a)g(b), & \text{if } \epsilon(g) = 1\\ g(b)g(a), & \text{if } \epsilon(g) = -1.\end{cases}$$

One sees immediately that involutive associative algebras are oriented associative algebras with $G = \{\pm1\}$ and $\epsilon = id$.

We fix a group G and a group homomorphism $\epsilon : G \to \{\pm1\}$ starting from now.

Definition 3.2. *A (G,ϵ)-**oriented Hom-associative algebra** is a Hom-associative algebra (A,α) together with a G-action $(g,a)\mapsto g(a)$ satisfying*

$$g(ab) = \begin{cases} g(a)g(b), & \text{if } \epsilon(g) = 1\\ g(b)g(a), & \text{if } \epsilon(g) = -1\end{cases}$$

and $g(\alpha(a)) = \alpha(g(a))$, for any $g \in G$ and $a,b \in A$.

*A (G,ϵ)-oriented Hom-associative algebra is **multiplicative** if the underlying Hom-algebra is multiplicative, that is, $\alpha(a\cdot b) = \alpha(a)\cdot\alpha(b)$ for any $a,b \in A$.*

Lemma 3.3. *Let A be a (G,ϵ)-oriented associative algebra and $\alpha : A \to A$ an algebra homomorphism such that it commutes with the orientation, i.e., $g(\alpha(a)) = \alpha(g(a))$ for any $a \in A$. Define $a \cdot_\alpha b := \alpha(a \cdot b) = \alpha(ab)$, then $(A, \cdot_\alpha, \alpha)$ is a (G,ϵ)-oriented Hom-associative algebra. This is called the* ***Yau twist*** *of the (G,ϵ)-oriented associative algebra A.*

Proof. One checks directly that

$$g(a \cdot_\alpha b) = g(\alpha(ab)) = \alpha(g(ab)) = \begin{cases} \alpha(g(a)g(b)) = g(a) \cdot_\alpha g(b), & \text{if } \epsilon(g) = 1 \\ \alpha(g(b)g(a)) = g(b) \cdot_\alpha g(a), & \text{if } \epsilon(g) = -1 \end{cases}$$

and $(A, \cdot_\alpha, \alpha)$ is a Hom-associative algebra. □

Definition 3.4. *Let A be a (G,ϵ)-oriented associative algebra, an* ***oriented A-bimodule*** *is a usual bimodule M together with a G-action on M satisfying*

$$g(am) = \begin{cases} g(a)g(m), & \text{if } \epsilon(g) = 1 \\ g(m)g(a), & \text{if } \epsilon(g) = -1 \end{cases}$$

$$g(ma) = \begin{cases} g(m)g(a), & \text{if } \epsilon(g) = 1 \\ g(a)g(m), & \text{if } \epsilon(g) = -1. \end{cases}$$

Similarly, an ***oriented Hom-bimodule*** *is a Hom-bimodule M for the underlying Hom-associative algebra (See Definition 2.3) together with a G-action on M satisfying the same conditions as above, and also $\alpha(g(m)) = g(\alpha(m))$ for any $m \in M$. It is obvious that an oriented Hom-associative algebra is naturally an oriented Hom-bimodule over itself.*

We will refer to oriented Hom-associative algebras simply as oriented Hom-algebras from now on.

Example 3.5. *Any involutive Hom-algebra is an oriented Hom-algebras with $G = \{\pm 1\}$ and $\epsilon = id$. In particular, the examples in Example 2.6 are oriented Hom-algebras.*

Let G be the cyclic group of order 2, $k = \mathbb{Z}$ and $A = \mathbb{Z}[\sqrt{5}] = \{m + \sqrt{5}n : m, n \in \mathbb{Z}\}$, define a G-action on A by $t(\sqrt{5}) = -\sqrt{5}$. This gives A an oriented G-associative algebra structure (See Example 3 [15]). Again, by Lemma 3.3 we obtain, for example, an oriented Hom-algebra $(\mathbb{Z}[\sqrt{5}], \cdot_\alpha, \alpha)$ by taking $\alpha := t$.

Let A be a multiplicative (G,ϵ)-oriented Hom-algebra and M an oriented Hom-bimodule, define a G-action on $C^n_\alpha(A, M)$ by

$$(g \cdot f)(a_1, \ldots, a_n) := \begin{cases} g(f(g^{-1}(a_1), \ldots, g^{-1}(a_n))), & \text{if } \epsilon(g) = 1 \\ (-1)^{\frac{(n-1)(n-2)}{2}} g(f(g^{-1}(a_n), \ldots, g^{-1}(a_1))), & \text{if } \epsilon(g) = -1 \end{cases}$$

so in particular the action is independent of $\epsilon(g)$ for $n = 1$. It's obvious that this defines a G-action on $C^n(A, M)$. To see that it's well-defined on $C^n_\alpha(A, M)$, i.e., $g \cdot f \in C^n_\alpha(A, M)$, one just needs to show $\alpha((g \cdot f)(a_1, \ldots, a_n)) = (g \cdot f)(\alpha(a_1), \ldots, \alpha(a_n))$ for any $f \in C^n_\alpha(A, M)$. Indeed, for $\epsilon(g) = 1$, we have

$$\alpha((g \cdot f)(a_1, \ldots, a_n)) = \alpha\Big(g(f(g^{-1}(a_1), \ldots, g^{-1}(a_n)))\Big) = g\Big(\alpha(f(g^{-1}(a_1), \ldots, g^{-1}(a_n)))\Big)$$

$$= g\Big(f(\alpha(g^{-1}(a_1)), \ldots, \alpha(g^{-1}(a_n)))\Big)$$

$$= g\Big(f(g^{-1}(\alpha(a_1)), \ldots, g^{-1}(\alpha(a_n)))\Big)$$

$$= (g \cdot f)(\alpha(a_1), \ldots, \alpha(a_n)),$$

as g, g^{-1} and f all commute with α. Similarly, for $\epsilon(g) = -1$, we get

$$\alpha((g\cdot f)(a_1,\dots,a_n)) = (-1)^{\frac{(n-1)(n-2)}{2}}\alpha\Big(g(f(g^{-1}(a_n),\dots,g^{-1}(a_1)))\Big)$$

$$= (-1)^{\frac{(n-1)(n-2)}{2}}g\Big(\alpha(f(g^{-1}(a_n),\dots,g^{-1}(a_1)))\Big)$$

$$= (-1)^{\frac{(n-1)(n-2)}{2}}g\Big(f(\alpha(g^{-1}(a_n)),\dots,\alpha(g^{-1}(a_1)))\Big)$$

$$= (-1)^{\frac{(n-1)(n-2)}{2}}g\Big(f(g^{-1}(\alpha(a_n)),\dots,g^{-1}(\alpha(a_1)))\Big)$$

$$= (g\cdot f)(\alpha(a_1),\dots,\alpha(a_n)).$$

Lemma 3.6. *The complex $(C^{\bullet}_{\alpha}(A,M), d_\alpha)$ is G-equivariant, that is, $d_\alpha(g\cdot f) = g\cdot(d_\alpha f)$ for any $f\in C^n_\alpha(A,M)$ and $g\in G$.*

Proof. First, suppose $\epsilon(g) = 1$, we have

$$d_\alpha(g\cdot f)(a_0,\dots,a_n) = \alpha^{n-1}(a_0)(g\cdot f)(a_1,\dots,a_n) + \sum_{k=1}^{n}(-1)^k(g\cdot f)(\alpha(a_0),\dots,a_{k-1}a_k,\dots,\alpha(a_n))$$
$$+(-1)^{n+1}(g\cdot f)(a_0,\dots,a_{n-1})\alpha^{n-1}(a_n)$$
$$= \alpha^{n-1}(a_0)\ g\Big(f(g^{-1}(a_1),\dots,g^{-1}(a_n))\Big) + \sum_{k=1}^{n}(-1)^k g\Big(f(g^{-1}(\alpha(a_0)),\dots,g^{-1}(a_{k-1}a_k),\dots,g^{-1}(\alpha(a_n))\Big)$$
$$+(-1)^{n+1}g\Big(f(g^{-1}(a_0),\dots,g^{-1}(a_{n-1}))\Big)\ \alpha^{n-1}(a_n)$$
$$= \alpha^{n-1}(a_0)g\Big(f(g^{-1}(a_1),\dots,g^{-1}(a_n))\Big)$$
$$+\sum_{k=1}^{n}(-1)^k g\Big(f(g^{-1}(\alpha(a_0)),\dots,g^{-1}(a_{k-1})g^{-1}(a_k),\dots,g^{-1}(\alpha(a_n))\Big)$$
$$+(-1)^{n+1}g\Big(f(g^{-1}(a_0),\dots,g^{-1}(a_{n-1}))\Big)\ \alpha^{n-1}(a_n)$$

At the same time, we have

$$(g\cdot(d_\alpha f))(a_0,\dots,a_n) = g\Big((d_\alpha f)(g^{-1}(a_0),\dots,g^{-1}(a_n))\Big)$$

$$= g\Big(\alpha^{n-1}(g^{-1}(a_0))f(g^{-1}(a_1),\dots,g^{-1}(a_n)) + \sum_{k=1}^{n}(-1)^k f(\alpha(g^{-1}(a_0)),\dots,g^{-1}(a_{k-1})g^{-1}(a_k),\dots,\alpha(g^{-1}(a_n)))$$
$$+(-1)^{n+1}f(g^{-1}(a_0),\dots,g^{-1}(a_{n-1}))\alpha^{n-1}(g^{-1}(a_n))\Big)$$
$$= g\Big(\alpha^{n-1}(g^{-1}(a_0))\Big)\ g\Big(f(g^{-1}(a_1),\dots,g^{-1}(a_n))\Big) + \sum_{k=1}^{n}(-1)^k g\Big(f(\alpha(g^{-1}(a_0)),\dots,g^{-1}(a_{k-1})g^{-1}(a_k),\dots,\alpha(g^{-1}(a_n)))\Big)$$
$$+(-1)^{n+1}g\Big(f(g^{-1}(a_0),\dots,g^{-1}(a_{n-1}))\Big)\ g\Big(\alpha^{n-1}(g^{-1}(a_n))\Big)$$

$$= \alpha^{n-1}(a_0)\ g\Big(f(g^{-1}(a_1),\dots,g^{-1}(a_n))\Big) + \sum_{k=1}^{n}(-1)^k g\Big(f(\alpha(g^{-1}(a_0)),\dots,g^{-1}(a_{k-1})g^{-1}(a_k),\dots,\alpha(g^{-1}(a_n)))\Big)$$

$$+(-1)^{n+1} g\Big(f(g^{-1}(a_0),\dots,g^{-1}(a_{n-1}))\Big)\alpha^{n-1}(a_n)$$

The above two expressions are equal as $g^{-1}(\alpha(-)) = \alpha(g^{-1}(-))$.

Similarly, when $\epsilon(g) = -1$, we have

$$d_\alpha(g\cdot f)(a_0,\dots,a_n) = \alpha^{n-1}(a_0)(g\cdot f)(a_1,\dots,a_n) + \sum_{k=1}^{n}(-1)^k (g\cdot f)(\alpha(a_0),\dots,a_{k-1}a_k,\dots,\alpha(a_n))$$

$$+(-1)^{n+1}(g\cdot f)(a_0,\dots,a_{n-1})\alpha^{n-1}(a_n)$$

$$= (-1)^{\frac{(n-1)(n-2)}{2}}\alpha^{n-1}(a_0)\ g\Big(f(g^{-1}(a_n),\dots,g^{-1}(a_1))\Big) + \sum_{k=1}^{n}(-1)^{\frac{(n-1)(n-2)}{2}+k} g\Big(f(g^{-1}(\alpha(a_n)),\dots,g^{-1}(a_{k-1}a_k),\dots,g^{-1}(\alpha(a_0))\Big)$$

$$+(-1)^{\frac{(n-1)(n-2)}{2}+(n+1)} g\Big(f(g^{-1}(a_{n-1}),\dots,g^{-1}(a_0))\Big)\ \alpha^{n-1}(a_n)$$

$$= (-1)^{\frac{(n-1)(n-2)}{2}}\alpha^{n-1}(a_0)\ g\Big(f(g^{-1}(a_n),\dots,g^{-1}(a_1))\Big) + \sum_{k=1}^{n}(-1)^{\frac{(n-1)(n-2)}{2}+k} g\Big(f(g^{-1}(\alpha(a_n)),\dots,g^{-1}(a_k)g^{-1}(a_{k-1}),\dots,g^{-1}(\alpha(a_0))\Big)$$

$$+(-1)^{\frac{(n-1)(n-2)}{2}+(n+1)} g\Big(f(g^{-1}(a_{n-1}),\dots,g^{-1}(a_0))\Big)\ \alpha^{n-1}(a_n)$$

meanwhile,

$$(g\cdot(d_\alpha f))(a_0,\dots,a_n) = (-1)^{\frac{n(n-1)}{2}} g\Big((d_\alpha f)(g^{-1}(a_n),\dots,g^{-1}(a_0))\Big)$$

$$= (-1)^{\frac{n(n-1)}{2}} g\Big(\alpha^{n-1}(g^{-1}(a_n))f(g^{-1}(a_{n-1}),\dots,g^{-1}(a_0)) + \sum_{k=1}^{n}(-1)^k f(\alpha(g^{-1}(a_n)),\dots,g^{-1}(a_{n-k+1})g^{-1}(a_{n-k}),\dots,\alpha(g^{-1}(a_0)))$$

$$+(-1)^{n+1} f(g^{-1}(a_n),\dots,g^{-1}(a_1))\alpha^{n-1}(g^{-1}(a_0))\Big)$$

$$= (-1)^{\frac{n(n-1)}{2}} g\Big(f(g^{-1}(a_{n-1}),\dots,g^{-1}(a_0))\Big)\ g\Big(\alpha^{n-1}(g^{-1}(a_n))\Big)$$

$$+\sum_{k=1}^{n}(-1)^{\frac{n(n-1)}{2}+k} g\Big(f(\alpha(g^{-1}(a_n)),\dots,g^{-1}(a_{n-k+1})g^{-1}(a_{n-k}),\dots,\alpha(g^{-1}(a_0)))\Big)$$

$$+(-1)^{\frac{n(n-1)}{2}+(n+1)} g\Big(\alpha^{n-1}(g^{-1}(a_0))\Big)\ g\Big(f(g^{-1}(a_n),\dots,g^{-1}(a_1))\Big)$$

$$= (-1)^{\frac{n(n-1)}{2}} g\Big(f(g^{-1}(a_{n-1}),\dots,g^{-1}(a_0))\Big)\ \alpha^{n-1}(a_n)$$

$$+\sum_{i=n}^{1}(-1)^{\frac{n(n-1)}{2}+(n-i+1)} g\Big(f(\alpha(g^{-1}(a_n)),\dots,g^{-1}(a_i)g^{-1}(a_{i-1}),\dots,\alpha(g^{-1}(a_0)))\Big)$$

$$+(-1)^{\frac{n(n-1)}{2}+(n+1)}\alpha^{n-1}(a_0)\ g\Big(f(g^{-1}(a_n),\dots,g^{-1}(a_1))\Big)$$

where $i := n - k + 1$ in the last step and one checks directly that the corresponding signs coincide, that is,

$$(-1)^{\frac{(n-1)(n-2)}{2}} = (-1)^{\frac{n(n-1)}{2}+(n+1)}$$

$$(-1)^{\frac{(n-1)(n-2)}{2}+j} = (-1)^{\frac{n(n-1)}{2}+(n-j+1)} \text{ for any } j$$

$$(-1)^{\frac{(n-1)(n-2)}{2}+(n+1)} = (-1)^{\frac{n(n-1)}{2}}. \quad \square$$

The complex $(C^\bullet_\alpha(A,M), d_\alpha)$ is a G-**complex** in the terminology of [15] and one can form the following bicomplex based on ideas of [15]

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle \partial''} & & \uparrow{\scriptstyle \partial''} & & \uparrow{\scriptstyle \partial''} & & \\
Maps(G^2, M) & \xrightarrow{\partial'} & Maps(G^2, C^1_\alpha(A,M)) & \xrightarrow{\partial'} & Maps(G^2, C^2_\alpha(A,M)) & \xrightarrow{\partial'} & \cdots \\
\uparrow{\scriptstyle \partial''} & & \uparrow{\scriptstyle \partial''} & & \uparrow{\scriptstyle \partial''} & & \\
Maps(G, M) & \xrightarrow{\partial'} & Maps(G, C^1_\alpha(A,M)) & \xrightarrow{\partial'} & Maps(G, C^2_\alpha(A,M)) & \xrightarrow{\partial'} & \cdots \\
\uparrow{\scriptstyle \partial''} & & \uparrow{\scriptstyle \partial''} & & \uparrow{\scriptstyle \partial''} & & \\
M & \xrightarrow{\partial'} & C^1_\alpha(A,M) & \xrightarrow{\partial'} & C^2_\alpha(A,M) & \xrightarrow{\partial'} & \cdots
\end{array}$$

with coboundary maps defined by

1. the horizontal coboundary maps are

$$(\partial' f)(g_1,\dots,g_m;a_1,\dots,a_{n+1}) := \alpha^{n-1}(a_1)f(g_1,\dots,g_m;a_2,\dots,a_{n+1}) + \sum_{i=1}^{n}(-1)^i f(g_1,\dots,g_m;\alpha(a_1),\dots,a_ia_{i+1},\dots,\alpha(a_{n+1}))$$
$$+(-1)^{n+1} f(g_1,\dots,g_m;a_1,\dots,a_n)\alpha^{n-1}(a_{n+1})$$

2. the first vertical maps are

$$(\partial'' f)(g_1,\dots,g_{m+1}) := g_1\Big(f(g_2,\dots,g_{m+1})\Big) + \sum_{i=1}^{m}(-1)^i f(g_1,\dots,g_ig_{i+1},\dots,g_{m+1}) + (-1)^{m+1} f(g_1,\dots,g_m)$$

3. the second vertical maps are

$$(\partial'' f)(g_1,\dots,g_{m+1};a) = g_1\Big(f(g_2,\dots,g_{m+1};g_1^{-1}a)\Big) + \sum_{i=1}^{m}(-1)^i f(g_1,\dots,g_ig_{i+1},\dots,g_{m+1};a) + (-1)^{m+1} f(g_1,\dots,g_m;a)$$

4. the third vertical maps, when $\epsilon(g_1) = 1$, are

$$(\partial'' f)(g_1,\dots,g_{m+1};a,b) := g_1\Big(f(g_2,\dots,g_{m+1};g_1^{-1}a,g_1^{-1}b)\Big) + \sum_{i=1}^{m}(-1)^i f(g_1,\dots,g_ig_{i+1},\dots,g_{m+1};a,b) + (-1)^{m+1} f(g_1,\dots,g_m;a,b)$$

when $\epsilon(g_1) = -1$, are

$$(\partial'' f)(g_1,\dots,g_{m+1};a,b) := g_1\Big(f(g_2,\dots,g_{m+1};g_1^{-1}b,g_1^{-1}a)\Big) + \sum_{i=1}^{m}(-1)^i f(g_1,\dots,g_ig_{i+1},\dots,g_{m+1};a,b) + (-1)^{m+1} f(g_1,\dots,g_m;a,b)$$

and the remaining vertical maps are defined in a similar way. For example, when $\epsilon(g_1) = 1$, are

$$(\partial'' f)(g_1, \dots, g_{m+1}; a_1, \dots, a_n) := g_1\Big(f(g_2, \dots, g_{m+1}; g_1^{-1}a_1, \dots, g_1^{-1}a_n)\Big)$$
$$+ \sum_{i=1}^{m} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{m+1}; a_1, \dots, a_n) + (-1)^{m+1} f(g_1, \dots, g_m; a_1, \dots, a_n)$$

Delete the first column and reindex the above bicomplex gives

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\uparrow{\scriptstyle \partial''} & & \uparrow{\scriptstyle \partial''} & & \\
Maps(G^2, C^1_\alpha(A, M)) & \xrightarrow{\partial'} & Maps(G^2, C^2_\alpha(A, M)) & \xrightarrow{\partial'} & \cdots \\
\uparrow{\scriptstyle \partial''} & & \uparrow{\scriptstyle \partial''} & & \\
Maps(G, C^1_\alpha(A, M)) & \xrightarrow{\partial'} & Maps(G, C^2_\alpha(A, M)) & \xrightarrow{\partial'} & \cdots \\
\uparrow{\scriptstyle \partial''} & & \uparrow{\scriptstyle \partial''} & & \\
C^1_\alpha(A, M) & \xrightarrow{\partial'} & C^2_\alpha(A, M) & \xrightarrow{\partial'} & \cdots
\end{array}$$

Definition 3.7. *The homologies of the associated total complex of the above complex is the* ***cohomologies*** *of the* (G, ϵ)*-oriented Hom-algebra* A *with coefficients in the oriented Hom-bimodule* M*. Denote the cohomologies by* $H^n_{G,\alpha}(A, M)$ *for* $n \geqslant 0$*.*

The coboundary maps for the total complex is $\partial(f) := \partial'' f + (-1)^i \partial' f$ if $f \in Maps(G^i, C^j_\alpha(A, M))$. Hence an element $(\gamma, f) \in Maps(G, C^1_\alpha(A, M)) \oplus C^2_\alpha(A, M)$ is a 2-cocycle if

$$\begin{array}{lll}
\partial''\gamma = 0 & & \\
\uparrow & & \\
\gamma & \longmapsto & -\partial'\gamma + \partial'' f = 0 \\
& & \uparrow \\
& & f \longmapsto \partial' f = 0
\end{array}$$

by definitions of the coboundary maps, this means exactly

(a) $(\partial''\gamma)(g, h; a) = g\gamma(h; g^{-1}a) - \gamma(gh; a) + \gamma(g; a) = 0.$

(b)
$$(\partial'\gamma)(g; a, b) = (\partial'' f)(g; a, b)$$
$$\Leftrightarrow a\gamma(g; b) - \gamma(g; ab) + \gamma(g; a)b = \begin{cases} g\Big(f(g^{-1}a, g^{-1}b)\Big) - f(a, b), & \text{if } \epsilon(g) = 1 \\ g\Big(f(g^{-1}b, g^{-1}a)\Big) - f(a, b), & \text{if } \epsilon(g) = -1 \end{cases}$$

(c) $(\partial' f)(a, b, c) = \alpha(a) f(b, c) - f(ab, \alpha(c)) + f(\alpha(a), bc) - f(a, b)\alpha(c) = 0.$

Similarly, (γ, f) is a 2-coboundary if there is an $\mu \in C^1_\alpha(A, M)$ such that $\partial''\mu = \gamma$ and $\partial'\mu = f$, that is, we have

(d) $\gamma(g; a) = g(\mu(g^{-1}a)) - \mu(a).$

(e) $f(a, b) = a\mu(b) - \mu(ab) + \mu(a)b$

4. Deformation

Definition 4.1. *Let $(A, \cdot = m, \alpha)$ be a (G, ϵ)-oriented Hom-algebra (here we use the letter m to represent the multiplication), a* ***formal one-parameter deformation*** *of A consists of two formal power series*

$$m_t = m_0 + m_1 t + m_2 t^2 + \cdots$$

$$\phi_t = \phi_0 + \phi_1 t + \phi_2 t^2 + \cdots$$

where $m_i : A \otimes A \to A$ and $\phi_i \in Maps(G, C^1_\alpha(A, A))$ with $m_0 = m$ and $\phi_0(g; a) = g(a)$ such that $(A[[t]], m_t, \alpha, \phi_t)$ is also a (G, ϵ)-oriented Hom-algebra.

Explicitly, by multiplying everything out and comparing the coefficients of t^n terms ($n \geqslant 0$), for each $n \geqslant 0$ and $a, b, c \in A$, we have

1. $\sum_{i+j=n} \Big(m_i(\alpha(a), m_j(b, c)) - m_i(m_j(a, b), \alpha(c)) \Big) = 0.$
2. $\phi_n(gh; a) = \sum_{i+j=n} \phi_i(g; \phi_j(h; a)).$
3. $\sum_{i+j=n} \phi_i(g; m_j(a, b)) = \begin{cases} \sum_{i+j+k=n} m_i(\phi_j(g; a), \phi_k(g; b)), & \text{if } \epsilon(g) = 1 \\ \sum_{i+j+k=n} m_i(\phi_j(g; b), \phi_k(g; a)), & \text{if } \epsilon(g) = -1 \end{cases}$
4. $\alpha(\phi_t(a, b)) = \phi_t(\alpha(a), \alpha(b)).$
5. $\alpha(m_t(a, b)) = m_t(\alpha(a), \alpha(b)).$

Let (m_t, ϕ_t) be a deformation of A, define $\xi_n \in Maps(G, C^1_\alpha(A, A))$ by $\xi_n(g; a) := \phi_n(g; g^{-1}a)$. We have

Proposition 4.2. *(ξ_1, m_1) is a 2-cocycle.*

Proof.
- Let $n = 1$ in equality (2): $\phi_1(gh; a) = g(\phi_1(h; a)) + \phi_1(g; ha) \quad (*)$, we get

$$\begin{aligned} g(\xi_1(h; g^{-1}a)) - \xi_1(gh; a) + \xi_1(g; a) &= g(\xi_1(h; g^{-1}a)) + \xi_1(g; a) - \xi_1(gh; a) \\ &= g(\phi_1(h; h^{-1}g^{-1}a)) + \phi_1(g; g^{-1}a) - \xi_1(gh; a) \\ &\overset{(*)}{=} \phi_1(gh; h^{-1}g^{-1}a) - \xi_1(gh; a) = 0 \end{aligned}$$

that is, $\partial'' \xi_1 = 0$ (See equality (a)).

- Let $n = 1$ in equality (3) gives

$$g\Big(m_1(a, b)\Big) + \phi_1(g; ab) = \begin{cases} m_1(ga, gb) + \phi_1(g; a) \cdot gb + (ga) \cdot \phi_1(g; b), & \text{if } \epsilon(g) = 1 \\ m_1(gb, ga) + \phi_1(g; b) \cdot ga + (gb) \cdot \phi_1(g; a), & \text{if } \epsilon(g) = -1 \end{cases}$$

that is,

$$g\Big(m_1(a, b)\Big) + \xi_1(g; g(ab)) = \begin{cases} m_1(ga, gb) + \xi_1(g; ga) \cdot gb + (ga) \cdot \xi_1(g; gb), & \text{if } \epsilon(g) = 1 \\ m_1(gb, ga) + \xi_1(g; gb) \cdot ga + (gb) \cdot \xi_1(g; ga), & \text{if } \epsilon(g) = -1 \end{cases}$$

therefore we get

$$g\Big(m_1(a, b)\Big) + \xi_1(g; g(ab)) = m_1(ga, gb) + \xi_1(g; ga) \cdot gb + (ga) \cdot \xi_1(g; gb)$$

$$\Leftrightarrow (ga) \cdot \xi_1(g; gb) - \xi_1(g; g(ab)) + \xi_1(g; ga) \cdot gb = g\Big(m_1(a, b)\Big) - m_1(ga, gb)$$

when $\epsilon(g) = 1$. The case for $\epsilon(g) = -1$ follows similarly and together give equality (b).

- Let $n = 1$ in equality (1) gives $\alpha(a) \cdot m_1(b, c) - m_1(a, b) \cdot \alpha(c) + m_1(\alpha(a), b \cdot c) - m_1(a \cdot b, \alpha(c)) = 0$ and this means exactly $\partial' m_1 = 0$ (See equality (c)).

Discussion below Definition 3.7 states that equality (a), (b) and (c) means exactly that (ξ_1, m_1) is a 2-cocycle. □

The pair (ξ_1, m_1) is called the **infinitesimal** of the deformation. Apparently, (ξ_n, m_n) is a 2-cocycle if $(\xi_1, m_1) = \cdots = (\xi_{n-1}, m_{n-1}) = 0$ and $(\xi_n, m_n) \neq 0$ for similar reasons and we call (ξ_n, m_n) the **nth infinitesimal** of the deformation.

Definition 4.3. *Two deformations (m_t, ϕ_t) and (m'_t, ϕ'_t) of an oriented Hom-algebra A is* ***equivalent*** *if there is a formal isomorphism $\psi_t = \psi_0 + \psi_1 t + \psi_2 t^2 + \cdots : A[[t]] \to A[[t]]$ with $\psi_0 = id_A, \psi_i : A \to A$ such that ψ_t defines an isomorphism between oriented Hom-algebras.*

That is to say, we have following identities for $n \geqslant 0, a, b \in A$ and $g \in G$:

(i) $\sum_{i+j=n} \psi_i(m'_j(a, b)) = \sum_{i+j+k=n} m_i(\psi_j(a), \psi_k(b))$.

(ii) $\sum_{i+j=n} \psi_i(\phi'_j(g; a)) = \sum_{i+j=n} \phi_i(g; \psi_j(a))$.

(iii) $\psi_i(\alpha(a)) = \alpha(\psi_i(a))$.

Proposition 4.4. *The infinitesimals of two equivalent deformations determine the same cohomology class.*

Proof. Let $n = 1$ in equalities (i) and (ii) give:

$$m'_1(a, b) + \psi_1(m'_0(a, b)) = m_1(a, b) + m_0(\psi_1(a), b) + m_0(a, \psi_1(b))$$

$$\Leftrightarrow m'_1(a, b) - m_1(a, b) = m_0(a, \psi_1(b)) - \psi_1(m'_0(a, b)) + m_0(\psi_1(a), b) = (\partial' \psi_1)(a, b)$$

and

$$\phi'_1(g; a) + \psi_1(ga) = g(\psi_1(a)) + \phi_1(g; a) \Leftrightarrow \xi'_1(g; ga) + \psi_1(ga) = g(\psi_1(a)) + \xi_1(g; ga)$$

$$\Leftrightarrow \xi'_1(g; ga) - \xi_1(g; ga) = g\Big(\psi_1\big(g^{-1}(ga)\big)\Big) - \psi_1(ga) = (\partial'' \psi_1)(g; ga)$$

which means there is an element $\psi_1 \in C^1_\alpha(A, A)$ such that (d) and (e) hold, that is, $\partial \psi_1 = (\xi'_1 - \xi_1, m'_1 - m_1)$. □

Therefore we have

Theorem 4.5. *There is a one-to-one correspondence between the space of equivalence classes of infinitesimal deformations and the second cohomology $H^2_{G,\alpha}(A, A)$.*

4.1. Rigidity

Definition 4.6. *A formal deformation (m_t, ϕ_t) of an oriented Hom-algebra A is* ***trivial*** *if it's equivalent to (m, ϕ_0). An oriented Hom-algebra is* ***rigid*** *if every formal deformation is trivial.*

Theorem 4.7. *An oriented Hom-algebra A is rigid if $H^2_{G,\alpha}(A, A) = 0$.*

Proof. The infinitesimal (ξ_1, m_1) of any deformation (m_t, ϕ_t) is a 2-cocycle so there is some $\psi_1 \in C^1_\alpha(A, A)$ such that

$$(\xi_1(g; a), m_1(b, c)) = -\partial \psi_1 = -(\quad g(\psi_1(a)) - \psi_1(ga) \quad , \quad b\psi_1(c) - \psi_1(bc) + \psi_1(b)c \quad) \qquad (**)$$

for any $g \in G$ and $a, b, c \in A$ as $H^2_{G,\alpha}(A, A) = 0$. Define $\psi_t := id + \psi_1 t$ and $\psi_t^{-1} = id - \psi_1 t + \psi_1^2 t^2 + \cdots$ the inverse, set

$$m'_t(b, c) := \psi_t^{-1}(m_t(\psi_t(b), \psi_t(c))) = \psi_t^{-1}(m_0(\psi_t(b), \psi_t(c)) + m_1(\psi_t(b), \psi_t(c))t + \cdots)$$

$$= \psi_t^{-1}(m_0(b+\psi_1(b), c+\psi_1(c)) + m_1(b+\psi_1(b), c+\psi_1(c))t + \cdots)$$

$$= \Big(m_0(b+\psi_1(b), c+\psi_1(c)) + m_1(b+\psi_1(b), c+\psi_1(c))t + \cdots\Big)$$

$$-\psi_1\Big(m_0(b+\psi_1(b), c+\psi_1(c)) + m_1(b+\psi_1(b), c+\psi_1(c))t + \cdots\Big)$$

$$+\psi_1^2\Big(m_0(b+\psi_1(b), c+\psi_1(c)) + m_1(b+\psi_1(b), c+\psi_1(c))t + \cdots\Big)$$

$$+ \quad \cdots$$

$$= bc + \Big(b\psi_1(c) + \psi_1(b)c + m_1(b,c) - \psi_1(bc)\Big)t + \cdots$$

$$\overset{(**)}{=} bc + (\cdots)t^2 + \cdots \quad (=: m_0 + m_2' t^2 + \cdots)$$

Similarly, set

$$\phi_t'(g;a) := \psi_t^{-1}(\phi_t(g;\psi_t(a))) = \psi_t^{-1}(\phi_t(g;a+\psi_1(a))) = \psi_t^{-1}\Big(\phi_t(g;a) + \phi_t(g;\psi_1(a))t\Big)$$

$$= \Big(\phi_t(g;a) + \phi_t(g;\psi_1(a))t\Big)$$

$$-\psi_1\Big(\phi_t(g;a) + \phi_t(g;\psi_1(a))t\Big)t$$

$$+\psi_1^2\Big(\phi_t(g;a) + \phi_t(g;\psi_1(a))t\Big)t^2$$

$$+ \quad \cdots$$

$$= ga + \Big(\phi_1(g;a) + g(\psi_1(a)) - \psi_1(ga)\Big)t + \cdots$$

$$\overset{(**)}{=} ga + (\cdots)t^2 + \cdots \quad (=: \phi_0 + \phi_2' t^2 + \cdots)$$

note that $\xi(g:a) := \phi_1(g;g^{-1}a)$ so the first component of (∗∗) indeed gives $\phi_1(g;a) + g(\psi_1(a)) - \psi_1(ga) = 0$.

To summarize, we demonstrated that the deformation (m_t, ϕ_t) is equivalent to a deformation (m_t', ϕ_t'), whose degree one terms vanish by the fact that $H^2_{G,\alpha}(A,A) = 0$. Repeating the above argument we can conclude that (m_t, ϕ_t) is equivalent to (m_0, ϕ_0). □

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