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The eigenvalues and Moore-Penrose inverse of reduced biquaternion matrices

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Abstract. The reduced biquaternions is a commutative algebra, which can be thought of as a two-dimension space over complex field. From this point of view, there are two complex representations of reduced biquaternion matrix A. We obtain the relationships among these sets of eigenvalues of A and its two complex representations. Our results show that each reduced biquaternion matrix A has infinite eigenvalues and different eigenvalues of A may have the same eigenvector. We also introduce the concepts of determinant and the Moore-Penrose inverse of reduced biquaternion matrices and obtain some properties of them. As applications, we solve some reduced biquaternion linear equations. Some algorithms with experimental examples are provided to support our theoretical results.

1. Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ be the field of real and complex numbers, respectively. As we all know the eigen-problem and the linear equations are fundamental problems in matrix algebra. Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. The determinant of A is defined by

$$\det(A) = \sum_{p_1p_2\cdots p_n} (-1)^{\tau} a_{1p_1}a_{2p_2}\cdots a_{np_n},$$

where τ is the inversion number of the permutation $p_1p_2\cdots p_n$. It is well known that complex matrix A has exactly n complex eigenvalues, counted with multiplicity, and the eigenvectors of different eigenvalues are linearly independent.

Let $A \in \mathbb{C}^{n\times m}, b \in \mathbb{C}^n$. The linear equation of

$$Ax = b$$

can be solved by Cramer's rule when $n = m$ and $det(A) \neq 0$. In the case of $n \neq m$ or $det(A) = 0$, such an equation can be solved by using of the Moore-Penrose inverse [3].

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There have been many attempts to find the analogues in some quaternionic algebras. The algebras of quaternions $\mathbb{H}$ and split quaternions $\mathbb{H}_s$ were introduced by Hamilton and Cockle [6, 13] in 1843 and 1849, respectively. Both quaternions and split quaternions are noncommutative algebras.

In quaternion algebra, the study of left eigenvalues stemmed from the Lee and Cohn's question [8, 23] whether left eigenvalue always exists. By using of topological method, Wood [29] confirmed that the left eigenvalue always exists. Zhang [30] has given a brief survey on quaternions and matrix of quaternions. Huang and So [14] reduced the computation of the left eigenvalues of quaternion matrix of order 2 to solve a quaternionic quadratic equation [15]. The concepts of determinant and the Moore-Penrose inverse are also generalized to quaternions [2, 7, 21].

Unlike the quaternion, the split quaternion algebra is a nondivision algebra. There are several attempts to understand properties of split quaternions and the eigenvalues of a matrix A over $\mathbb{H}_s$ [1, 9, 10, 16, 20, 22].

After the discovery of quaternions by Hamilton, Segre proposed modified quaternions so that commutative property in multiplication is possible [28]. Hans-Dieter Schutte and Jorg Wenzel [27] called such an algebra as reduced biquaternions and used it in digital signal processing. From then on, the reduced biquaternion has been extensively used in image processing and physical field [5, 11, 12, 24–26].

Kosal and Tosun investigated some algebraic properties of commutative quaternion matrices and consider some linear equations in [17–19]. We will go on the research in this direction.

In this paper we mainly focus on the eigenvalues and eigenvectors, determinant and Moore-Penrose inverse of reduced biquaternion matrices.

We firstly recall some basic notations in [17] . The reduced biquaternions are elements of a 4-dimensional associative and commutative algebra which can be represented as

$$\mathbb{H}_r = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, q_i \in \mathbb{R}, i = 0, 1, 2, 3\},$$

where $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are basis of $\mathbb{H}_r$ with the following multiplication rules:

	1	**i**	**j**	**k**
1	1	**i**	**j**	**k**
i	**i**	-1	**k**	-**j**
j	**j**	**k**	1	**i**
k	**k**	-**j**	**i**	-1

We can view $\mathbb{R} = span\{1\}$ and $\mathbb{C} = span\{1, \mathbf{i}\}$. According to the multiplication rules, a reduced biquaternion can be written as

$$q = (q_0 + q_1\mathbf{i}) + (q_2 + q_3\mathbf{i})\mathbf{j} = c_1 + c_2\mathbf{j} = c_1 \cdot 1 + c_2\mathbf{j}, c_1, c_2 \in \mathbb{C}. \tag{1}$$

Besides $1, \mathbf{j}$, there are also two very important elements in reduced biquaternions, which are

$$e_1 = \frac{1+\mathbf{j}}{2}, \ \ e_2 = \frac{1-\mathbf{j}}{2}. \tag{2}$$

Note that

$$e_1^n = e_1^{n-1} = \cdots = e_1, \ e_2^n = e_2^{n-1} = \cdots = e_2, \ e_1e_2 = 0, \ e_1 + e_2 = 1, \tag{3}$$

$$e_1q = (c_1 + c_2)e_1, \ e_2q = (c_1 - c_2)e_2 \tag{4}$$

and

$$q = (c_1 + c_2\mathbf{j})(e_1 + e_2) = (c_1 + c_2)e_1 + (c_1 - c_2)e_2, \forall q = c_1 + c_2\mathbf{j}. \tag{5}$$

By (1) and (5), $\mathbb{H}_r$ can be thought of as a linear space over $\mathbb{C}$ with two bases $\{1, \mathbf{j}\}$ and $\{e_1, e_2\}$, respectively. It is obvious that

$$(e_1, e_2) = (1, \mathbf{j})\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \ \ (1, \mathbf{j}) = (e_1, e_2)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{6}$$

This implies that

$$\mathbb{H}_r = \mathbb{C}e_1 + \mathbb{C}e_2 = \mathbb{C} + \mathbb{C}\mathbf{j}. \tag{7}$$

There exist three kinds of conjugate of $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$:

$$\overline{q} = q_0 + q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k},$$
$$q^* = q_0 - q_1\mathbf{i} + q_2\mathbf{j} - q_3\mathbf{k},$$
$$\widehat{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} + q_3\mathbf{k}.$$

The norm of q is defined by

$$\|q\|^4 = q \cdot \overline{q} \cdot q^* \cdot \widehat{q} = ((q_0 + q_2)^2 + (q_1 + q_3)^2)((q_0 - q_2)^2 + (q_1 - q_3)^2).$$

Let

$$N(q) = \overline{q}q = q\overline{q} = q_0^2 - q_1^2 - q_2^2 + q_3^2 + 2(q_0q_1 - q_2q_3)\mathbf{i}.$$

If $N(q) = 0$, then

$$q_0^2 - q_1^2 - q_2^2 + q_3^2 = 0,\ q_0q_1 - q_2q_3 = 0. \tag{8}$$

For $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, if $N(q) \neq 0$, then $p = \frac{\overline{q}}{N(q)}$ is the inverse of q. That is

$$q^{-1} = \frac{\overline{q}}{N(q)}.$$

Unlike the Hamilton quaternion algebra, the reduced biquaternions contain nontrivial zero divisors. The set of nontrivial zero divisors is

$$Z(\mathbb{H}_r) = \{q \in \mathbb{H}_r : N(q) = 0\}. \tag{9}$$

The zero divisors can be represented as follows.

Proposition 1.1. *([4, Proposition 2.4])*

$$\begin{aligned} Z(\mathbb{H}_r) &= \{q \in \mathbb{H}_r : q = c_1 \pm c_1\mathbf{j}, c_1 \in \mathbb{C}\} \\ &= \{q \in \mathbb{H}_r : q = ye_1 \text{ or } q = ye_2, \forall y \in \mathbb{H}_r\} \\ &= \{q \in \mathbb{H}_r : q = ye_1 \text{ or } q = ye_2, \forall y \in \mathbb{C}\}. \end{aligned}$$

The main obstacles in the study of reduced biquaternion matrices, as expected come from the existance of nontrivial zero divisors. The effective approaches of studying reduced biquaternion matrices may be the methods of converting reduced biquaternion matrices into a pair of complex matrices. It follows from (5) that any matrix $A \in \mathbb{H}_r^{n\times m}$ can be decomposed by

$$A = A_1 + A_2\mathbf{j} = Q_1e_1 + Q_2e_2,\ \text{where } A_i, Q_i \in \mathbb{C}^{n\times m}, i = 1, 2 \tag{10}$$

with

$$Q_1 = A_1 + A_2, Q_2 = A_1 - A_2. \tag{11}$$

For $A \in \mathbb{H}_r^{n\times m}$, Kosal and Pei et al [17, 25, 26] introduced the following two representations of reduced biquaternion matrices:

$$A^J = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix} \in \mathbb{C}^{2n\times 2m}$$

and

$$A^E = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \in \mathbb{C}^{2n\times 2m}.$$

Let $\sigma(A^J)$, $\sigma(A^E)$ and $\sigma_r(A)$ be the set of eigenvalues of A^J, A^E and A, respectively.

We will obtain the relationships among $\sigma(A^J)$, $\sigma(A^E)$ and $\sigma_r(A)$. Our results show that each reduced biquaternion matrix A has infinite eigenvalues and different eigenvalues of A may have the same eigenvector.

Since the reduced biquaternion $\mathbb{H}_r$ is an associative and commutative algebra, we can define its determinant as in the complex case. We will consider the determinant of matrices in $\mathbb{H}_r^{n\times n}$ and introduce the Moore-Penrose inverse of matrices in $\mathbb{H}_r^{n\times m}$ to solve some linear equations in $\mathbb{H}_r$.

The paper is organized as follows. In Section 2, we introduce two complex representations of reduced biquaternion matrices. We also obtain some properties of them. In Section 3, we find the the relationship between the eigenvalues of $\sigma_r(A)$, $\sigma(A^E)$ and $\sigma(A^J)$. In Section 4, we will introduce the concept of determinant of reduced biquaternion matrices and obtain the Cramer's rule of linear equation $Ax = b$. In Section 5, we introduce the concept of the Moore-Penrose inverse in reduced biquaternions and solve the linear equation $AXB = C$. Some algorithms with experimental examples are provided to support our theoretical results in Sections 3 to 5.

2. Two complex representations

In this section, we will introduce two complex representations of reduced biquaternion matrices. We recall that the scalar multiplication is defined as

$$qA = (qa_{ij}) = Aq,\ \forall q \in \mathbb{H}_r,\ A = (a_{ij}) \in \mathbb{H}_r^{n\times m}.$$

Obviously, we have the following proposition.

Proposition 2.1. *Let $A_i, Q_i \in \mathbb{C}^{n\times m}, i = 1, 2$. Then*
(1) $A_1 + A_2\mathbf{j} = 0$ if and only if $A_1 = 0, A_2 = 0$;
(2) $Q_1e_1 + Q_2e_2 = 0$ if and only if $Q_1 = 0, Q_2 = 0$.

Let I_n be the identity matrix of order n and $A \in \mathbb{H}_r^{n\times m}$. Then A can be represented as

$$A = A_1 + A_2\mathbf{j} = (I_n, I_n\mathbf{j})\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{12}$$

and

$$A = Q_1e_1 + Q_2e_2 = (I_ne_1, I_ne_2)\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}. \tag{13}$$

By Proposition 2.1, the set $\mathbb{H}_r^{n\times m}$ can be thought of as a linear space over $\mathbb{C}^{n\times m}$ with bases $\{I_n, I_n\mathbf{j}\}$ and $\{I_ne_1, I_ne_2\}$. Let

$$P = \begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix}. \tag{14}$$

Then

$$P^{-1} = \frac{1}{2}\begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix} = \frac{1}{2}P. \tag{15}$$

It is obvious that P is the transformation matrix from $\{I_ne_1, I_ne_2\}$ to $\{I_n, I_n\mathbf{j}\}$. That is

$$(I_n, I_n\mathbf{j}) = (I_ne_1, I_ne_2)P,\ (I_ne_1, I_ne_2) = (I_n, I_n\mathbf{j})\frac{P}{2}. \tag{16}$$

Let $A = A_1 + A_2\mathbf{j} \in \mathbb{H}_r^{n\times m}, B = B_1 + B_2\mathbf{j} \in \mathbb{H}_r^{m\times s}$ and $\overrightarrow{A} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$. Then

$$AB = (A_1 + A_2\mathbf{j})(B_1 + B_2\mathbf{j}) = (A_1B_1 + A_2B_2) + (A_2B_1 + A_1B_2)\mathbf{j}.$$

Hence

$$\overrightarrow{AB} = \begin{pmatrix} A_1B_1 + A_2B_2 \\ A_2B_1 + A_1B_2 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}. \tag{17}$$

The above formula induces an isomorphism form $\mathbb{H}_r^{n\times m}$ to $\mathbb{C}^{2n\times 2m}$ as follows:

$$A^J = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix} \in \mathbb{C}^{2n\times 2m}. \tag{18}$$

Similarly, the following formula

$$AB = (Q_1e_1 + Q_2e_2)(R_1e_1 + R_2e_2) = Q_1R_1e_1 + Q_2R_2e_2$$

induces an isomorphism form $\mathbb{H}_r^{n\times m}$ to $\mathbb{C}^{2n\times 2m}$ as follows:

$$A^E = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \in \mathbb{C}^{2n\times 2m}. \tag{19}$$

The following two propositions can be verified directly.

Proposition 2.2. *Let $A, B \in \mathbb{H}_r^{n\times m}, C \in \mathbb{H}_r^{m\times s}$. Then the two complex representation matrices of reduced biquaternion matrices have the following properties:*

$$(A + B)^J = A^J + B^J,\ \ (AC)^J = A^JC^J; \tag{20}$$

$$(A + B)^E = A^E + B^E,\ \ (AC)^E = A^EC^E. \tag{21}$$

Proposition 2.3. *Let $A \in \mathbb{H}_r^{n\times n}$. Then*

$$PA^JP^{-1} = P^{-1}A^JP = A^E. \tag{22}$$

3. Eigenvalues of A and its two complex representations

In this section, we will consider the relationship among the eigenvalues of A and its complex representations A^E and A^J. To make the problem more precise, we give the following definitions.

Definition 3.1. *Let $A \in \mathbb{H}_r^{n\times n}$ and $\lambda \in \mathbb{H}_r$. If λ holds the equation*

$$Ax = \lambda x,\ \ 0 \neq x \in \mathbb{H}_r^n,$$

then λ is called an eigenvalue of A. The set of distinct eigenvalues is called the spectrum of A, denoted

$$\sigma_r(A) = \{\lambda \in \mathbb{H}_r : Ax = \lambda x, x \neq 0\}.$$

Let

$$V_r(A)_\lambda = \{x \in \mathbb{H}_r^n : Ax = \lambda x\}.$$

The subscript "r" means we consider the eigen-problem in reduced biquaternions. It is obvious that the nonzero vector of $V_r(A)_\lambda$ is an eigenvector corresponding to λ. For any $\alpha, \beta \in V_r(A)_\lambda$, we have

$$\mu_1\alpha + \mu_2\beta \in V_r(A)_\lambda, \forall \mu_1, \mu_2 \in \mathbb{H}_r. \tag{23}$$

This shows that $V_r(A)_\lambda$ is an eigenvector space of λ.

In the complex field $\mathbb{C}$, we follow the conventional definition.

Definition 3.2. *Let $A \in \mathbb{C}^{n\times n}, \lambda \in \mathbb{C}$. If λ holds the equation*

$$Ax = \lambda x,\ \ 0 \neq x \in \mathbb{C}^n,$$

then λ is called an eigenvalue of A. The set of distinct eigenvalues is called the spectrum of A, denoted

$$\sigma(A) = \{\lambda \in \mathbb{C} : Ax = \lambda x,\ \ x \neq 0\}.$$

Accordingly, we denote

$$V(A)_\lambda = \{x \in \mathbb{C}^n : Ax = \lambda x\}.$$

Note that $\mathbb{H}_r$ is commutative quaternion algebra. Due to the restriction of dimension, we remark that $Ax = \lambda x$ should be viewed as $Ax = (\lambda I_n)x$ or $Ax = x\lambda$ when someone applies Proposition 2.2.

By Propositions 2.1 and 2.2, we have the following lemma.

Lemma 3.1. *Let $A = A_1 + A_2\mathbf{j} = Q_1e_1 + Q_2e_2 \in \mathbb{H}_r^{n\times n}$, $x = x_1 + x_2\mathbf{j} \in \mathbb{H}_r^n$, $\lambda = \lambda_1 + \lambda_2\mathbf{j} \in \mathbb{H}_r$. Then the following statements are equivalent:*

$$Ax = x\lambda. \tag{24}$$

$$A^J x^J = x^J \lambda^J. \tag{25}$$

$$A^E x^E = x^E \lambda^E. \tag{26}$$

$$\begin{cases} A_1x_1 + A_2x_2 = \lambda_1x_1 + \lambda_2x_2, \\ A_1x_2 + A_2x_1 = \lambda_1x_2 + \lambda_2x_1. \end{cases} \tag{27}$$

$$\begin{cases} (A_1 + A_2)(x_1 + x_2) = (\lambda_1 + \lambda_2)(x_1 + x_2), \\ (A_1 - A_2)(x_1 - x_2) = (\lambda_1 - \lambda_2)(x_1 - x_2). \end{cases} \tag{28}$$

The following theorem describes how to obtain $\sigma_r(A)$ from $\sigma(A^E) = \sigma(Q_1) \cup \sigma(Q_2)$.

Theorem 3.1. *Let $A = Q_1e_1 + Q_2e_2 \in \mathbb{H}_r^{n\times n}$.*

(1) If $\lambda_1 \in \sigma(Q_1), \lambda_2 \in \sigma(Q_2)$ with $\gamma_1 \in V(Q_1)_{\lambda_1}, \gamma_2 \in V(Q_2)_{\lambda_2}$, then

$$\lambda_1e_1 + \lambda_2e_2 \in \sigma_r(A) \tag{29}$$

and

$$\gamma_1e_1 + \gamma_2e_2 \in V_r(A)_{\lambda_1e_1+\lambda_2e_2}. \tag{30}$$

(2) If $\lambda_1 \in \sigma(Q_1)$ with $\gamma_1 \in V(Q_1)_{\lambda_1}$, then for any $\mu \in \mathbb{C}$,

$$\lambda_1e_1 + \mu e_2 \in \sigma_r(A) \tag{31}$$

with

$$\gamma_1e_1 \in V_r(A)_{\lambda_1e_1+\mu e_2}. \tag{32}$$

(3) If $\lambda_2 \in \sigma(Q_2)$ with $\gamma_2 \in V(Q_2)_{\lambda_1}$, then for any $\mu \in \mathbb{C}$,

$$\mu e_1 + \lambda_2e_2 \in \sigma_r(A) \tag{33}$$

and

$$\gamma_2e_2 \in V_r(A)_{\mu e_1+\lambda_2e_2}. \tag{34}$$

Proof. Note that

$$e_1 = e_1^2,\ e_2 = e_2^2,\ e_1e_2 = 0.$$

If $\lambda_1 \in \sigma(Q_1), \lambda_2 \in \sigma(Q_2)$ with $\gamma_1 \in V(Q_1)_{\lambda_1}, \gamma_2 \in V(Q_2)_{\lambda_2}$, then

$$Q_1\gamma_1 = \gamma_1\lambda_1, Q_2\gamma_2 = \gamma_2\lambda_2.$$

Thus we have

$$(Q_1e_1 + Q_2e_2)(\gamma_1e_1 + \gamma_2e_2) = (\gamma_1e_1 + \gamma_2e_2)(\lambda_1e_1 + \lambda_2e_2).$$

This proves (1). We mention that (1) is just restatement of [17, Theorem 3.7] and [26, Section II].

If $\lambda_1 \in \sigma(Q_1)$ with $\gamma_1 \in V(Q_1)_{\lambda_1}$, then

$$Q_1\gamma_1 = \gamma_1\lambda_1,\ Q_1\gamma_1e_1 = \gamma_1\lambda_1e_1.$$

Hence

$$(Q_1e_1 + Q_2e_2)\gamma_1e_1 = \gamma_1e_1\lambda_1e_1 = \gamma_1e_1(\lambda_1e_1 + \mu e_2), \forall\mu \in \mathbb{C}.$$

By Definition 3.1, we have $\lambda_1e_1 + \mu e_2 \in \sigma_r(A)$ and $\gamma_1e_1 \in V_r(A)_{\lambda_1e_1+\mu e_2}$. This proves (2).

If $\lambda_2 \in \sigma(Q_2)$ with $\gamma_2 \in V(Q_2)_{\lambda_2}$, then

$$Q_2\gamma_2 = \gamma_2\lambda_2,\ Q_2\gamma_2e_2 = \gamma_2\lambda_2e_2.$$

Hence

$$(Q_1e_1 + Q_2e_2)\gamma_2e_2 = \gamma_2e_2\lambda_2e_2 = \gamma_2e_2(\mu e_1 + \lambda_2e_2), \forall\mu \in \mathbb{C}.$$

By Definition 3.1, we have $\mu e_1 + \lambda_2e_2 \in \sigma_r(A)$ and $\gamma_2e_2 \in V_r(A)_{\mu e_1+\lambda_2e_2}$. This proves (3). □

Remark 3.1. *We remark that for any eigenvalue $\lambda = \lambda_1e_1 + \lambda_2e_2 \in \sigma_r(A)$ with eigenvector $x = \gamma_1e_1 + \gamma_2e_2 \in V_r(A)_{\lambda_1e_1+\lambda_2e_2}$, we can obtain them by combining Theorem 3.1 (2) and (3). In fact, it follows from*

$$(Q_1e_1 + Q_2e_2)(\gamma_1e_1 + \gamma_2e_2) = (\gamma_1e_1 + \gamma_2e_2)(\lambda_1e_1 + \lambda_2e_2)$$

that

$$Q_1\gamma_1 = \gamma_1\lambda_1, Q_2\gamma_2 = \gamma_2\lambda_2.$$

Applying Theorem 3.1 (2) with $\mu = \lambda_2$, we have

$$(Q_1e_1 + Q_2e_2)\gamma_1e_1 = \gamma_1e_1(\lambda_1e_1 + \lambda_2e_2). \tag{35}$$

Applying Theorem 3.1 (3) with $\mu = \lambda_1$, we have

$$(Q_1e_1 + Q_2e_2)\gamma_2e_2 = \gamma_2e_2(\lambda_1e_1 + \lambda_2e_2). \tag{36}$$

Therefore both γ_1e_1 and γ_2e_2 belong to $V_r(A)_{\lambda_1e_1+\lambda_2e_2}$. Hence $\gamma_1e_1 + \gamma_2e_2$ belong to $V_r(A)_{\lambda_1e_1+\lambda_2e_2}$. This implies that we can reconstruct any eigenvalues of A, as well as the corresponding eigenvectors by Theorem 3.1 (2) and (3).

It is well known that $A \in \mathbb{C}^{n\times n}$ has exactly n complex eigenvalues, counted with multiplicity and if $\lambda, \mu \in \sigma(A)$ with $\lambda \neq \mu$, then $V(A)_\lambda \cap V(A)_\mu = \{0\}$. In contrast to complex matrices, Theorem 3.1 implies that any matrix $A \in \mathbb{H}_r^{n\times n}$ has infinite eigenvalues an different eigenvalues may have the same eigenvector.

By Theorem 3.1, we have the following relationship of $\sigma(A)$ and $\sigma_r(A)$ when $A \in \mathbb{C}^{n\times n}$ is thought of as a reduced biquaternion matrix.

Corollary 3.1. *Let $A \in \mathbb{C}^{n\times n} \subset \mathbb{H}_r^{n\times n}$. Then we have the following properties:*

(1) If $\lambda_1 \in \sigma(A), \lambda_2 \in \sigma(A)$ with $\gamma_1 \in V(A)_{\lambda_1}, \gamma_2 \in V(A)_{\lambda_2}$, then

$$\lambda_1e_1 + \lambda_2e_2 \in \sigma_r(A) \tag{37}$$

and

$$\gamma_1 e_1 + \gamma_2 e_2 \in V_r(A)_{\lambda_1 e_1 + \lambda_2 e_2}. \tag{38}$$

(2) If $\lambda_1 \in \sigma(A)$ with $\gamma_1 \in V(A)_{\lambda_1}$, then for any $\mu \in \mathbb{C}$,

$$\lambda_1 e_1 + \mu e_2 \in \sigma_r(A) \tag{39}$$

with

$$\gamma_1 e_1 \in V_r(A)_{\lambda_1 e_1 + \mu e_2}. \tag{40}$$

(3) If $\lambda_2 \in \sigma(A)$ with $\gamma_2 \in V(A)_{\lambda_2}$, then for any $\mu \in \mathbb{C}$,

$$\mu e_1 + \lambda_2 e_2 \in \sigma_r(A) \tag{41}$$

and

$$\gamma_2 e_2 \in V_r(A)_{\mu e_1 + \lambda_2 e_2}. \tag{42}$$

For two sets M and N, the difference set $M - N$ is defined as

$$M - N = \{x \in M \text{ and } x \notin N\}.$$

Lemma 3.2. *Let $A = Q_1 e_1 + Q_2 e_2 \in \mathbb{H}_r^{n\times n}$ and $A^E = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$. Then we have the following properties:*

(1) If $\lambda \in \sigma(Q_1) - \sigma(Q_2)$, then $\lambda \in \sigma(A^E)$ and

$$V(A^E)_\lambda = \begin{pmatrix} V(Q_1)_\lambda \\ 0 \end{pmatrix} := \left\{ \begin{pmatrix} \gamma_1 \\ 0 \end{pmatrix} : \forall\, \gamma_1 \in V(Q_1)_\lambda \right\}. \tag{43}$$

(2) If $\lambda \in \sigma(Q_2) - \sigma(Q_1)$, then $\lambda \in \sigma(A^E)$ and

$$V(A^E)_\lambda = \begin{pmatrix} 0 \\ V(Q_2)_\lambda \end{pmatrix} := \left\{ \begin{pmatrix} 0 \\ \gamma_2 \end{pmatrix} : \forall\, \gamma_2 \in V(Q_2)_\lambda \right\}. \tag{44}$$

(3) If $\lambda \in \sigma(Q_1) \cap \sigma(Q_2)$, then $\lambda \in \sigma(A^E)$ and

$$V(A^E)_\lambda = \begin{pmatrix} V(Q_1)_\lambda \\ V(Q_2)_\lambda \end{pmatrix} := \left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} : \forall\, \gamma_1 \in V(Q_1)_\lambda, \gamma_2 \in V(Q_2)_\lambda \right\}. \tag{45}$$

Proof. Note that if $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in V(A^E)_\lambda$, then

$$Q_1\gamma_1 = \lambda\gamma_1,\ Q_2\gamma_2 = \lambda\gamma_2.$$

The above observation concludes the proof. □

Based on the above observation, we can refine Theorem 3.1 as follows.

Theorem 3.2. *Let $A = Q_1 e_1 + Q_2 e_2 \in \mathbb{H}_r^{n\times n}$. Then we have*

(1) If $\lambda \in \sigma(Q_1) \cap \sigma(Q_2) \subset \sigma(A^E)$ with $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in V(A^E)_\lambda$, then

$$\lambda \in \sigma_r(A),\ \ \gamma_1 e_1 + \gamma_2 e_2 \in V_r(A)_\lambda. \tag{46}$$

(2) If $\lambda \in (\sigma(Q_1) - \sigma(Q_2)) \subset \sigma(A^E)$ *with* $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in V(A^E)_\lambda$*, then*

$$\lambda e_1 + \mu e_2 \in \sigma_r(A),\ \gamma_1 e_1 \in V_r(A)_{\lambda e_1 + \mu e_2},\ \forall \mu \in \mathbb{C}. \tag{47}$$

(3)If $\mu \in (\sigma(Q_2) - \sigma(Q_1)) \subset \sigma(A^E)$ *with* $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in V(A^E)_\mu$*, then*

$$\lambda e_1 + \mu e_2 \in \sigma_r(A),\ \beta_2 e_2 \in V_r(A)_{\lambda e_1 + \mu e_2},\ \forall \lambda \in \mathbb{C}. \tag{48}$$

(4) If $\lambda \in (\sigma(Q_1) - \sigma(Q_2)) \subset \sigma(A^E)$ *with* $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in V(A^E)_\lambda$ *and* $\mu \in (\sigma(Q_2) - \sigma(Q_1)) \subset \sigma(A^E)$ *with* $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in V(A^E)_\mu$*, then*

$$\lambda e_1 + \mu e_2 \in \sigma_r(A),\ \gamma_1 e_1 + \beta_2 e_2 \in V_r(A)_{\lambda e_1 + \mu e_2}. \tag{49}$$

The following theorem describes how to get $\sigma(Q_1), \sigma(Q_2)$ from $\sigma_r(A)$.

Theorem 3.3. *Let* $A = Q_1 e_1 + Q_2 e_2 \in \mathbb{H}_r^{n\times n}$ *having eigenvalue* $\lambda = \lambda_1 e_1 + \lambda_2 e_2$ *with eigenvector* $\gamma = \gamma_1 e_1 + \gamma_2 e_2$*. Then*
(1) $\lambda_1 \in \sigma(Q_1)$ *provided* $\gamma_1 \neq 0$.
(2) $\lambda_2 \in \sigma(Q_2)$ *provided* $\gamma_2 \neq 0$.

Proof. It follows from

$$(Q_1 e_1 + Q_2 e_2)(\gamma_1 e_1 + \gamma_2 e_2) = (\gamma_1 e_1 + \gamma_2 e_2)(\lambda_1 e_1 + \lambda_2 e_2)$$

that

$$(Q_1 e_1 + Q_2 e_2)e_1(\gamma_1 e_1 + \gamma_2 e_2)e_1 = (\gamma_1 e_1 + \gamma_2 e_2)e_1(\lambda_1 e_1 + \lambda_2 e_2)e_1.$$

That is

$$Q_1 \gamma_1 e_1 = \gamma_1 \lambda_1 e_1,$$

which implies that

$$Q_1 \gamma_1 = \gamma_1 \lambda_1.$$

Similarly, we have

$$Q_2 \gamma_2 = \gamma_2 \lambda_2.$$

□

The following theorem describes how to obtain $\sigma_r(A)$ from $\sigma(A^J)$.

Theorem 3.4. *Let* $A = A_1 + A_2\mathbf{j} \in \mathbb{H}_r^{n\times n}$*. Then we have*
(1) If $\lambda \in \sigma(A_1 + A_2) \cap \sigma(A_1 - A_2) \subset \sigma(A^J)$ *with* $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in V(A^J)_\lambda$*, then*

$$\lambda \in \sigma_r(A),\ (\gamma_1 + \gamma_2)e_1 + (\gamma_1 - \gamma_2)e_2 \in V_r(A)_\lambda. \tag{50}$$

(2) If $\lambda \in (\sigma(A_1 + A_2) - \sigma(A_1 - A_2)) \subset \sigma(A^J)$ *with* $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in V(A^J)_\lambda$*, then*

$$\lambda e_1 + \mu e_2 \in \sigma_r(A),\ (\gamma_1 + \gamma_2)e_1 \in V_r(A)_{\lambda e_1 + \mu e_2},\ \forall \mu \in \mathbb{C}. \tag{51}$$

(3) If $\mu \in (\sigma(A_1 - A_2) - \sigma(A_1 - A_2)) \subset \sigma(A^J)$ *with* $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in V(A^J)_\mu$*, then*

$$\lambda e_1 + \mu e_2 \in \sigma_r(A),\ (\beta_1 - \beta_2)e_2 \in V_r(A)_{\lambda e_1 + \mu e_2},\ \forall \lambda \in \mathbb{C}. \tag{52}$$

(4) If $\lambda \in (\sigma(A_1+A_2)-\sigma(A_1-A_2)) \subset \sigma(A^J)$ *with* $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in V(A^J)_\lambda$ *and* $\mu \in (\sigma(A_1-A_2)-\sigma(A_1-A_2)) \subset \sigma(A^J)$ *with* $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in V(A^J)_\mu$*, then*

$$\lambda e_1 + \mu e_2 \in \sigma_r(A),\ (\gamma_1+\gamma_2)e_1 + (\beta_1-\beta_2)e_2 \in V_r(A)_{\lambda e_1+\mu e_2}. \tag{53}$$

Proof. By Proposition 2.3, we have

$$P^{-1}A^E P = A^J, P = \begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix}. \tag{54}$$

Hence

$$\sigma(A^J) = \sigma(A^E).$$

If $\lambda \in \sigma(A^J)$ with $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in V(A^J)_\lambda$, then

$$A^E P \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = P \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \lambda. \tag{55}$$

That is

$$A^E \begin{pmatrix} \gamma_1+\gamma_2 \\ \gamma_1-\gamma_2 \end{pmatrix} = \begin{pmatrix} \gamma_1+\gamma_2 \\ \gamma_1-\gamma_2 \end{pmatrix} \lambda. \tag{56}$$

Applying Theorem 3.2 gets this result. □

Based on Theorem 3.2, we provide an algorithm to find the eigenvalues of a given $A \in \mathbb{H}_r^{n\times n}$ as follows.

Algorithm 3.1. *Find the eigenvalues of a given* $A \in \mathbb{H}_r^{n\times n}$.

step 1: Find the matrix Q_1 *and* Q_2 *and complex representation matrix* $A^E = diag(Q_1, Q_2)$.

step 2: Find all the different eigenvalues of Q_1 *and* Q_2. *Suppose that*

$$\sigma(Q_1) - \sigma(Q_2) = \{\lambda_1, \cdots, \lambda_t\},\ \sigma(Q_2) - \sigma(Q_1) = \{\mu_1, \cdots, \mu_s\}$$

and

$$\sigma(Q_1) \cap \sigma(Q_2) = \{\theta_1, \cdots, \theta_k\},\ \sigma(A^E) = \sigma(Q_1) \cup \sigma(Q_2).$$

step 3: Find the eigenvalues of A and corresponding eigenvectors of A:

(1) For any $\theta_i \in \sigma(Q_1) \cap \sigma(Q_2)$ *with* $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in V(A^E)_{\theta_i}$*, we have* $\theta_i \in \sigma_r(A),\ \gamma_1 e_1 + \gamma_2 e_2 \in V_r(A)_{\theta_i}$.

(2) For any $\lambda_i \in (\sigma(Q_1) - \sigma(Q_2))$ *with* $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in V(A^E)_{\lambda_i}$*, we have* $\lambda_i e_1 + \mu e_2 \in \sigma_r(A)$, $\gamma_1 e_1 \in V_r(A)_{\lambda_i e_1 + \mu e_2}$, $\forall \mu \in \mathbb{C}$.

(3) For any $\mu_i \in (\sigma(Q_2) - \sigma(Q_1))$ *with* $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in V(A^E)_{\mu_i}$*, we have* $\lambda e_1 + \mu e_2 \in \sigma_r(A)$, $\beta_2 e_2 \in V_r(A)_{\lambda e_1 + \mu_i e_2}$, $\forall \lambda \in \mathbb{C}$.

(4) For any $\lambda_i \in (\sigma(Q_1) - \sigma(Q_2))$ *with* $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in V(A^E)_{\lambda_i}$ *and* $\mu_j \in (\sigma(Q_2) - \sigma(Q_1))$ *with* $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in V(A^E)_{\mu_j}$*, we have* $\lambda e_1 + \mu e_2 \in \sigma_r(A)$, $\gamma_1 e_1 + \beta_2 e_2 \in V_r(A)_{\lambda e_1 + \mu e_2}$.

Example 3.1. *For*

$$A = \begin{pmatrix} 7+4\mathbf{i}+4\mathbf{j}+6\mathbf{k} & -6-6\mathbf{k} \\ 8+7\mathbf{i}+7\mathbf{j}+8\mathbf{k} & -5-3\mathbf{i}-3\mathbf{j}-6\mathbf{k} \end{pmatrix}.$$

By Algorithm 3.1, we have

$$Q_1 = \begin{pmatrix} 11+10\mathbf{i} & -6-6\mathbf{i} \\ 15+15\mathbf{i} & -8-9\mathbf{i} \end{pmatrix}; Q_2 = \begin{pmatrix} 3-2\mathbf{i} & -6+6\mathbf{i} \\ 1-\mathbf{i} & -2+3\mathbf{i} \end{pmatrix}, A^E = \begin{pmatrix} Q_1 & \\ & Q_2 \end{pmatrix}.$$

$$\sigma(Q_1) = \{2+\mathbf{i}, 1\},\ \sigma(Q_2) = \{\mathbf{i}, 1\},\ \sigma(A^E) = \{\mathbf{i}, 2+\mathbf{i}, 1\}.$$

$$V(A^E)_{2+\mathbf{i}} = \{k_1\gamma_1\}, \gamma_1 = (2,3,0,0)^T, \forall k_1 \in \mathbb{C};\ V(A^E)_{\mathbf{i}} = \{k_2\gamma_2\}, \gamma_2 = (0,0,2,1)^T, \forall k_2 \in \mathbb{C};$$

$$V(A^E)_1 = \{k_3\gamma_3 + k_4\gamma_4\}, \gamma_3 = (3,5,0,0)^T, \gamma_4 = (0,0,3,1)^T, \forall k_3, k_4 \in \mathbb{C}.$$

(1) Since $1 \in \sigma(Q_1) \cap \sigma(Q_2)$ *with multiplicity* 2 *with eigenvectors* $\gamma_3, \gamma_4 \in V(A^E)_1$*, we have* $1 \in \sigma_r(A)$ *and* $k_3(3,5)^T e_1 + k_4(3,1)^T e_2 \in V_r(A)_1, \forall k_3, k_4 \in \mathbb{C}$.

(2) Since $2+\mathbf{i} \in \sigma(Q_1) - \sigma(Q_2)$ *with* $(2,3,0,0)^T \in V(A^E)_{2+\mathbf{i}}$*, we have* $(2+\mathbf{i})e_1 + \mu e_2 \in \sigma_r(A)$ *and* $k_1(2,3)^T e_1 \in V_r(A)_{2+\mathbf{i}}, \forall k_1 \in \mathbb{C}$.

(3) Since $\mathbf{i} \in \sigma(Q_2) - \sigma(Q_1)$ *with* $(0,0,2,1)^T \in V(A^E)_{\mathbf{i}}$*, we have* $\mu e_1 + \mathbf{i}e_2 \in \sigma_r(A)$ *and* $k_2(2,1)^T e_2 \in V_r(A)_{\mathbf{i}}, \forall k_2 \in \mathbb{C}$.

(4) Since $2+\mathbf{i} \in (\sigma(Q_1) - \sigma(Q_2))$ *with* $(2,3,0,0)^T \in V(A^E)_{2+\mathbf{i}}$ *and* $\mathbf{i} \in \sigma(Q_2) - \sigma(Q_1)$ *with* $(0,0,2,1)^T \in V(A^E)_{\mathbf{i}}$*, we have* $(2+\mathbf{i})e_1 + \mathbf{i}e_2 \in \sigma_r(A)$*,* $k_1(2,3)^T e_1 + k_2(2,1)^T e_2 \in V_r(A)_{(2+\mathbf{i})e_1+\mathbf{i}e_2}, \forall k_1, k_2 \in \mathbb{C}$.

4. The determinant and Cramer's rule

Since the reduced biquaternions $\mathbb{H}_r$ is an associative and commutative algebra, we can define its determinant as follows.

Definition 4.1. *Let* $A = (a_{ij}) \in \mathbb{H}_r^{n\times n}$*. The determinant of A is defined by*

$$\det(A) = \sum_{p_1p_2\cdots p_n} (-1)^\tau a_{1p_1}a_{2p_2}\cdots a_{np_n},$$

where τ *is the inversion number of the permutation* $p_1p_2\cdots p_n$.

Definition 4.2. *For* $A = (a_{ij}) \in \mathbb{H}_r^{n\times n}$*, the minor* M_{ij} *of* a_{ij} *is the determinant of the matrix obtained by deleting both the i-th row and the j-th column of A. The cofactor* A_{ij} *of* a_{ij} *is defined by*

$$A_{ij} = (-1)^{i+j} M_{ij}. \tag{57}$$

The adjoint of A is defined to be the transpose of the cofactor matrix (A_{ij})*. That is*

$$adj(A) = (A_{ji}). \tag{58}$$

Theorem 4.1. *Let* $A = (a_{ij}) \in \mathbb{H}_r^{n\times n}$*. Then*

$$\det(A) = \sum_{j=1}^{n} a_{ij}A_{ij},\ i = 1, 2, \cdots, n \tag{59}$$

and

$$\det(A) = \sum_{i=1}^{n} a_{ij}A_{ij},\ j = 1, 2, \cdots, n. \tag{60}$$

Corollary 4.1. *Let* $A = (a_{ij}) \in \mathbb{H}_r^{n\times n}$*. Then*

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = 0, i \neq j \tag{61}$$

and

$$a_{1i}A_{1j} + a_{2i}A_{2j} + \cdots + a_{ni}A_{nj} = 0, i \neq j. \tag{62}$$

Theorem 4.2. *Let* $A \in \mathbb{H}_r^{n\times n}$*. Then*

$$A\mathrm{adj}(A) = \mathrm{adj}(A)A = \det(A)I_n. \tag{63}$$

Proposition 4.1. *([17, Theorem 3.2.]) Let* $A, B \in \mathbb{H}_r^{n\times n}$*. If* $AB = I_n$*, then* $BA = I_n$*.*

Definition 4.3. *Let* $A \in \mathbb{H}_r^{n\times n}$*. If there exists a matrix* $B \in \mathbb{H}_r^{n\times n}$ *such that* $AB = I_n$*, then* A *is invertible and* B *is the inverse of* A*.*

Proposition 4.2. *Let* $A = Q_1e_1 + Q_2e_2 \in \mathbb{H}_r^{n\times n}$ *with* $Q_i \in \mathbb{C}^{n\times n}, i = 1, 2$*. Then*

$$\det(A) = \det(Q_1)e_1 + \det(Q_2)e_2. \tag{64}$$

Proof. Let $Q_1 = (q^1_{ij})$ and $Q_2 = (q^2_{ij})$. Note that

$$e_1 = e_1^2 = \cdots = e_1^n,\ e_2 = e_2^2 = \cdots = e_2^n,\ e_1e_2 = 0,\ e_1 + e_2 = 1$$

and $a_{ip_i} = a_{ip_i}e_1 + a_{ip_i}e_2$. Then

$$a_{ip_i}e_1 = q^1_{ip_i}e_1,\ a_{ip_i}e_2 = q^2_{ip_i}e_2.$$

Hence

$$\begin{aligned}\det(A) &= \sum_{p_1p_2\cdots p_n}(-1)^\tau a_{1p_1}a_{2p_2}\cdots a_{np_n} = \sum_{p_1p_2\cdots p_n}(-1)^\tau(a_{1p_1}e_1 + a_{1p_1}e_2)\cdots(a_{np_n}e_1 + a_{np_n}e_2)\\ &= \sum_{p_1p_2\cdots p_n}(-1)^\tau(a_{1p_1}e_1)(a_{2p_2}e_1)\cdots(a_{np_n}e_1) + \sum_{p_1p_2\cdots p_n}(-1)^\tau(a_{1p_1}e_2)(a_{2p_2}e_2)\cdots(a_{np_n}e_2)\\ &= \sum_{p_1p_2\cdots p_n}(-1)^\tau q^1_{1p_1}q^1_{2p_2}\cdots q^1_{np_n}e_1 + \sum_{p_1p_2\cdots p_n}(-1)^\tau q^2_{1p_1}q^2_{2p_2}\cdots q^2_{np_n}e_2\\ &= \det(Q_1)e_1 + \det(Q_2)e_2.\end{aligned}$$

□

By Theorem 4.2, we obtain the following result.

Theorem 4.3. $A = (a_{ij}) \in \mathbb{H}_r^{n\times n}$ *is invertible if and only if* $\det(A) \in \mathbb{H}_r - Z(\mathbb{H}_r)$*. In this case,*

$$A^{-1} = \det(A)^{-1}\mathrm{adj}(A).$$

Therefore we have the following corollary.

Corollary 4.2. *(cf.[26, Section II])* $A = Q_1e_1 + Q_2e_2$ *is invertible if and only if* Q_1 *and* Q_2 *are invertible. If* A *is invertible, then*

$$A^{-1} = Q_1^{-1}e_1 + Q_2^{-1}e_2.$$

Proof. By Theorem 4.3, if $A = Q_1e_1 + Q_2e_2$ is invertible, then $\det(A) = \det(Q_1)e_1 + \det(Q_2)e_2 \in \mathbb{H}_r - Z(\mathbb{H}_r)$. This implies that $\det(Q_1) \neq 0$, $\det(Q_2) \neq 0$ and

$$\det(A)^{-1} = \det(Q_1)^{-1}e_1 + \det(Q_2)^{-1}e_2.$$

It follows from Definition 4.1 that

$$\mathrm{adj}(Q_1e_1 + Q_2e_2) = \mathrm{adj}(Q_1)e_1 + \mathrm{adj}(Q_2)e_2.$$

Therefore

$$A^{-1} = \det(A)^{-1}\mathrm{adj}(A) = \Big(\det(Q_1)^{-1}e_1 + \det(Q_2)^{-1}e_2\Big)\Big(\mathrm{adj}(Q_1)e_1 + \mathrm{adj}(Q_2)e_2\Big).$$

This implies that $A^{-1} = Q_1^{-1}e_1 + Q_2^{-1}e_2$. □

As a by-production, we have the following corollary.

Corollary 4.3. $A^J = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix} \in \mathbb{C}^{2n\times 2m}$ *is invertible if and only if* $Q_1 = A_1 + A_2$ *and* $Q_2 = A_1 - A_2$ *is invertible. In this case, we have*

$$\begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} Q_1^{-1} + Q_2^{-1} & Q_1^{-1} - Q_2^{-1} \\ Q_1^{-1} - Q_2^{-1} & Q_1^{-1} + Q_2^{-1} \end{pmatrix}.$$

Proof. By Proposition 2.2, we have

$$I_{2n} = (AA^{-1})^J = A^J(A^{-1})^J.$$

Hence $(A^J)^{-1} = (A^{-1})^J$. By Corollary 4.2, we have $A^{-1} = Q_1^{-1}e_1 + Q_2^{-1}e_2$. Thus $(A^{-1})^E = \begin{pmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{pmatrix}$. By Proposition 2.3, we have

$$(A^{-1})^J = P(A^{-1})^E P^{-1} = \frac{1}{2}\begin{pmatrix} Q_1^{-1} + Q_2^{-1} & Q_1^{-1} - Q_2^{-1} \\ Q_1^{-1} - Q_2^{-1} & Q_1^{-1} + Q_2^{-1} \end{pmatrix}.$$

□

Theorem 4.4. *Let* $A, B \in \mathbb{H}_r^{n\times n}$. *Then*

$$\det(AB) = \det(A)\det(B).$$

Proof. Let $A = Q_1e_1 + Q_2e_2$ and $B = P_1e_1 + P_2e_2$ with $Q_i, P_i \in \mathbb{C}^{n\times n}$. Then we have

$$AB = Q_1P_1e_1 + Q_2P_2e_2.$$

Note that for complex matrices, we have

$$\det(Q_iP_i) = \det(Q_i)\det(P_i),\ i = 1, 2.$$

By Proposition 4.2, we have

$$\det(AB) = \det(Q_1P_1)e_1 + \det(Q_2P_2)e_2 = \Big(\det(Q_1)e_1 + \det(Q_2)e_2\Big)\Big(\det(P_1)e_1 + \det(P_2)e_2\Big) = \det(A)\det(B).$$

□

Let $A(i \to b)$ be the matrix obtained by replacing the i-th column of A with b.

Theorem 4.5 (Cramer's Rule)**.** *Let* $A = Q_1e_1 + Q_2e_2 \in \mathbb{H}_r^{n\times n}, b = b_1e_1 + b_2e_2 \in \mathbb{H}_r^n$. *The equation* $Ax = b$ *has a unique solution if and only if* $\det(A) \in \mathbb{H}_r - Z(\mathbb{H}_r)$. *In this case, the solution* $x = (x_1, x_2, \cdots, x_n)$ *can be expressed as*

$$x_i = \det(A)^{-1}D_i, i = 1, 2, \cdots, n,$$

where $D_i = \det(A(i \to b))$. *Moreover, we have*

$$x_i = \frac{\det(Q_1(i \to b_1))}{\det(Q_1)}e_1 + \frac{\det(Q_2(i \to b_2)}{\det(Q_2)}e_2.$$

Proof. Let $A = (a_{ij}) = (a_1, a_2, \cdots, a_n)$ and e_i be the i-th column of I_n. Note that

$$AI_n = A(e_1, \cdots, e_{i-1}, e_i, e_{i+1}, \cdots, e_n) = (a_1, \cdots, a_{i-1}, a_i, a_{i+1}, \cdots, a_n). \tag{65}$$

If $Ax = b$ and $\det(A) \in \mathbb{H}_r - Z(\mathbb{H}_r)$, then A is invertible and we have a unique solution $x = A^{-1}b$. Replacing the i-th column with Ax and b in (65), we have

$$A(e_1, \cdots, e_{i-1}, x, e_{i+1}, \cdots, e_n) = (a_1, \cdots, a_{i-1}, b, a_{i+1}, \cdots, a_n).$$

Therefore

$$\det\Big(A(e_1, \cdots, e_{i-1}, x, e_{i+1}, \cdots, e_n)\Big) = \det\Big((a_1, \cdots, a_{i-1}, b, a_{i+1}, \cdots, a_n)\Big).$$

By Theorem 4.1, we have

$$\det\Big((e_1, \cdots, e_{i-1}, x, e_{i+1}, \cdots, e_n)\Big) = x_i.$$

Applying Theorem 4.4, we have

$$\det(A)\det\Big((e_1, \cdots, e_{i-1}, x, e_{i+1}, \cdots, e_n)\Big) = \det\Big((a_1, \cdots, a_{i-1}, b, a_{i+1}, \cdots, a_n)\Big),$$

that is

$$x_i = \det(A)^{-1}D_i, i = 1, 2, \cdots, n.$$

Hence

$$\begin{aligned} x_i &= \det(A)^{-1}D_i = \Big(\frac{1}{\det(Q_1)}e_1 + \frac{1}{\det(Q_2)}e_2\Big)\Big(\det(Q_1(i \to b_1))e_1 + \det(Q_2(i \to b_2))e_2\Big) \\ &= \frac{\det(Q_1(i \to b_1))}{\det(Q_1)}e_1 + \frac{\det(Q_2(i \to b_2))}{\det(Q_2)}e_2. \end{aligned} \tag{66}$$

□

Based on Theorem 4.5, we provide an algorithm to find the solution of linear equation $Ax = b$ as follows.

Algorithm 4.1. *Solve the equation $Ax = b$ by Crame's rule.*

step 1: Input $A \in \mathbb{H}_r^{n\times n}, b \in \mathbb{H}_r^n$. Find the matrix Q_1 and Q_2 and b_1, b_2.

step 2: Calculate

$$x_i = \frac{\det(Q_1(i \to b_1))}{\det(Q_1)}e_1 + \frac{\det(Q_2(i \to b_2))}{\det(Q_2)}e_2, i = 1, \cdots, n.$$

step 3: Output the result $x = (x_1, x_2)^T$.

Example 4.1. *For*

$$A = \begin{pmatrix} 7 + 4\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} & -6 - 6\mathbf{k} \\ 8 + 7\mathbf{i} + 7\mathbf{j} + 8\mathbf{k} & -5 - 3\mathbf{i} - 3\mathbf{j} - 6\mathbf{k} \end{pmatrix}, b = \begin{pmatrix} 3 + \mathbf{i} + 2\mathbf{j} \\ 4 + 5\mathbf{i} + 3\mathbf{k} \end{pmatrix}.$$

By Algorithm 4.1, we have

$$Q_1 = \begin{pmatrix} 11 + 10\mathbf{i} & -6 - 6\mathbf{i} \\ 15 + 15\mathbf{i} & -8 - 9\mathbf{i} \end{pmatrix}; Q_2 = \begin{pmatrix} 3 - 2\mathbf{i} & -6 + 6\mathbf{i} \\ 1 - \mathbf{i} & -2 + 3\mathbf{i} \end{pmatrix}, b_1 = \begin{pmatrix} 5 + \mathbf{i} \\ 4 + 8\mathbf{i} \end{pmatrix}, b_2 = \begin{pmatrix} 1 + \mathbf{i} \\ 4 + 2\mathbf{i} \end{pmatrix}.$$

$$\det(Q_1) = 2 + \mathbf{i},\ \det(Q_2) = \mathbf{i};$$

$$\det(Q_1(1 \to b_1)) = -55 + 19\mathbf{i},\ \det(Q_2(1 \to b_2)) = 33 - 11\mathbf{i};$$

$$\det(Q_1(2 \to b_1)) = -96 + 38\mathbf{i},\ \det(Q_2(2 \to b_2)) = 14 - 2\mathbf{i}.$$

$$x_1 = (-18.2 + 18.6\mathbf{i})e_1 + (-11 - 31\mathbf{i})e_2,\ x_2 = (-30.8 + 34.4\mathbf{i})e_1 + (-2 - 14\mathbf{i})e_2.$$

Therefore the solution is

$$\begin{aligned} x &= ((-18.2 + 18.6\mathbf{i})e_1 + (-11 - 31\mathbf{i})e_2, (-30.8 + 34.4\mathbf{i})e_1 + (-2 - 14\mathbf{i})e_2)^T \\ &= (-14.6 - 6.2\mathbf{i} - 3.6\mathbf{j} + 24.8\mathbf{k}, -16.4 + 10.2\mathbf{i} - 14.4\mathbf{j} + 24.2\mathbf{k})^T. \end{aligned}$$

5. The Moore-Penrose inverse

We recall that the Moore-Penrose inverse [3] of $A \in \mathbb{C}^{n\times m}$ is the unique complex matrix X satisfying the following equations:

$$AXA = A, XAX = X, (AX)^* = AX, (XA)^* = XA,$$

where * is the conjugate transpose of a matrix. We denote the Moore-Penrose inverse of A by $A^\dagger$.

For $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = c_1 + c_2\mathbf{j}$, we call

$$q^* = q_0 - q_1\mathbf{i} + q_2\mathbf{j} - q_3\mathbf{k} = \overline{c_1} + \overline{c_2}\mathbf{j} \tag{67}$$

the complex conjugate of q. Then we have

$$q^*q = qq^* = q_0^2 + q_1^2 + q_2^2 + q_3^2 + 2(q_0q_2 + q_1q_3)\mathbf{j} = |c_1|^2 + |c_2|^2 + (c_1\overline{c_2} + c_2\overline{c_1})\mathbf{j}.$$

For $A = (a_{ij}) \in \mathbb{H}_r^{n\times m}$, let A^* be the complex conjugate transpose of A. The following proposition can be verified directly.

Proposition 5.1.

$$e_1^* = e_1,\ e_2^* = e_2; \tag{68}$$

$$(pq)^* = q^*p^* = p^*q^*,\ \forall p, q \in H_r; \tag{69}$$

$$(AB)^* = B^*A^*,\ \forall A \in \mathbb{H}_r^{n\times m},\ B \in \mathbb{H}_r^{m\times s}. \tag{70}$$

Cao and Tang have given the following definition of reduced biquaternions.

Definition 5.1. *([4, Definition 2.6]) The Moore-Penrose inverse of $a = c_1 + c_2\mathbf{j} \in \mathbb{H}_r$ is defined to be*

$$a^\dagger = \begin{cases} 0, & \text{if } a=0; \\ \frac{\bar{a}}{N(a)}, & \text{if } c_1 \pm c_2 \neq 0; \\ \frac{a^*}{4|c_1|^2}, & \text{if } c_1 + c_2 = 0 \text{ or } c_1 - c_2 = 0. \end{cases}$$

Note that for $a = c_1 + c_1\mathbf{j} = 2c_1e_1 \neq 0$,

$$a^\dagger = \frac{1}{2c_1}e_1$$

and for $a = c_1 - c_1\mathbf{j} = 2c_1e_2 \neq 0$,

$$a^\dagger = \frac{1}{2c_1}e_2.$$

As in the complex case, we have the following theorem.

Theorem 5.1. *Let $A = Q_1e_1 + Q_2e_2 \in \mathbb{H}_r^{n\times m}$. Then there exists a unique $X \in \mathbb{H}_r^{m\times n}$ satisfying the following equation:*

$$AXA = A, XAX = X, (AX)^* = AX, (XA)^* = XA. \tag{71}$$

Proof. Let $Q_1^\dagger$ and $Q_2^\dagger$ be the Moore-Penrose inverses of Q_1 and Q_2, respectively. Let

$$X = Q_1^\dagger e_1 + Q_2^\dagger e_2. \tag{72}$$

Then we have

$$AXA = (Q_1e_1 + Q_2e_2)(Q_1^\dagger e_1 + Q_2^\dagger e_2)(Q_1e_1 + Q_2e_2) = Q_1Q_1^\dagger Q_1e_1 + Q_2Q_2^\dagger Q_2e_2 = A$$

and $AX = Q_1Q_1^\dagger e_1 + Q_2Q_2^\dagger e_2$. Hence

$$(AX)^* = (Q_1Q_1^\dagger e_1 + Q_2Q_2^\dagger e_2)^* = AX.$$

Similarly, we have $XAX = X, (XA)^* = XA$.

If there exists a $Y \in \mathbb{H}_r^{m\times n}$ such that

$$AYA = A, YAY = Y, (AY)^* = AY, (YA)^* = YA. \tag{73}$$

Then

$$\begin{aligned} Y &= YAY = Y(AXA)Y = Y(AX)(AY) = Y(AX)^*(AY)^* \\ &= Y(AYAX)^* = Y(AX)^* = Y(AX) = YAXAX = (YA)^*(XA)^*X \\ &= (XAYA)^*X = (XA)^*X = XAX = X. \end{aligned}$$

□

Based on the above theorem, we introduce the Moore-Penrose inverse in reduced biquaternion matrices.

Definition 5.2. *For $A = Q_1e_1 + Q_2e_2 \in \mathbb{H}_r^{n\times m}$, the Moore-Penrose inverse of A is defined to be*

$$A^\dagger = Q_1^\dagger e_1 + Q_2^\dagger e_2. \tag{74}$$

We remark that in the case of A being a reduced biquaternion, the above definition is the same as Definition 5.1. Also, when we restrict our Definition 5.2 in complex case, it is identical to the Moore-Penrose inverse of complex matrices.

Theorem 5.2. *Let $A \in \mathbb{H}_r^{n\times m}, b \in \mathbb{H}_r^n$. The linear equation $Ax = b$ is solvable if and only if*

$$AA^\dagger b = b. \tag{75}$$

In this case, the general solution can be expressed as

$$x = A^\dagger b + (I_m - A^\dagger A)z, \ \forall z \in \mathbb{H}_r^m. \tag{76}$$

Proof. If the linear equation $Ax = b$ has a solution y, then $b = Ay$. Therefore $AA^\dagger b = AA^\dagger Ay = Ay = b$. Also, if $AA^\dagger b = b$, then $x = A^\dagger b$ is a solution of $Ax = b$.

By the definition of the Moore-Penrose inverse, each x given by (76) is a solution of $Ax = b$. For any solution x of the equation $Ax = b$, it is obvious that $x = A^\dagger b + (I_m - A^\dagger A)x$. That is any solution can be expressed as the form of (76). This concludes the proof. □

We remark that for $A = Q_1e_1 + Q_2e_2$, the condition $AA^\dagger b = b$ is equivalent to the condition

$$Q_1Q_1^\dagger b_1 = b_1, \ Q_2Q_2^\dagger b_2 = b_2. \tag{77}$$

Theorem 5.3. *Let $A \in \mathbb{H}_r^{n\times m}, B \in \mathbb{H}_r^{s\times t}, C \in \mathbb{H}_r^{n\times t}$. The matrix equation $AXB = C$ is solvable if and only if*

$$AA^\dagger CB^\dagger B = C. \tag{78}$$

In this case, the general solution can be expressed as

$$X = A^\dagger CB^\dagger + Y - A^\dagger AYBB^\dagger, \ \forall \, Y \in \mathbb{H}_r^{m\times s}. \tag{79}$$

Proof. If the linear equation $AXB = C$ has a solution Y, then $C = AYB$. Therefore

$$AA^\dagger CB^\dagger B = AA^\dagger(AYB)B^\dagger B = AA^\dagger AYBB^\dagger B = AYB = C. \tag{80}$$

On the other hand, if $AA^\dagger CB^\dagger B = C$, then $X = A^\dagger CB^\dagger$ is a solution of $AXB = C$.

By the definition of the Moore-Penrose inverse, for each X given by (79) , we have

$$AXB = A(A^\dagger CB^\dagger + Y - A^\dagger AYBB^\dagger)B = AA^\dagger CB^\dagger B + AYB - AA^\dagger AYBB^\dagger B = C.$$

That is X is a solution of $AXB + C$ for each specific Y. For any solution X of the equation $AXB = C$, it is obvious that

$$X = A^\dagger CB^\dagger + X - A^\dagger AXBB^\dagger.$$

That is any solution can be expressed as the form of (79). This concludes the proof. □

Based on Theorem 5.2 and Definition 5.2, we provide an algorithm to solve equation $Ax = b$ by using of the Moore Penrose inverse of $A \in \mathbb{H}_r^{n\times m}$ as follows.

Algorithm 5.1. *Solve the equation $Ax = b$ by using of Moore-Penrose inverse.*

step 1: Input $A \in \mathbb{H}_r^{n\times m}, b \in \mathbb{H}_r^{n}$. Find the matrices Q_1 and Q_2 and b_1, b_2.

step 2: Calculate the Moore Penrose inverse of complex matrices Q_1 and Q_2, and obtain $A^\dagger = Q_1^\dagger e_1 + Q_2^\dagger e_2$.

step 3: If the condition $AA^\dagger b = b$ is satisfied, then go to step 4; otherwise the equation $Ax = b$ is unsolvable.

step 4: Calculate $A^\dagger b$ and $I_m - A^\dagger A$, output the result $x = A^\dagger b + (I_m - A^\dagger A)z$.

Example 5.1. *For*

$$A = \begin{pmatrix} 1+\mathbf{i}+\mathbf{k} & 1+2\mathbf{i}-\mathbf{j} & 3+2\mathbf{i}+2\mathbf{k} \\ 2+3\mathbf{i}-\mathbf{j}+\mathbf{k} & 4+\mathbf{i}-\mathbf{j}-\mathbf{k} & 1+3\mathbf{i}-\mathbf{j}+2\mathbf{k} \end{pmatrix}, b = \begin{pmatrix} 6+14\mathbf{i}-5\mathbf{j}-\mathbf{k} \\ 6+26\mathbf{i}-10\mathbf{j}-9\mathbf{k} \end{pmatrix}.$$

By Algorithm 5.1, we have

$$Q_1 = \begin{pmatrix} 1+2\mathbf{i} & 2\mathbf{i} & 3+4\mathbf{i} \\ 1+4\mathbf{i} & 3 & 5\mathbf{i} \end{pmatrix}; Q_2 = \begin{pmatrix} 1 & 2+2\mathbf{i} & 3 \\ 3+2\mathbf{i} & 5+2\mathbf{i} & 2+\mathbf{i} \end{pmatrix}, b_1 = \begin{pmatrix} 1+13\mathbf{i} \\ -4+17\mathbf{i} \end{pmatrix}, b_2 = \begin{pmatrix} 11+15\mathbf{i} \\ 16+35\mathbf{i} \end{pmatrix}.$$

$$Q_1^\dagger = \begin{pmatrix} -0.0285+0.0039\mathbf{i} & 0.0350-0.0868\mathbf{i} \\ -0.1127-0.1749\mathbf{i} & 0.1606+0.0751\mathbf{i} \\ 0.1269-0.0764\mathbf{i} & -0.0557-0.0272\mathbf{i} \end{pmatrix},$$

$$Q_2^\dagger = \begin{pmatrix} -0.0633+0.1551\mathbf{i} & 0.0981-0.1171\mathbf{i} \\ -0.0601-0.1361\mathbf{i} & 0.1329+0.0253\mathbf{i} \\ 0.3038+0.0791\mathbf{i} & -0.1044-0.0665\mathbf{i} \end{pmatrix}.$$

We can verified that

$$Q_1Q_1^\dagger b_1 = b_1, Q_2Q_2^\dagger b_2 = b_2.$$

$$Q_1^\dagger b_1 = \begin{pmatrix} 1.2565+0.5751\mathbf{i} \\ 0.2409+0.7902\mathbf{i} \\ 1.8057+0.7358\mathbf{i} \end{pmatrix}, Q_2^\dagger b_2 = \begin{pmatrix} 2.6456+2.3165\mathbf{i} \\ 2.6203+2.6582\mathbf{i} \\ 2.8101+0.7089\mathbf{i} \end{pmatrix}.$$

$$I_3 - Q_1^\dagger Q_1 = \begin{pmatrix} 0.6541 & -0.0972+0.3174\mathbf{i} & -0.3329-0.0725\mathbf{i} \\ -0.0972-0.3174\mathbf{i} & 0.1684 & 0.0142+0.1723\mathbf{i} \\ -0.3329+0.0725\mathbf{i} & 0.0142-0.1723\mathbf{i} & 0.1775 \end{pmatrix}$$

$$I_3 - Q_2^\dagger Q_2 = \begin{pmatrix} 0.5348 & -0.2880+0.2057\mathbf{i} & -0.1234-0.3291\mathbf{i} \\ -0.2880-0.2057\mathbf{i} & 0.2342 & -0.0601+0.2247\mathbf{i} \\ -0.1234+0.3291\mathbf{i} & -0.0601-0.2247\mathbf{i} & 0.2310 \end{pmatrix}$$

Thus the solutions x can be expressed as

$$x = \Big(Q_1^\dagger b_1 + (I_3 - Q_1^\dagger Q_1)z\Big)e_1 + \Big(Q_2^\dagger b_2 + (I_3 - Q_2^\dagger Q_2)w\Big)e_2, \forall z, w \in \mathbb{C}^3.$$

Particularly, if we take $z = (1, \mathbf{i}, 2+\mathbf{i})^T$ and $w = (1+\mathbf{i}, 3+4\mathbf{i}, 4)^T$ in the above formula, then

$$x = \begin{pmatrix} 1 \\ \mathbf{i} \\ 2+\mathbf{i} \end{pmatrix} e_1 + \begin{pmatrix} 1+\mathbf{i} \\ 3+4\mathbf{i} \\ 4 \end{pmatrix} e_2 = \begin{pmatrix} 1+\frac{1}{2}\mathbf{i}-\frac{1}{2}\mathbf{k} \\ \frac{3}{2}+\frac{5}{2}\mathbf{i}-\frac{3}{2}\mathbf{j}-\frac{3}{2}\mathbf{k} \\ 3+\frac{1}{2}\mathbf{i}-\mathbf{j}+\frac{1}{2}\mathbf{k} \end{pmatrix}$$

is a solution of $Ax = b$.

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