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Nonlinear maps preserving mixed products on von Neumann algebras

Meilian Gao[a], Xingpeng Zhao[a,*], Kaimin Teng[a]

[a]*Department of Mathematics, Taiyuan University of Technology, China.*

Abstract. Let $\mathcal{M}$ be a von Neumann algebra with no central summands of type I_1 and $\Phi : \mathcal{M} \to \mathcal{M}$ be a nonlinear bijective map preserving mixed products satisfying that $\Phi([a \bullet b, c]) = [\Phi(a) \bullet \Phi(b), \Phi(c)]$ for all $a, b, c \in \mathcal{M}$. Then there exists $z \in \mathcal{Z}_{\mathcal{M}}$ with $z^2 = I$ such that Φ is of the form $\Phi = z\Psi$, where $\Psi : \mathcal{M} \to \mathcal{M}$ is the sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism.

1. Introduction and preliminaries

Let $\mathcal{A}$ be a $*$-algebra over the complex number field $\mathbb{C}$. For all $a, b \in \mathcal{M}$, define the Lie product $[a, b] = ab - ba$, the skew Lie product $[a, b]_* = ab - ba^*$ and the jordan $*$-product $a \bullet b = ab + ba^*$. Recently, inspired by the question that when a multiplicative map is additive raised by Martindale [1], more and more authors are committed to the research on product preserving problems on certain algebras, including corresponding 2-local mappings. For example, we can refer to [2–9] on Lie product preserving problems, [10–12] on skew Lie product preserving problems and [13, 14] on jordan $*$-products preserving problems.

Recently, nonlinear maps preserving the products of the mixture of (skew) Lie products and Jordan $*$-product have received a fair amount of attention. We can refer to [15–20]. For example, Let $\mathcal{A}$ and $\mathcal{B}$ be two factors with $\dim\mathcal{A} > 4$. Zhao, Li and Chen [15] give the characterization of a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfying $\Phi([a \bullet b, c]) = [\Phi(a) \bullet \Phi(b), \Phi(c)]$ for all $a, b, c \in \mathcal{A}$. They proved that Φ is a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism.

In the article, we shall study nonlinear maps discussed in [15] between von Neumann algebras with no central summands of type I_1. Due to the significant differences in the properties of factors and von Neumann algebras with no central summands of type I_1, we need to adopt different methods and techniques to prove the main result. Let $\mathcal{M}$ be a von Neumann algebra with no central summands of type I_1 and $\Phi : \mathcal{M} \to \mathcal{M}$ be a nonlinear bijective map preserving mixed products satisfying that $\Phi([a \bullet b, c]) = [\Phi(a) \bullet \Phi(b), \Phi(c)]$ for all $a, b, c \in \mathcal{M}$. Then we show that there exists $z \in \mathcal{Z}_{\mathcal{M}}$ with $z^2 = I$ such that Φ is of the form $\Phi = z\Psi$, where $\Psi : \mathcal{M} \to \mathcal{M}$ is the sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism.

Before embarking on the proof, we need some notations and preliminaries. Let $\mathcal{H}$ be a complex separable Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be

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* Corresponding author: Xingpeng Zhao
Email addresses: gaomeilian0402@sina.com (Meilian Gao), zhaoxingpeng@tyut.edu.cn (Xingpeng Zhao), tengkaimin2013@163.com (Kaimin Teng)

a von Neumann algebra. The set $\mathcal{Z}_{\mathcal{M}}=\{s \in \mathcal{M} \big| st = ts \text{ for all } t \in \mathcal{M}\}$ is called the center of $\mathcal{M}$. For $a \in \mathcal{M}$, the center carrier of a, denoted by $\overline{a}$, is the intersection of all central projections $p \in \mathcal{M}$ such that $pa = a$. It is well known that the central carrier of a is the projection with the range $[\mathcal{M}a(\mathcal{H})]$, the closed linear span of $\{ma(x) \big| m \in \mathcal{M}, x \in \mathcal{H}\}$. For each self-adjoint operator $a \in \mathcal{M}$, we define the core of a, denoted by $\underline{a}$, to be $\sup\{s \in \mathcal{Z}_{\mathcal{M}} \big| s = s^*, s \le a\}$. If p is a projection and $\underline{p} = 0$, we call p a core-free projection. Clearly, one has $a - \underline{a} \ge 0$. Further if $s \in \mathcal{Z}_{\mathcal{M}}$ and $a - \underline{a} \ge S \ge 0$, then $s = 0$. If p is a projection, it is clear that $\underline{p}$ is the largest central projection$\le p$. It is to see that $\underline{p} = 0$ if and only if $\overline{I - p} = I$, where $\overline{I - p}$ denotes the central carrier of $I - p$. To complete the proof of the main theorem, we will use frequently several fundamental properties of von Neumann algebras. We list them in the following proposition.

Proposition 1.1. *[5, 21–23] Let $\mathcal{M}$ be a von Neumann algebra.*

(i)If p is a projection, then $\mathcal{Z}_{p\mathcal{M}p} = p\mathcal{Z}_{\mathcal{M}}$.

(ii) If $\mathcal{M}$ has no central summands of type I_1, then each nonzero central projection of $\mathcal{M}$ is the central carrier of a core-free projection of $\mathcal{M}$.

(iii)If p is a core-free projection in $\mathcal{M}$, then $p\mathcal{M}p \cap \mathcal{Z}_{\mathcal{M}} = \{0\}$.

(iv) If $t \in \mathcal{M}$ and p is a projection in $\mathcal{M}$ with $\overline{p} = I$, then $tmp = 0$ for all $m \in \mathcal{M}$ implies $t = 0$. Consequently, if $z \in \mathcal{Z}_{\mathcal{M}}$, then $zp = 0$ implies $z = 0$.

(v)If $\mathcal{M}$ is a von Neumann algebra with no central summands of type I_1 and $c \in \mathcal{Z}_{\mathcal{M}}$ such that $c\mathcal{M} \subseteq \mathcal{Z}_{\mathcal{M}}$, then $c = 0$.

By Proposition 1.1 (ii), if $\mathcal{M}$ has no central summands of type I_1, then there exists a core-free projection with central carrier I, denoted by p_1, that is $\overline{p_1} = I$ and $\underline{p_1} = 0$. Clearly, $p_1 \ne 0, I$. Throughout the article, p_1 is fixed. Denote $p_2 = I - p_1$. By the definition of core and central carrier, p_2 is core-free and $\overline{p_2} = I$. Denote $\mathcal{M}_{ij} = p_i\mathcal{M}p_j, i, j = 1, 2$. Then we may write $\mathcal{M} = \mathcal{M}_{11} + \mathcal{M}_{12} + \mathcal{M}_{21} + \mathcal{M}_{22}$. And for each element $t \in \mathcal{M}$, we may write $t = \sum_{i,j=1}^{2} t_{ij}$. In all that follows, when we write t_{ij}, it indicates that $t_{ij} \in \mathcal{M}_{ij}$.

In addition, the following conclusion will play an important role in our proof of the main result.

Proposition 1.2. *[23] Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras with no central summands of type I_1 or I_2. Let θ be a bijective additive mapping. If θ preserves commutativity in both directions then it is of the form*

$$\theta(x) = c\varphi(x) + f(x)$$

where c is an invertible element in $\mathcal{Z}_{\mathcal{N}}$, $\varphi : \mathcal{M} \to \mathcal{N}$ is a jordan isomorphism of $\mathcal{M}$ onto $\mathcal{N}$ and f is an additive mapping of $\mathcal{M}$ into $\mathcal{Z}_{\mathcal{N}}$.

By using the proof method of Proposition 1.2 in [24], we can obtain the following result.

Proposition 1.3. *Let $\mathcal{M}$ be von Neumann algebras with no central summands of type I_1. If $a_{11}b_{12} + b_{12}a_{22} = 0$ for all $b_{12} \in \mathcal{M}_{12}$, there exists $z \in \mathcal{Z}_{\mathcal{M}}$ such that $a_{11} = zp_1$ and $a_{22} = -zp_2$.*

Proposition 1.4. *Let $\mathcal{M}$ be von Neumann algebras with no central summands of type I_1. If $[a, b] = z \in \mathcal{Z}_{\mathcal{M}}$ for all $a, b \in \mathcal{M}$, then $z = 0$.*

Throughout the article, $\mathcal{Z}_{\mathcal{A}}$ and $\mathcal{A}_{sa}$ denote the center of $\mathcal{A}$ and the set of self-adjoint operators of an algebra $\mathcal{A}$ respectively.

2. Additivity

Let $\mathcal{M}$ be a von Neumann algebra with no central summands of type I_1 and $\Phi : \mathcal{M} \to \mathcal{M}$ be a nonlinear bijective map preserving mixed products satisfying that $\Phi([a \bullet b, c]) = [\Phi(a) \bullet \Phi(b), \Phi(c)]$ for all $a, b, c \in \mathcal{M}$. In this section, we will first consider the additivity of Φ. The main result reads as follows.

Theorem 2.1. *Let $\mathcal{M}$ be a von Neumann algebra with no central summands of type I_1 and $\Phi : \mathcal{M} \to \mathcal{M}$ be a nonlinear bijective map preserving mixed products satisfying that $\Phi([a \bullet b, c]) = [\Phi(a) \bullet \Phi(b), \Phi(c)]$ for all $a, b, c \in \mathcal{M}$. Then Φ is additive.*

In the following, we will prove Theorem 2.1 by checking several Lemmas.

Lemma 2.2. $\Phi(0) = 0$.

Proof. Since Φ is surjective, there exists $a \in \mathcal{M}$ such that $\Phi(a) = 0$. It follows that

$$\Phi(0) = \Phi([0 \bullet 0, a]) = [\Phi(0) \bullet \Phi(0), \Phi(a)] = [\Phi(0) \bullet \Phi(0), 0] = 0.$$

□

Lemma 2.3. $\Phi(a_{12} + b_{21}) = \Phi(a_{12}) + \Phi(b_{21})$ *for all* $a_{12} \in \mathcal{M}_{12}$ *and* $b_{21} \in \mathcal{M}_{21}$.

Proof. Denote $t = a_{12} + b_{21} - \Phi^{-1}(\Phi(a_{12}) + \Phi(b_{21}))$. It follows from $[a_{12} \bullet p_1, p_1] = [b_{21} \bullet p_2, p_2] = 0$ for all $a_{12} \in \mathcal{M}_{12}$ and $b_{21} \in \mathcal{M}_{21}$ and Lemma 2.2 that

$$\begin{aligned}[\Phi(a_{12} + b_{21}) \bullet \Phi(p_1), \Phi(p_1)] =&\Phi([a_{12} + b_{21} \bullet p_1, p_1])\\ =&\Phi([a_{12} \bullet p_1, p_1]) + \Phi([b_{21} \bullet p_1, p_1])\\ =&[\Phi(a_{12}) + \Phi(b_{21}) \bullet \Phi(p_1), \Phi(p_1)]\end{aligned}$$

and

$$\begin{aligned}[\Phi(a_{12} + b_{21}) \bullet \Phi(p_2), \Phi(p_2)] =&\Phi([a_{12} + b_{21} \bullet p_2, p_2])\\ =&\Phi([a_{12} \bullet p_2, p_2]) + \Phi([b_{21} \bullet p_2, p_2])\\ =&[\Phi(a_{12}) + \Phi(b_{21}) \bullet \Phi(p_2), \Phi(p_2)].\end{aligned}$$

Then we have $\Phi([t \bullet p_1, p_1]) = [\Phi(t) \bullet \Phi(p_1), \Phi(p_1)] = 0$ and $\Phi([t \bullet p_2, p_2]) = [\Phi(t) \bullet \Phi(p_2), \Phi(p_2)] = 0$. Thus $[t \bullet p_1, p_1] = [t \bullet p_2, p_2] = 0$ and then $t_{12} = t_{21} = 0$.

For every $c_{kl} \in \mathcal{M}_{kl}$ for $1 \le k \ne l \le 2$, we have from $[c_{kl} \bullet a_{12}, p_k] = [c_{kl} \bullet b_{21}, p_k] = 0$ that

$$\begin{aligned}[\Phi(c_{kl}) \bullet \Phi(a_{12} + b_{21}), \Phi(p_k)] =&\Phi([c_{kl} \bullet (a_{12} + b_{21}), p_k])\\ =&\Phi([c_{kl} \bullet a_{12}, p_k]) + \Phi([c_{kl} \bullet b_{21}, p_k])\\ =&[\Phi(c_{kl}) \bullet (\Phi(a_{12}) + \Phi(b_{21})), \Phi(p_k)].\end{aligned}$$

Thus $\Phi([c_{kl} \bullet t, p_k]) = [\Phi(c_{kl}) \bullet \Phi(t), \Phi(p_k)] = 0$ and then $[c_{kl} \bullet t, p_k] = 0$, which implies that $c_{kl}t_{ll} = 0$ for all $c_{kl} \in \mathcal{M}_{kl}$. It follows from Proposition 1.1 (iv) that $t_{ll} = 0$ for $l = 1, 2$. Therefore, we have $t = 0$. The proof is completed. □

Lemma 2.4. *For all* $a_{11} \in \mathcal{M}_{11}$, $b_{12} \in \mathcal{M}_{12}$, $c_{21} \in \mathcal{M}_{21}$ *and* $d_{22} \in \mathcal{M}_{22}$, *we have*

$$\Phi(a_{11} + b_{12} + c_{21} + d_{22}) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21}) + \Phi(d_{22}).$$

Proof. Denote $t = a_{11} + b_{12} + c_{21} + d_{22} - \Phi^{-1}(\Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21}) + \Phi(d_{22}))$. Noticing that $\Phi([p_1 \bullet a_{11}, p_2]) = \Phi([p_1 \bullet d_{22}, p_2]) = 0$, it follows from Lemmas 2.1 and 2.3 that

$$\begin{aligned}&[\Phi(p_1) \bullet \Phi(a_{11} + b_{12} + c_{21} + d_{22}), \Phi(p_2)]\\ =&\Phi([p_1 \bullet (a_{11} + b_{12} + c_{21} + d_{22}), p_2])\\ =&\Phi([p_1 \bullet (b_{12} + c_{21}), p_2])\\ =&\Phi([p_1 \bullet (b_{12} + c_{21}), p_2]) + \Phi([p_1 \bullet a_{11}, p_2]) + \Phi([p_1 \bullet d_{22}, p_2])\\ =&[\Phi(p_1) \bullet (\Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21}) + \Phi(d_{22})), \Phi(p_2)],\end{aligned}$$

which implies that $[p_1 \bullet t, p_2] = 0$ and then $p_1 t p_2 = p_2 t p_1 = 0$.

On the other hand, for all $e_{ij} \in \mathcal{M}_{ij}$ with $i \neq j$, we obtain that

$$\begin{aligned}
&[\Phi(e_{12}) \bullet \Phi(a_{11} + b_{12} + c_{21} + d_{22}), \Phi(p_1)] \\
=&\Phi([e_{12} \bullet (a_{11} + b_{12} + c_{21} + d_{22}), p_1]) \\
=&\Phi([e_{12} \bullet d_{22}, p_1]) \\
=&\Phi([e_{12} \bullet d_{22}, p_1]) + \Phi([e_{12} \bullet a_{11}, p_1]) + \Phi([e_{12} \bullet b_{12}, p_1]) + \Phi([e_{12} \bullet c_{21}, p_1]) \\
=&[\Phi(e_{12}) \bullet (\Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21}) + \Phi(d_{22})), \Phi(p_1)]
\end{aligned}$$

and

$$\begin{aligned}
&[\Phi(e_{21}) \bullet \Phi(a_{11} + b_{12} + c_{21} + d_{22}), \Phi(p_2)] \\
=&\Phi([e_{21} \bullet (a_{11} + b_{12} + c_{21} + d_{22}), p_2]) \\
=&\Phi([e_{21} \bullet a_{11}, p_2]) \\
=&\Phi([e_{21} \bullet d_{22}, p_2]) + \Phi([e_{21} \bullet a_{11}, p_2]) + \Phi([e_{21} \bullet b_{12}, p_1]) + \Phi([e_{21} \bullet c_{21}, p_2]) \\
=&[\Phi(e_{21}) \bullet (\Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21}) + \Phi(d_{22})), \Phi(p_2)].
\end{aligned}$$

Then $[e_{ij} \bullet t, p_i] = 0$. Thus $e_{ij} t p_j = 0$ for all $e_{ij} \in \mathcal{M}_{ij}$ with $i \neq j$. It follows from $\overline{p_i} = I$ and Proposition 1.1(iv) that $p_j t p_j = 0$ for $j = 1, 2$. In all, we have $t = 0$. The proof is completed. □

Lemma 2.5. *$\Phi(a_{ij} + b_{ij}) = \Phi(a_{ij}) + \Phi(b_{ij})$ for all $a_{ij}, b_{ij} \in \mathcal{M}_{ij}$ with $i \neq j$.*

Proof. It follows from Lemma 2.4 that

$$\begin{aligned}
&\Phi(a_{ij} + b_{ij}) = \Phi([\frac{I}{2} \bullet (p_i + a_{ij}), p_j + b_{ij}]) \\
=&[\Phi(\frac{I}{2}) \bullet \Phi(p_i + a_{ij}), \Phi(p_j + b_{ij})] \\
=&[\Phi(\frac{I}{2}) \bullet \Phi(p_i) + \Phi(a_{ij}), \Phi(p_j) + \Phi(b_{ij})] \\
=&\Phi([\frac{I}{2} \bullet p_i, p_j]) + \Phi([\frac{I}{2} \bullet p_i, b_{ij}]) + \Phi([\frac{I}{2} \bullet a_{ij}, p_j]) + \Phi([\frac{I}{2} \bullet a_{ij}, b_{ij}]) \\
=&\Phi(a_{ij}) + \Phi(b_{ij}).
\end{aligned}$$

□

Lemma 2.6. *$\Phi(a_{ii} + b_{ii}) = \Phi(a_{ii}) + \Phi(b_{ii})$ for all $a_{ii}, b_{ii} \in \mathcal{M}_{ii}$.*

Proof. Denote $t = a_{ii} + b_{ii} - \Phi^{-1}(\Phi(a_{ii}) + \Phi(b_{ii}))$. It follows from Lemmas 2.4 and 2.5 that

$$\begin{aligned}
&[\Phi(c_{ji}) \bullet \Phi(a_{ii} + b_{ii}), \Phi(p_i)] \\
=&\Phi([c_{ji} \bullet (a_{ii} + b_{ii}), p_i]) \\
=&\Phi(c_{ji} a_{ii}) + \Phi(c_{ji} b_{ii}) - \Phi(a_{ii} c_{ji}^*) - \Phi(b_{ii} c_{ji}{}^*) \\
=&\Phi([c_{ji} \bullet a_{ii}, p_i]) + \Phi([c_{ji} \bullet b_{ii}, p_i]) \\
=&[\Phi(c_{ji}) \bullet \Phi(a_{ii}), \Phi(p_i)] + [\Phi(c_{ji}) \bullet \Phi(b_{ii}), \Phi(p_i)] \\
=&[\Phi(c_{ji}) \bullet (\Phi(a_{ii}) + \Phi(b_{ii})), \Phi(p_i)]
\end{aligned}$$

for any $c_{ji} \in \mathcal{M}_{ji}$ with $i \neq j$. It follows that $[c_{ji} \bullet t, p_i] = 0$. That is $c_{ji} t p_i - p_i t c_{ji}^* = 0$. Thus $c_{ji} t p_i = 0$ for any $c_{ji} \in \mathcal{M}_{ji}$ with $i \neq j$. It follows from $\overline{p_j} = I$ and Proposition 1.1(iv) that $p_i t p_i = 0$ for $i = 1, 2$.

On the other hand, it is obvious that

$$\begin{aligned}&[\Phi(p_i)\bullet\Phi(a_{ii}+b_{ii}),\Phi(p_i)]\\=&\Phi([p_i\bullet(a_{ii}+b_{ii}),p_i])\\=&\Phi([p_i\bullet a_{ii},p_i])+\Phi([p_i\bullet b_{ii},p_i])\\=&[\Phi(p_i)\bullet(\Phi(a_{ii})+\Phi(b_{ii})),\Phi(p_i)]).\end{aligned}$$

Thus $[p_i\bullet t,p_i]=0$, which implies that $t_{12}=t_{21}=0$. In all, we have $t=0$. The proof is completed. □

Up to now, we give the proof of Theorem 2.1 in the following.

Proof. For any $a=\sum_{i,j=1}^{2}a_{ij}$ and $b=\sum_{i,j=1}^{2}b_{ij}$, where $a_{ij},b_{ij}\in\mathcal{M}_{ij}$, it follows from Lemmas 2.4, 2.5 and 2.6 that

$$\begin{aligned}\Phi(a+b)=&\Phi(\sum_{i,j=1}^{2}a_{ij}+\sum_{i,j=1}^{2}b_{ij})=\Phi(\sum_{i,j=1}^{2}(a_{ij}+b_{ij}))\\=&\sum_{i,j=1}^{2}\Phi(a_{ij}+b_{ij})=\sum_{i,j=1}^{2}(\Phi(a_{ij})+\Phi(b_{ij}))\\=&\Phi(\sum_{i,j=1}^{2}a_{ij})+\Phi(\sum_{i,j=1}^{2}b_{ij})=\Phi(a)+\Phi(b).\end{aligned}$$

□

3. Structure

In this section, we shall study the characterization of Φ mentioned in Theorem 2.1. The main result reads as follows.

Theorem 3.1. *Let $\mathcal{M}$ be a von Neumann algebra with no central summands of type I_1 and $\Phi:\mathcal{M}\to\mathcal{M}$ be a nonlinear bijective map preserving mixed products satisfying that $\Phi([a\bullet b,c])=[\Phi(a)\bullet\Phi(b),\Phi(c)]$ for all $a,b,c\in\mathcal{M}$. Then there exists $z\in\mathcal{Z}_{\mathcal{M}}$ with $z^2=I$ such that Φ is of the form $\Phi=z\Psi$, where $\Psi:\mathcal{M}\to\mathcal{M}$ is the sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism.*

In the following, we will prove Theorem 3.1 by checking several lemmas.

Lemma 3.2. $\Phi(\mathcal{Z}_{\mathcal{M}})=\mathcal{Z}_{\mathcal{M}}$.

Proof. Since Φ is surjective, there exists $b\in\mathcal{M}$ such that $\Phi(b)=I$. Then for all $z\in\mathcal{Z}_{\mathcal{M}}$, we have

$$0=\Phi([b\bullet c,z])=[\Phi(b)\bullet\Phi(c),\Phi(z)]=2[\Phi(c),\Phi(z)]$$

for all $c\in\mathcal{M}$. It follows from the surjectivity of Φ that $\Phi(z)\in\mathcal{Z}_{\mathcal{M}}$, which means that $\Phi(\mathcal{Z}_{\mathcal{M}})\subseteq\mathcal{Z}_{\mathcal{M}}$. By considering Φ^{-1}, we can obtain that $\Phi(\mathcal{Z}_{\mathcal{M}})=\mathcal{Z}_{\mathcal{M}}$. □

Lemma 3.3. *There exists an element $z\in\mathcal{Z}_{\mathcal{M}}$ with $z^2=I$ such that*

$$\Phi([a,b])=z[\Phi(a),\Phi(b)]$$

for all $a,b\in\mathcal{M}$.

Proof. For all $a, b \in \mathcal{M}$, we have from Lemma 3.2 and the additivity of Φ that

$$\begin{aligned} 2\Phi([a,b]) =&\Phi(2[a,b]) = \Phi([I \bullet a, b]) \\ =&[\Phi(I) \bullet \Phi(a), \Phi(b)] \\ =&(\Phi(I) + \Phi(I)^*)[\Phi(a), \Phi(b)]. \end{aligned} \tag{1}$$

Then $\Phi([a,b]) = \frac{\Phi(I)+\Phi(I)^*}{2}[\Phi(a), \Phi(b)]$. Denote $z = \frac{\Phi(I)+\Phi(I)^*}{2} \in \mathcal{Z}_{\mathcal{M}}$ by Lemma 3.2. In the following, we will prove that $z^2 = I$, which implies that z is invertible.

For each $a \in \mathcal{M}$ with $a = -a^*$, we have from Equation (1) that

$$[\Phi(a) \bullet \Phi(b), \Phi(c)] = \Phi([a \bullet b, c]) = \Phi([[a, b], c]) = z^2[[\Phi(a), \Phi(b)], \Phi(c)] \tag{2}$$

for all $b, c \in \mathcal{M}$. Thus we have

$$(I - z^2)\Phi(a)\Phi(b) + \Phi(b)(z^2\Phi(a) + \Phi(a)^*) \in \mathcal{Z}_{\mathcal{M}} \tag{3}$$

for all $b \in \mathcal{M}$ and $a \in \mathcal{M}$ with $a = -a^*$. For for convenience, denote $s = (I - z^2)\Phi(a)$, $t = \Phi(b)$ and $r = z^2\Phi(a) + \Phi(a)^*$. Then $st + tr \in \mathcal{Z}_{\mathcal{M}}$. Since Φ is surjective and b is arbitrary in $\mathcal{M}$, $\Phi(b)$ can retrieve all the elements in $\mathcal{M}$. In the following, we will prove $\Phi(a^*) = -\Phi(a)^*$ for all $a \in \mathcal{M}$ with $a = -a^*$ by taking different values of Φ.

(1) Take $t = p_1$. Then $sp_1 + p_1r \in \mathcal{Z}_{\mathcal{M}}$ and thus $p_2sp_1 = p_1rp_2 = 0$.

(2) Take $t = p_2$. Then $sp_2 + p_2r \in \mathcal{Z}_{\mathcal{M}}$ and thus $p_1sp_2 = p_2rp_1 = 0$.

Therefore $s = s_{11} + s_{22}, r = r_{11} + r_{22}$.

(3) For any $a_{12} \in \mathcal{M}_{12}$, take $t = a_{12}$. Then $s_{11}a_{12} + a_{12}r_{22} \in \mathcal{Z}_{\mathcal{M}}$ and thus $s_{11}a_{12} + a_{12}r_{22} = 0$. By Proposition 1.3, there exists $z_1 \in \mathcal{Z}_{\mathcal{M}}$ such that $s_{11} = z_1p_1$ and $r_{22} = -z_1p_2$.

(4) For any $a_{21} \in \mathcal{M}_{21}$, take t as a_{21}. Then $s_{22}a_{21} + a_{21}r_{11} \in \mathcal{Z}_{\mathcal{M}}$ and thus $s_{22}a_{21} + a_{21}r_{11} = 0$. By Proposition 1.3, there exists $z_2 \in \mathcal{Z}_{\mathcal{M}}$ such that $s_{22} = z_2p_2$ and $r_{11} = -z_2p_1$. Therefore we have from Equation (2) that

$$st + tr = z_1p_1t + z_2p_2t - tz_1p_2 - tz_2p_1 \in \mathcal{Z}_{\mathcal{M}}. \tag{4}$$

Multiplying Equation (4) on both sides by p_1, we have from Proposition 1.1 (i) that

$$(z_1 - z_2)p_1tp_1 \in p_1\mathcal{Z}_{\mathcal{M}} = \mathcal{Z}_{p_1\mathcal{M}p_1}$$

for all $t \in \mathcal{M}$. Thus

$$(z_1 - z_2)p_1\mathcal{M}p_1 \subseteq p_1\mathcal{Z}_{\mathcal{M}} = \mathcal{Z}_{p_1\mathcal{M}p_1}.$$

Noting that $p_1\mathcal{M}p_1$ is also von Neumann algebra with no central summands of type I_1, it follows from Proposition 1.1(iv) that $z_1 = z_2$. For convenience, denote $z_0 = z_1 = z_2 \in \mathcal{Z}_{\mathcal{M}}$. Then $s = s_{11} + s_{22} = z_0p_1 + z_0p_2 = z_0 \in \mathcal{Z}_{\mathcal{M}}$. Thus

$$(I - z^2)\Phi(a) = z_0. \tag{5}$$

Similarly, we can obtain

$$z^2\Phi(a) + \Phi(a)^* = -z_0. \tag{6}$$

Then adding Equations (5) and (6) yields

$$\Phi(a^*) = -\Phi(a) \tag{7}$$

for any $a \in \mathcal{M}$ with $a = -a^*$. Let $\mathcal{A} = \{a|a^* = -a\}$. Thus $\Phi(\mathcal{A}) \subseteq \mathcal{A}$. Since Φ is bijective, $\Phi(\mathcal{A}) = \mathcal{A}$. Then combining this with Equation (2), we have

$$[\Phi(a) \bullet \Phi(b), \Phi(c)] = [[\Phi(a), \Phi(b)], \Phi(c)] = z^2[[\Phi(a), \Phi(b)], \Phi(c)]$$

for any $a \in \mathcal{M}$ with $a = -a^*$ and all $b, c \in \mathcal{M}$. Since Φ is bijective and c is arbitrary, we have

$$(I - z^2)[\Phi(a), \Phi(b)] \in \mathcal{Z}_{\mathcal{M}}$$

for all $b \in \mathcal{M}$ and $a \in \mathcal{M}$ with $a = -a^*$. Then by Proposition 1.4,

$$(I - z^2)[\Phi(a), \Phi(b)] = 0$$

for all $b \in \mathcal{M}$ and $a \in \mathcal{M}$ with $a = -a^*$. For any $a \in \mathcal{M}$ with $a = -a^*$, we have from Equation (7) that $(i\Phi(a))^* = -i\Phi(a)^* = i\Phi(a)$, which implies that $i\Phi(a) \in \mathcal{M}_{sa}$ for any $a \in \mathcal{M}$ with $a = -a^*$. Then

$$(I - z^2)[i\Phi(a), \Phi(b)] = 0$$

for all $b \in \mathcal{M}$ and $a \in \mathcal{M}$ with $a = -a^*$. Since $\Phi(\mathcal{A}) = \mathcal{A}$, we have

$$(I - z^2)[m, \Phi(b)] = 0$$

for all $b, m \in \mathcal{M}$. Take $\Phi(b) = p_1$ and then we have

$$(I - z^2)mp_1 - (I - z^2)p_1 m = 0 \tag{8}$$

for all $m \in \mathcal{M}$. Multiplying on the left by p_2 and on the right by p_1 of Equation (8), it concludes that $(I - z^2)p_2\mathcal{M}p_1 = \{0\}$. Then it follows from Proposition 1.1(iv) that $(I - z^2)p_2 = 0$. Since $I - z^2 \in \mathcal{Z}_{\mathcal{M}}$ and $\overline{p_2} = I$, we have from Proposition 1.1(iv) that $I - z^2 = 0$, which implies that $z^2 = I$. The proof is finished. □

Remark 3.4. *Let z be as above and define $\Psi = z\Phi$. It follows from Lemma 3.3 that $\Psi([a, b]) = [\Psi(a), \Psi(b)]$ for all $a, b \in \mathcal{M}$. It is clear that $\Psi : \mathcal{M} \to \mathcal{M}$ is an additive bijection that preserves commutativity in both directions. There by Proposition 1.2, there exists an invertible element $z_0 \in \mathcal{Z}_{\mathcal{M}}$ such that $\Psi(a) = z_0\theta(a) + f(a)$ for any $a \in \mathcal{M}$, where $\theta : \mathcal{M} \to \mathcal{M}$ is an additive Jordan isomorphism and $f : \mathcal{M} \to \mathcal{Z}_{\mathcal{M}}$ is an additive map.*

Lemma 3.5. $z_0 = I$.

Proof. For all $a, b \in \mathcal{M}$, it follows from Remark 3.4 that

$$z_0\theta([a, b]) + f([a, b]) = z_0^2[\theta(a), \theta(b)]. \tag{9}$$

Since $\theta : \mathcal{M} \to \mathcal{M}$ is an additive Jordan isomorphism, θ can be decomposed as the direct sum of an additive isomorphism and an additive anti-isomorphim from $\mathcal{M}$ to $\mathcal{M}$. That is $\theta = \theta_1 \bigoplus \theta_2$, where $\theta_1 : \mathcal{M} \to \mathcal{M}$ is an additive isomorphism and $\theta_2 : \mathcal{M} \to \mathcal{M}$ is an additive anti-isomorphism. It follows from Equation (9) that

$$z_0\theta_1(ab - ba) + z_0\theta_2(ab - ba) + f([a, b]) = z_0^2[\theta_1(a) + \theta_2(a), \theta_1(b) + \theta_2(b)].$$

By simple calculation, we have

$$(z_0 - z_0^2)(\theta_1(a)\theta_1(b) - \theta_1(b)\theta_1(a)) + (z_0 + z_0^2)(\theta_2(b)\theta_2(a) - \theta_2(a)\theta_2(b)) \in \mathcal{Z}_{\mathcal{M}} \tag{10}$$

for all $a, b \in \mathcal{M}$. Since θ_1 is surjective, there exists $s \in \mathcal{M}$ such that $\theta_1(s) = p_1$ and then $\theta_1(I - s) = p_2$. Taking $a = s$ in Equation (10), we obtain

$$(z_0 - z_0^2)(p_1\theta_1(b) - \theta_1(b)p_1) + (z_0 + z_0^2)(\theta_2(b)\theta_2(p_1) - \theta_2(p_1)\theta_2(b)) \in \mathcal{Z}_{\mathcal{M}} \tag{11}$$

for all $b \in \mathcal{M}$. Multiplying on the left by $\theta_1(s) = p_1$ and on the right by $\theta_1(I - s) = p_2$ of Equation (11), it concludes that $(z_0 - z_0^2)p_1\theta_1(b)p_2 = 0$. Since $z_0 - z_0^2 \in \mathcal{Z}_{\mathcal{M}}$ and $\overline{p_2} = I$, we have from Proposition 1.1(iv) that $(z_0 - z_0^2)p_1 = 0$ and thus $z_0 - z_0^2 = z_0(z_0 - I) = 0$ by Proposition 1.1(iv). Therefore $z_0 = I$ for z_0 is invertible. The proof is completed. □

Lemma 3.6. *$\theta : \mathcal{M} \to \mathcal{M}$ in Remark 3.4 must be an additive isomorphism.*

Proof. From the above, $\theta = \theta_1 \bigoplus \theta_2$, where $\theta_1 : \mathcal{M} \to \mathcal{M}$ is an additive isomorphism and $\theta_2 : \mathcal{M} \to \mathcal{M}$ is an additive anti-isomorphism. Therefore we need to show that $\theta_2 \equiv 0$. Otherwise, assume that $\theta : \mathcal{M} \to \mathcal{M}$ is an additive anti-isomorphism for convenience. Then by Lemma 3.5, we have $\Psi = \theta + f$, where $\theta : \mathcal{M} \to \mathcal{M}$ is an additive anti-isomorphism and $f : \mathcal{M} \to \mathcal{Z}_{\mathcal{M}}$ is an additive map. Then there exists $z_1 \in \mathcal{Z}_{\mathcal{M}}$ such that

$$\Psi([a \bullet b, c]) = \theta([a \bullet b, c]) + z_1 = [\theta(c), \theta(b)\theta(a) + \theta(a^*)\theta(b)] + z_1 \tag{12}$$

for all $a, b, c \in \mathcal{M}$. On the other hand,

$$\begin{aligned}\Psi([a \bullet b, c]) =& [\Psi(a) \bullet \Psi(b), \Psi(c)] \\ =& [(\theta(a) + f(a)) \bullet (\theta(b) + f(b)), \theta(c)] \\ =& [\theta(a) \bullet \theta(b) + \theta(a) \bullet f(b) + f(a) \bullet \theta(b), \theta(c)]\end{aligned} \tag{13}$$

for all $a, b, c \in \mathcal{M}$.

Combing Equations (12) and (13), we have from Proposition 1.4 that $z_1 = 0$ and then

$$(\theta(a^*) + \theta(a))\theta(b) + (\theta(b) + f(b))(\theta(a)^* + \theta(a)) + (f(a) + f(a)^*)\theta(b) \in \mathcal{Z}_{\mathcal{M}} \tag{14}$$

for all $a, b \in \mathcal{M}$. Taking $b = p_1$ in Equation (14) and multiplying Equation (3.14) on the left side by $\theta(p_1)$ and on the right side by $\theta(p_2)$, we can obtain

$$(I + f(p_1))\theta(p_1)(\theta(a)^* + \theta(a))\theta(p_2) = 0 \tag{15}$$

for all $a \in \mathcal{M}$. Replacing $\theta(a)$ by $i\theta(a)$ in Equation (15), we have

$$(I + f(p_1))\theta(p_1)(\theta(a)^* - \theta(a))\theta(p_2) = 0 \tag{16}$$

for all $a \in \mathcal{M}$. Then we have from Equations (15) and (16) that

$$(I + f(p_1))\theta(p_1)\theta(a)\theta(p_2) = 0$$

for all $a \in \mathcal{M}$. Thus

$$\theta^{-1}(I + f(p_1))p_2 a p_1 = 0$$

for all $a \in \mathcal{M}$. It follows from Proposition 1.1(iv) that $\theta^{-1}(I + f(p_1)) = 0$ and then $f(p_1) = -I$. Now take $b = p_1$ in Equation (14) and multiplying Equation (14) on the right side by $\theta(p_2)$, we have

$$\theta(p_2)(\theta(a)^* + \theta(a))\theta(p_2) \in \mathcal{Z}_{\mathcal{M}}\theta(p_2) = \theta(\mathcal{Z}_{\mathcal{M}}p_2) \tag{17}$$

for all $a \in \mathcal{M}$. Replacing $\theta(a)$ by $i\theta(a)$ in Equation (17), it follows that

$$\theta(p_2 a p_2) = \theta(p_2)\theta(a)\theta(p_2) \in \theta(\mathcal{Z}_{\mathcal{M}}p_2)$$

for all $a \in \mathcal{M}$. Thus $p_2\mathcal{M}p_2 = \mathcal{Z}_{\mathcal{M}}p_2$. Similarly, $p_1\mathcal{M}p_1 = \mathcal{Z}_{\mathcal{M}}p_1$. Since $\mathcal{Z}_{\mathcal{M}} \subseteq \mathcal{Z}_{p_1\mathcal{M}p_1} = \mathcal{Z}_{\mathcal{M}}p_1$ by Proposition 1.1(i), we have $\mathcal{Z}_{\mathcal{M}} \subseteq p_1\mathcal{M}p_1$. It follows from Proposition 1.1(iii) that $\mathcal{Z}_{\mathcal{M}} = \{0\}$, which is a contradiction. Therefore $\theta_2 \equiv 0$. The proof is completed. □

Lemma 3.7. *θ is an additive $*$-isomorphism and $f(a) = 0$ for all $a \in \mathcal{M}$.*

Proof. By Lemma 3.6, we have obtained that $\Psi = \theta + f$, where $\theta : \mathcal{M} \to \mathcal{M}$ is an additive isomorphism and $f : \mathcal{M} \to \mathcal{Z}_{\mathcal{M}}$ is an additive map. Thus there exists $z \in \mathcal{Z}_{\mathcal{M}}$ such that

$$\Psi([a \bullet b, c]) = \theta([a \bullet b, c]) + z = [\theta(a)\theta(b) + \theta(b)\theta(a^*), \theta(c)] + z$$

for all $a, b, c \in \mathcal{M}$. On the other hand, we have

$$\begin{aligned}\Psi([a \bullet b, c]) =& ([\Psi(a) \bullet \Psi(b), \Psi(c)]) \\ =& [(\theta(a) + f(a)) \bullet (\theta(b) + f(b)), \theta(c)] \\ =& [\theta(a) \bullet \theta(b) + f(a) \bullet \theta(b) + \theta(a) \bullet f(b), \theta(c)]\end{aligned}$$

for all $a, b, c \in \mathcal{M}$. It follows from Proposition 1.4 that $z = 0$. Thus we have from the surjectivity of θ that

$$\theta(b)(\theta(a)^* - \theta(a^*)) + \theta(b)(f(a)^* + f(a)) + f(b)(\theta(a)^* + \theta(a)) \in \mathcal{Z}_{\mathcal{M}} \tag{18}$$

for all $a, b \in \mathcal{M}$. Take $b \in p_1$ in Equation (18) and multiplying Equation (18) on the left side by $\theta(p_2)$ and on the right side by $\theta(p_1)$, we have

$$f(p_1)\theta(p_2)\theta(a)\theta(p_1) = 0$$

for all $a \in \mathcal{M}$. It follows that

$$\theta^{-1}(f(p_1))p_2 a p_1 = 0$$

for all $a \in \mathcal{M}$. Thus we have from Proposition 1.1(iv) that $\theta^{-1}(f(p_1)) = 0$ and then $f(p_1) = 0$. Similarly, $f(p_2) = 0$ and then $f(I) = 0$. Take $b = I$ in Equation (18) and then we have

$$\theta(a)^* - \theta(a^*) \in \mathcal{Z}_{\mathcal{M}} \tag{19}$$

for all $a \in \mathcal{M}$. In the following, we show $f(a) = 0$. Take $b = x_{11}$ in Equation (3.18) and multiplying Equation (18) on the left side by $\theta(p_2)$ and on the right side by $\theta(p_1)$, we have

$$\theta(p_2)f(x_{11})(\theta(a) + \theta(a)^*)\theta(p_1) = 0$$

for all $a \in \mathcal{M}$. Noticing that $\theta(a) + \theta(a)^* \in \mathcal{M}_{sa}$ and θ is surjective, we have

$$\theta(p_2)f(x_{11})\theta(a)\theta(p_1) = 0$$

for all $a \in \mathcal{M}$. Then

$$p_2\theta^{-1}(f(x_{11}))a p_1 = 0$$

for all $a \in \mathcal{M}$. It follows from Proposition 1.1(iv) that $\theta^{-1}(f(x_{11})) = 0$ and then $f(x_{11}) = 0$. Repeating the similar process, we can show that $f(x_{ij}) = 0$ for $1 \le i, j \le 2$. Since f is additive, it follows that $f(a) = 0$ for all $a \in \mathcal{M}$. Combining this with Equation (18), we have

$$\theta(b)(\theta(a)^* - \theta(a^*)) \in \mathcal{Z}_{\mathcal{M}}$$

for all $a \in \mathcal{M}$. Then by Proposition 1.1(v) and Equation (19), we have $\theta(a)^* = \theta(a^*)$ for all $a \in \mathcal{M}$. The proof is finished. □

Remark 3.8. *It follows from Remark 3.4 and Lemma 3.7 that Ψ is an additive $*$-isomorphism.*

Lemma 3.9. *There exists a central projection $e \in \mathcal{M}$ such that the restriction of Ψ to $\mathcal{M}e$ is a linear $*$-isomorphism and the restriction of Ψ to $\mathcal{M}(I - e)$ is a conjugate linear $*$-isomorphism.*

Proof. For each rational number q, we have $\Psi(qI) = q\Psi(I)$. In fact, since q is a rational number, there exists two integers r and s such that $q = \frac{r}{s}$. Since $\Psi(I) = I$ and Ψ is additive, it follows that

$$\Psi(qI) = \Psi(\frac{r}{s}I) = r\Psi(\frac{1}{s}) = \frac{r}{s}\Psi(I) = qI.$$

Now we show that Ψ is real linear. Let $a \in \mathcal{M}$ be a positive element and then $a = b^2$ for some self-adjoint element $b \in \mathcal{M}$. Thus $\Psi(a) = \Psi(b)^2$. Since $\Psi(b) \in \mathcal{M}_{sa}$ by Remark 3.8, we have $\Psi(a)$ is a positive element in

$\mathcal{M}$, which shows that Ψ preserves positive elements. Let $\lambda \in \mathbb{R}$. Choose sequences $\{a_n\}$ and $\{b_n\}$ of rational numbers such that $a_n \leq \lambda \leq b_n$ for all n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lambda$. It follows from

$$a_n I \leq \lambda I \leq b_n I$$

that

$$a_n I \leq \Psi(\lambda I) \leq b_n I.$$

Taking the limit, we have $\Psi(\lambda I) = \lambda I$. Hence for all $a \in \mathcal{M}$, it follows that

$$\Psi(\lambda a) = \Psi((\lambda I)a) = \Psi(\lambda I)\Psi(a) = \lambda\Psi(a).$$

Therefore Ψ is real linear.

Let $f = \frac{I - i\Psi(iI)}{2}$. It is easy to verify that f is a central projection in $\mathcal{M}$ for Ψ is an additive isomorphism and $\Psi(\mathcal{Z}_{\mathcal{M}}) = \mathcal{Z}_{\mathcal{M}}$. Since $\Psi(iI) = i(2f - I)$, we have

$$f\theta(iI) = if, (I - f)\Psi(iI) = i(f - I).$$

Let $e = \Psi^{-1}(f)$. Then e is also a central projection in $\mathcal{M}$. Therefore, for all $a \in \mathcal{M}$, we have

$$\Psi(iae) = \Psi(a)\Psi(e)\Psi(iI) = i\Psi(a)f = i\Psi(ae)$$

and

$$\Psi(ia(I - e)) = \Psi(a)\Psi(I - e)\Psi(iI) = -i\Psi(a)(I - f) = -i\Psi(a(I - e)).$$

Therefore, the restriction of Ψ to $\mathcal{M}e$ is linear and the restriction of Ψ to $\mathcal{M}(I - e)$ is conjugate linear. The proof is finished.

□

Finally, we give the proof of Theorem 3.1.

Proof. Obviously, Theorem 3.1 can be easily obtained by Remark 3.4 and 3.8, and 3.9. □

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