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New results in gradual normed linear spaces in light of generalized statistical convergence via ideal

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Abstract. This article introduces the novel concept of rough $\mathcal{I}-$statistical convergence of order α in gradual normed linear spaces (GNLS). The paper presents noteworthy findings utilizing the $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})-$ limit set, which exhibits several fundamental properties of the concept. Moreover, a necessary and sufficient condition for $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})-$convergence of a sequence in a GNLS is established.

1. Introduction

The concept of statistical convergence was initially introduced by Fast [18]. The key principle underlying statistical convergence is the notion of natural density. The natural density of a set $A \subseteq \mathbb{N}$ is defined and denoted as

$$d(A) = \lim_n \frac{1}{n} |\{k \in A : k \leq n\}|,$$

where the vertical bars represent the cardinality of the enclosed set. It is evident that $d(\mathbb{N} \setminus A) + d(A) = 1$, and $A \subseteq B$ implies $d(A) \leq d(B)$. Additionally, it is clear that if A is a finite set, then $d(A) = 0$. A sequence of real numbers $u = (u_k)$ is said to be statistically convergent to the value u_0 if, for every $\eta > 0$,

$$d(\{k \in \mathbb{N} : |u_k - u_0| \geq \eta\}) = 0.$$

Subsequently, statistical convergence received significant attention and was explored from the perspective of sequence spaces by researchers such as Connor [10], Hazarika et al. [21], Khan et al. [23, 24], Mohiuddine et al. [29, 31], Tripathy [40, 41], and various others.

In 2001, the concept of $\mathcal{I}-$convergence was introduced by Kostyrko et al. [26] as a generalization of both usual and statistical convergence. An ideal $\mathcal{I}$ on a non-empty set Y is a family of subsets $\mathcal{I} \subset 2^Y$ satisfying the properties that for any $R, S \in \mathcal{I}$, $R \cup S \in \mathcal{I}$ and for any $R \in \mathcal{I}$ with $S \subset R$, $S \in \mathcal{I}$. An ideal $\mathcal{I}$ is considered non-trivial if $\mathcal{I} \neq \emptyset$ and $Y \notin \mathcal{I}$. Furthermore, a non-trivial ideal $\mathcal{I} \subset 2^Y$ is called an admissible

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ideal in Y if $\mathcal{I} \supset \{\{u\} : u \in Y\}$. For an ideal $\mathcal{I}$ in a set Y, the class $\mathcal{F}(\mathcal{I}) = \{Y - K : K \in \mathcal{I}\}$ is referred to as the filter associated with the ideal $\mathcal{I}$. When considering $Y = \mathbb{N}$, a real-valued sequence $u = (u_k)$ is said to be $\mathcal{I}$-convergent to u_0 if, for every $\eta > 0$,

$$\{k \in \mathbb{N} : |u_k - u_0| \geq \eta\} \in \mathcal{I} \text{ (or } \{k \in \mathbb{N} : |u_k - u_0| < \eta\} \in \mathcal{F}(\mathcal{I})).$$

It is evident that when considering $\mathcal{I} = \mathcal{I}_f = \{A \subset \mathbb{N} : |A| < \infty\}$, the aforementioned definition reduces to the definition of usual convergence. Similarly, when $\mathcal{I} = \mathcal{I}_d = A \subset \mathbb{N} : d(A) = 0$, the given definition corresponds to the definition of statistical convergence. Therefore, in a sense, $\mathcal{I}-$convergence offers a comprehensive framework for studying various types of convergence within a unified context. To delve deeper into the topic of $\mathcal{I}-$convergence, interested readers can explore the research conducted by Hazarika [20], Hazarika and Esi [22], Kostyrko et al. [25], Mohiuddine et al. [30], Nabiev et al. [33], Tripathy and Hazarika [42], and other related works. These references provide further insights and additional sources for comprehensive understanding.

The combination of statistical convergence and $\mathcal{I}-$convergence led Savaş and Das [38] to introduce a novel concept called $\mathcal{I}-$statistical convergence. Building upon this concept, Das and Savaş [11] further extended it to $\mathcal{I}-$statistical convergence of order α for $0 < \alpha \leq 1$ three years later. In addition to investigating fundamental properties, they established interesting implications. Subsequently, researchers such as Debnath and Choudhury [12], Debnath and Rakshit [13], Mursaleen et al. [32], and others conducted various studies in this direction.

The concept of rough convergence was independently introduced by Burgin [5] and Phu [35]. Although their ideas were similar, Burgin explored it in the fuzzy setting, while Phu studied it in the context of finite-dimensional normed spaces.

Let r be a non-negative real number. In a normed linear space $(Y, |\cdot|)$, a sequence $u = (u_k)$ is said to exhibit rough convergence to $u_0 \in Y$ with a roughness degree of r if, for every $\eta > 0$, there exists an $N = (N_\eta)$ such that for all $k \geq N$,

$$||u_k - u_0|| < r + \eta.$$

Symbolically, it is represented as $u_k \xrightarrow{r-||\cdot||} u_0$.

Phu [36] extensively investigated the properties of the set $LIM^r x$, demonstrating that it is bounded, convex, and closed. This remarkable concept of rough convergence finds natural applications in numerical analysis. Building upon the notion of rough convergence and statistical convergence, Aytar [3] introduced the concept of rough statistical convergence. Furthermore, independent extensions were made to rough ideal convergence by Pal et al. [34] and Dündar and Çakan [15]. The examination of rough $\mathcal{I}-$statistical convergence was carried out by Malik et al. [28] and Savaş et al. [39]. For a comprehensive exploration of this field, interested readers may refer to works such as [2, 4], which provide additional references for further study.

The concept of fuzzy sets (FS) was introduced by Zadeh [43] and has found wide-ranging applications in various fields of engineering and science. Within the framework of FS theory, the concept of "fuzzy numbers (FNs)" plays a significant role. FNs serve as a generalization of intervals rather than traditional numbers. However, FNs do not possess all the algebraic properties of well-known numbers, which has sparked debate among researchers regarding their behavior. As an alternative, many researchers widely employ the concept of "fuzzy intervals" instead of FNs due to their more consistent behavior.

To address this discrepancy, Fortin et al. [19] proposed the concept of "gradual real numbers (GRNs)" as elements of fuzzy intervals. GRNs are characterized by their respective assignment function, defined on the interval $(0, 1]$. Each element of $\mathbb{R}$ can be represented as a gradual number with a constant assignment function. Importantly, GRNs encompass all the algebraic properties of traditional $\mathbb{R}$ numbers and have found applications in optimization problems and computation. This notion of GRNs helps resolve the confusion surrounding the use of FNs and provides a framework that aligns with the desired algebraic properties of real numbers.

The concept of gradual normed linear spaces (GNLS) was first introduced by Sadeqi and Azari [37], who extensively investigated its significant features from both topological and algebraic perspectives.

Subsequently, Aiche and Dubois [1], Ettefagh et al. [16], and others have made notable contributions to the advancement of GNLS. For a comprehensive study on gradual real numbers (GRNs), references such as [14, 27] provide valuable insights and additional references.

It should be noted that a GNLS is different from normed linear space (NLS). **For example,** the space of real continuous function $C((0,1])$ is a gradual normed linear space with respect to the gradual norm

$$\|\cdot\|_{\mathcal{G}} : C((0,1]) \to \mathcal{G}^*(\mathbb{R})$$

defined by

$$\left\|f\right\|_{\mathcal{G}}(\psi) = \left|f(\psi)\right| \text{ for } \psi \in (0,1] \text{ and } f \in C((0,1]),$$

but it is not classical normable.

The convergence of sequences in GNLS has been studied by Ettefagh et al. [17]. Recently, Choudhury and Debnath [6, 8, 9] extended this research by incorporating ideals into $\mathcal{I}-$convergence and $\mathcal{I}-$statistical convergence. For further information, readers can refer to [7], which provides additional references for in-depth exploration of the topic.

Influenced by these noteworthy findings and the fundamental properties exhibited by the $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})-$limit set, the main objective of this work is to investigate the concept of rough $\mathcal{I}-$statistical convergence in gradual normed linear spaces (GNLS). Furthermore, the study aims to establish a necessary and sufficient condition for the $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})-$convergence of a sequence in a GNLS.

2. Preliminaries

In this section, we provide an overview of relevant definitions and results that form the foundation for our subsequent findings.

Definition 2.1. ([19]) A gradual real number (GRN) $\tilde{u}$ is defined by an assignment function $\mathcal{S}_{\tilde{u}} : (0,1] \to \mathbb{R}$. We say that $\tilde{u}$ is non-negative if for every $\psi \in (0,1]$, $\mathcal{S}_{\tilde{u}}(\psi) \geq 0$. The set of all GRNs and non-negative GRNs is denoted by $\mathcal{G}(\mathbb{R})$ and $\mathcal{G}^*(\mathbb{R})$, respectively.

The gradual operations between the elements of $\mathcal{G}(\mathbb{R})$ were defined as follows in [19]:

Definition 2.2. Let $\diamond$ be an arbitrary operation on real numbers, and consider $\tilde{u}_1, \tilde{u}_2 \in \mathcal{G}(\mathbb{R})$ with assignment functions $\mathcal{S}_{\tilde{u}_1}$ and $\mathcal{S}_{\tilde{u}_2}$, respectively. Then, the operation $\tilde{u}_1 \diamond \tilde{u}_2$ in $\mathcal{G}(\mathbb{R})$ is defined with the assignment function $\mathcal{S}_{\tilde{u}_1 \diamond \tilde{u}_2}$ given by:

$$\mathcal{S}_{\tilde{u}_1 \diamond \tilde{u}_2}(\psi) = \mathcal{S}_{\tilde{u}_1}(\psi) \diamond \mathcal{S}_{\tilde{u}_2}(\psi), \forall \psi \in (0,1].$$

In particular, the gradual addition $\tilde{u}_1 + \tilde{u}_2$ and the gradual scalar multiplication $p\tilde{u}$ (where $p \in \mathbb{R}$) are defined as follows:

$$\mathcal{S}_{\tilde{u}_1+\tilde{u}_2}(\psi) = \mathcal{S}_{\tilde{u}_1}(\psi) + \mathcal{S}_{\tilde{u}_2}(\psi) \quad \text{and} \quad \mathcal{S}_{p\tilde{u}}(\psi) = p\mathcal{S}_{\tilde{u}}(\psi), : \forall \psi \in (0,1].$$

The constant gradual real number (GRN) $\tilde{p}$ is formally defined for any $p \in \mathbb{R}$ using the constant assignment function $\mathcal{S}_{\tilde{p}}(\psi) = p$ for $\psi \in (0,1]$. Specifically, $\tilde{0}$ and $\tilde{1}$ represent constant GRNs, where $\mathcal{S}_{\tilde{0}}(\psi) = 0$ and $\mathcal{S}_{\tilde{1}}(\psi) = 1$ respectively. It is straightforward to verify that the gradual multiplication and addition operations on $\mathcal{G}(\mathbb{R})$ establish a real vector space, as stated in [19].

Definition 2.3. ([37]) Consider Y as a real vector space. A function $\|\cdot\|_{\mathcal{G}} : Y \to \mathcal{G}^*(\mathbb{R})$ is defined as a gradual norm on Y if, for every $\psi \in (0,1]$, the following conditions hold for any $w, v \in Y$:

1. $\mathcal{S}_{\|w\|_{\mathcal{G}}}(\psi) = \mathcal{S}_{\tilde{0}}(\psi)$ iff $w = 0$;
2. $\mathcal{S}_{\|\mu w\|_{\mathcal{G}}}(\psi) = |\mu|\mathcal{S}_{\|w\|_{\mathcal{G}}}(\psi)$ for any $\mu \in \mathbb{R}$;

3. $\mathcal{S}_{\|w+v\|_{\mathcal{G}}}(\psi) \leq \mathcal{S}_{\|w\|_{\mathcal{G}}}(\psi) + \mathcal{S}_{\|v\|_{\mathcal{G}}}(\psi)$.

The pair $(Y, \|\cdot\|_{\mathcal{G}})$ is called an GNLS.

Definition 2.4. ([17]) Let $u = (u_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, u is called to be gradual bounded provided that for each $\psi \in (0,1]$, there is an $M = M(\psi) > 0$ so that $\mathcal{S}_{\|u_k\|_{\mathcal{G}}} \leq M$ for all $k \in \mathbb{N}$.

Definition 2.5. ([37]) Let $u = (u_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, (u_k) is called to be gradual convergent to $u_0 \in Y$, provided that for each $\psi \in (0,1]$ and $\eta > 0$, there is an $N(= N_\eta(\psi)) \in \mathbb{N}$ so that for each $k \geq N$,

$$\mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) < \eta.$$

Symbolically, it is represented as $u_k \xrightarrow{\mathcal{G}} u_0$.

Example 2.6. ([37]) Take $Y = \mathbb{R}^l$ and for $u = (u_1, u_2, ..., u_\alpha) \in \mathbb{R}^l, \psi \in (0,1]$, determine $\|\cdot\|_{\mathcal{G}}$ as

$$\mathcal{S}_{\|u\|_{\mathcal{G}}}(\psi) = e^{\psi} \sum_{j=1}^{l} |u_j|.$$

Here, $\|\cdot\|_{\mathcal{G}}$ is a gradual norm on $\mathbb{R}^l$ and $(\mathbb{R}^l, \|\cdot\|_{\mathcal{G}})$ is a GNLS.

Definition 2.7. ([11]) A real-valued sequence $u = (u_k)$ is said to be $\mathcal{I}-$ statistically convergent of order α $(0 < \alpha \leq 1)$ to u_0, provided that for each $\eta, \gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha} \left|\{k \leq n : |u_k - u_0| \geq \eta\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

When a sequence $u = (u_k)$ is $\mathcal{I}-$statistically convergent of order α $(0 < \alpha \leq 1)$ to l, then it is demonstrated by $u_k \xrightarrow{\mathcal{I}_{st}^{\alpha}} l$.

Especially, for $\alpha = 1$, the above definition reduces to the definition of $\mathcal{I}-$statistical convergence given in [38].

Definition 2.8. ([32]) A real number u_0 is named to be an $\mathcal{I}-$ statistical cluster point of a real-valued sequence $u = (u_k)$, provided that for each $\eta, \gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : |u_k - u_0| \geq \eta\}\right| < \gamma\right\} \notin \mathcal{I}.$$

Definition 2.9. ([34]) A sequence $u = (u_k)$ in a normed linear space $(Y, \|\cdot\|)$ is called to be rough $\mathcal{I}-$convergent to $u_0 \in Y$ with roughness degree r, provided that for each $\eta > 0$,

$$\{k \in \mathbb{N} : \|u_k - u_0\| \geq r + \eta\} \in \mathcal{I}.$$

Symbolically, it is represented as $u_k \xrightarrow{\mathcal{I}^r - \|\cdot\|} u_0$.

For $\mathcal{I} = \mathcal{I}_f$, the above description reduces to the description of rough convergence [35] and for $\mathcal{I} = \mathcal{I}_d$, we get the definition of rough statistical convergence [3].

Definition 2.10. ([39]) A sequence $u = (u_k)$ is called to be rough $\mathcal{I}-$statistically convergent to $u_0 \in Y$ with roughness degree r, provided that for each $\eta, \gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : \|u_k - u_0\| \geq r + \eta\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

Symbolically, it is indicated as $u_k \xrightarrow{\mathcal{I}_{st}^r - \|\cdot\|} u_0$.

Definition 2.11. ([6]) Let $u = (u_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, u is called to be gradually $\mathcal{I}-$ convergent to $u_0 \in Y$, provided that for each $\psi \in (0,1]$ and $\eta > 0$,

$$\left\{k \in \mathbb{N} : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \geq \eta\right\} \in \mathcal{I}.$$

Symbolically, $u_k \xrightarrow{\mathcal{I}(\mathcal{G})} u_0$.

Definition 2.12. Let $u = (u_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, u is said to be gradually $\mathcal{I}-$statistical convergent of order α $(0 < \alpha \leq 1)$ (in short $\mathcal{I}_{st}^{\alpha}(\mathcal{G})-$ convergent) to $u_0 \in Y$, provided that for each $\psi \in (0,1]$ and $\eta > 0, \gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left|\left\{k \leq n : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \geq \eta\right\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

Symbolically it is represented as $u_k \xrightarrow{\mathcal{I}_{st}^{\alpha}(\mathcal{G})} u_0$.

Especially, for $\alpha = 1$, the above description reduces to the description of $\mathcal{I}-$statistical convergence in a *GNLS* given in [8].

3. Main results

In this section, we put forward the main results of the paper. Throughout the paper, $\mathcal{I}$ will demonstrate a non-trivial admissible ideal in $\mathbb{N}$.

Definition 3.1. Let $u = (u_k) \in (Y, \|\cdot\|_{\mathcal{G}})$ and r be a non-negative real number. Then, u is called to be $\mathcal{I}-$statistically rough convergent to $u_0 \in Y$ (in short $\mathcal{I}_{st}^{r}(\mathcal{G})-$convergent) with roughness degree r, if for every $\psi \in (0,1]$ and $\eta, \gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{k \leq n : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \geq r + \eta\right\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

In this case, u_0 is referred to as the $\mathcal{I}_{st}^{r}(\mathcal{G})$-limit of the sequence u, denoted by $u_k \xrightarrow{\mathcal{I}_{st}^{r}(\mathcal{G})} u_0$. The set denoted as $\mathcal{I}_{st}^{r}(\mathcal{G})$ represents the collection of all $\mathcal{I}_{st}^{r}(\mathcal{G})$-convergent sequences with a roughness degree of r, where $r \geq 0$.

Definition 3.2. Let $u = (u_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, u is said to be $\mathcal{I}-$statistically rough convergent of order $\alpha(0 < \alpha \leq 1)$ with roughness degree r, to $u_0 \in Y$ (in short $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})-$convergent) provided that for each $\psi \in (0,1]$ and $\eta, \gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left|\left\{k \leq n : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \geq r + \eta\right\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

In this scenario, u_0 is referred to as the $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})$ limit of the sequence u, denoted as $u_k \xrightarrow{\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})} u_0$. The set of all $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})$-convergent sequences with roughness degree $r \geq 0$ is denoted as $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})$.

It's worth mentioning that Definition 3.2 holds true when $0 < \alpha \leq 1$. However, if $\alpha > 1$, it becomes invalid when $r = 0$. This can be observed in the following example.

Example 3.3. Let $Y = \mathbb{R}^l$ and $\|.\|_{\mathcal{G}}$ denote the gradual norm as given in Example 2.6. Consider

$$\mathcal{I} = \mathcal{I}_f = \{A \subseteq \mathbb{N} : |A| < \infty\} \text{ and } \alpha > 1.$$

Define the sequence $u = (u_k)$ in $\mathbb{R}^l$ by

$$u_k = \begin{cases} (0,0,...,0,1), & \text{if } k \text{ is even} \\ (0,0,...,0,0), & \text{if } k \text{ is odd.} \end{cases}$$

Then, for all $\psi \in (0,1]$ and $\eta, \gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left|\left\{k \leq n : \mathcal{S}_{\|u_k-(0,0,\ldots,0,1)\|_{\mathcal{G}}}(\psi) \geq \eta\right\}\right| \geq \gamma\right\} \in \mathcal{I}$$

and

$$\left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left|\left\{k \leq n : \mathcal{S}_{\|u_k-(0,0,\ldots,0,0)\|_{\mathcal{G}}}(\psi) \geq \eta\right\}\right| \geq \gamma\right\} \in \mathcal{I},$$

which clearly contradicts the uniqueness of limit for $r = 0$.

Clearly, when $r = 0$, Definition 3.1 and Definition 3.2 correspond to the definitions of $\mathcal{I}$-statistical convergence and $\mathcal{I}$-statistical convergence of order α, respectively, in an GNLS. Therefore, our main focus is on the case when $r > 0$. There are several reasons for this particular interest. In practical scenarios, a sequence $w = (w_k)$ that is $\mathcal{I}_{st}^{\alpha}(\mathcal{G})$-convergent to u_0 may not be easily calculable or precisely measurable. Hence, one often has to work with an approximate sequence $u = (u_k)$ that satisfies

$$\mathcal{S}_{\|u_k-w_k\|_{\mathcal{G}}}(\psi) < r \text{ for all } k \in \mathbb{N}.$$

Subsequently, it cannot be guaranteed that x converges in $\mathcal{I}_{st}^{\alpha}(\mathcal{G})$. However, for any $\eta > 0$, the following inclusion holds:

$$\left\{n \in \mathbb{N} : \tfrac{1}{n^{\alpha}} \left|\left\{k \leq n : \mathcal{S}_{\|w_k-u_0\|_{\mathcal{G}}}(\psi) \geq \eta\right\}\right| \geq \gamma\right\} \supseteq \left\{n \in \mathbb{N} : \tfrac{1}{n^{\alpha}} \left|\left\{k \leq n : \mathcal{S}_{\|u_k-u_0\|_{\mathcal{G}}}(\psi) \geq r+\eta\right\}\right| \geq \gamma\right\}$$

holds, one can certainly assure the $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})-$convergence of x.

We provide the following example to illustrate the aforementioned fact more accurately.

Example 3.4. Let $Y = \mathbb{R}^l$ and $\|.\|_{\mathcal{G}}$ be the gradual norm given in Example 2.6. Consider $\mathcal{I} = \mathcal{I}_f$ and $\alpha = 1$. Define the sequence $u = (u_k)$ in $\mathbb{R}^l$ by

$$u_k = \begin{cases} \left(0,0,\ldots,0,0.5+2.\frac{(-1)^k}{k}\right), & \text{if } k \text{ is a perfect square} \\ (0,0,\ldots,0,0), & \text{otherwise.} \end{cases}$$

Then, we obtain

$$\mathcal{S}_{\|u_k-(0,0,\ldots,0,0.5)\|_{\mathcal{G}}}(\psi) = \begin{cases} \frac{2e^{\psi}}{k}, & \text{if } k \text{ is a perfect square} \\ 0, & \text{otherwise.} \end{cases}$$

Hence, for any $\eta > 0$, the following inclusion

$$\left\{k \in \mathbb{N} : \mathcal{S}_{\|u_k-(0,0,\ldots,0,0.5)\|_{\mathcal{G}}}(\psi) \geq \eta\right\} \subseteq \{1,4,9,\ldots\}$$

holds and eventually $u_k \xrightarrow{\mathcal{I}_{st}^{\alpha}(\mathcal{G})} (0,0,\ldots,0,0.5)$. However, for sufficiently large k, it becomes computationally infeasible to calculate (u_k) exactly. Instead, it is rounded to the nearest value. Thus, for simplicity, let's approximate (u_k) by $w_k = (0,0,\ldots,0,t)$ at non-perfect square positions, where t is the integer satisfying $t-0.5 < u_k < t+0.5$. Then, the sequence (w_k) does not $\mathcal{I}_{st}^{\alpha}(\mathcal{G})$-converges anymore. Nonetheless, by the definition $w_k \xrightarrow{\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})} (0,0,\ldots,0,0.5)$ for $r = 0.5$.

It is clear that for $r > 0$, the $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})-$limit of a sequence is not generally unique. Therefore, we construct $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})-$limit set of a sequence $u = (u_k)$ denoted and defined by:

$$\mathcal{I}_{st}^{r,\alpha}(\mathcal{G}) - LIMu = \left\{u_0 \in Y : u_k \xrightarrow{\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})} u_0\right\}.$$

For $\alpha = 1$, the above limit set reduces to

$$\mathcal{I}^{r}_{st}(\mathcal{G}) - LIMu = \left\{u_0 \in Y : u_k \xrightarrow{\mathcal{I}^{r}_{st}(\mathcal{G})} u_0\right\}.$$

It is obvious that a sequence $u = (u_k)$ in a GNLS $(Y, \|\cdot\|_{\mathcal{G}})$ is $\mathcal{I}^{r}_{st}(\mathcal{G})-$convergent or $\mathcal{I}^{r,\alpha}_{st}(\mathcal{G})-$convergent according as $\mathcal{I}^{r}_{st}(\mathcal{G}) - LIMu \neq \emptyset$ and $\mathcal{I}^{r,\alpha}_{st}(\mathcal{G}) - LIMu \neq \emptyset$ for some $r \geq 0$.

Theorem 3.5. *Every $\mathcal{I}^{r,\alpha}_{st}(\mathcal{G})-$convergent sequence is also $\mathcal{I}^{r}_{st}(\mathcal{G})-$convergent for $0 < \alpha \leq 1$.*

Proof. Let $u = (u_k)$ be a sequence in the $(Y, \|\cdot\|_{\mathcal{G}})$ so that $u_k \xrightarrow{\mathcal{I}^{r,\alpha}_{st}(\mathcal{G})} u_0$ holds. Then, according to the definition, for each $\psi \in (0, 1]$ and $\eta, \gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\left\{k \leq n : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \geq r + \eta\right\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

Since $0 < \alpha \leq 1$, so the following inclusion

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \geq r + \eta\right\}\right| \geq \gamma\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\left\{k \leq n : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \geq r + \eta\right\}\right| \geq \gamma\right\}$$

holds and the result follows. □

Theorem 3.6. *Let $u = (u_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then,*

$$diam\left(\mathcal{I}^{r,\alpha}_{st}(\mathcal{G}) - LIMu\right) = \sup\left\{\mathcal{S}_{\|y-z\|_{\mathcal{G}}}(\psi) : y, z \in \mathcal{I}^{r,\alpha}_{st}(\mathcal{G}) - LIMu, \psi \in [0, 1)\right\} \leq 2r.$$

In general, $diam\left(\mathcal{I}^{r,\alpha}_{st}(\mathcal{G}) - LIMu\right)$ has no smaller bound.

Proof. If possible, let us assume that $diam\left(\mathcal{I}^{r,\alpha}_{st}(\mathcal{G}) - LIMu\right) > 2r$. Then, there exists $y_0, z_0 \in \mathcal{I}^{r,\alpha}_{st}(\mathcal{G}) - LIMu$ such that $\mathcal{S}_{\|y_0 - z_0\|_{\mathcal{G}}}(\psi) > 2r$. Choose $\eta > 0$ in such a manner that $\eta < \frac{\mathcal{S}_{\|y_0 - z_0\|_{\mathcal{G}}}(\psi)}{2} - r$. Suppose $P = \{k \in \mathbb{N} : \mathcal{S}_{\|u_k - y_0\|_{\mathcal{G}}}(\psi) \geq r + \eta\}$ and $Q = \{k \in \mathbb{N} : \mathcal{S}_{\|u_k - z_0\|_{\mathcal{G}}}(\psi) \geq r + \eta\}$. Then, the following inequality

$$\frac{1}{n^\alpha}\left|\{k \leq n : k \in P \cup Q\}\right| \leq \frac{1}{n^\alpha}\left|\{k \leq n : k \in P\}\right| + \frac{1}{n^\alpha}\left|\{k \leq n : k \in Q\}\right|$$

holds and we have

$$\mathcal{I} - \lim_{n\to\infty}\frac{1}{n^\alpha}\left|\{k \leq n : k \in P \cup Q\}\right| \leq \mathcal{I} - \lim_{n\to\infty}\frac{1}{n^\alpha}\left|\{k \leq n : k \in P\}\right| + \mathcal{I} - \lim_{n\to\infty}\frac{1}{n^\alpha}\left|\{k \leq n : k \in Q\}\right|.$$

By our assumptions, the right-hand side of the above inequality vanishes and so is the left-hand side. Thus, for any $\gamma > 0$,

$$K(\gamma) = \left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\{k \leq n : k \in P \cup Q\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

Let $n_0 \in \mathbb{N} \setminus K(\frac{1}{2})$. Then, we have

$$\frac{1}{n_0^\alpha}\left|\{k \leq n_0 : k \in P \cup Q\}\right| < \frac{1}{2} \text{ i.e., } \frac{1}{n_0^\alpha}\left|\{k \leq n_0 : k \notin P \cup Q\}\right| \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

This means that $\{k \in \mathbb{N} : k \notin P \cup Q\} \neq \emptyset$. Let $k_0 \in \mathbb{N}$ be such that $k_0 \notin P \cup Q$. Then, $k_0 \in (\mathbb{N} \setminus P) \cap (\mathbb{N} \setminus Q)$ and eventually,

$$\mathcal{S}_{\|u_{k_0} - y_0\|_{\mathcal{G}}}(\psi) < r + \eta \text{ and } \mathcal{S}_{\|u_{k_0} - z_0\|_{\mathcal{G}}}(\psi) < r + \eta.$$

Therefore,

$$\mathcal{S}_{\|y_0 - z_0\|_{\mathcal{G}}}(\psi) \leq \mathcal{S}_{\|u_{k_0} - y_0\|_{\mathcal{G}}}(\psi) + \mathcal{S}_{\|u_{k_0} - z_0\|_{\mathcal{G}}}(\psi) < 2(r + \eta) < \mathcal{S}_{\|y_0 - z_0\|_{\mathcal{G}}}(\psi),$$

which is a contradiction. Hence, $diam\left(\mathcal{I}^{r,\alpha}_{st}(\mathcal{G}) - LIMu\right) \leq 2r$.

To prove the second part, let $u = (u_k)$ be a sequence in $(Y, \|\cdot\|_{\mathcal{G}})$ so that $u_k \xrightarrow{\mathcal{I}^{\alpha}_{st}(\mathcal{G})} u_0$. Then, for any $\psi \in (0,1]$ and $\eta, \gamma > 0$,

$$P = \left\{n \in \mathbb{N} : \tfrac{1}{n^\alpha} \left|\left\{k \leq n : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \geq \eta\right\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

Let $n \notin P$. Then, we have

$$\tfrac{1}{n^\alpha} \left|\left\{k \leq n : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \geq \eta\right\}\right| < \gamma$$

which further gives

$$\frac{1}{n^\alpha} \left|\left\{k \leq n : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) < \eta\right\}\right| \geq 1 - \gamma. \tag{1}$$

Let for $n \in \mathbb{N}$, P_n denote the set $\left\{k \leq n : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) < \eta\right\}$. Then, for each

$$y_0 \in (u_0 + \bar{N}(r, \psi)) = \left\{u \in Y : \mathcal{S}_{\|u_0 - u\|_{\mathcal{G}}}(\psi) \leq r\right\},$$

the following inequation

$$\mathcal{S}_{\|u_k - y_0\|_{\mathcal{G}}}(\psi) \leq \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) + \mathcal{S}_{\|u_0 - y_0\|_{\mathcal{G}}}(\psi) < r + \eta,$$

holds whenever $k \in P_n$. This proves that the inclusion

$$P_n \subseteq \left\{k \leq n : \mathcal{S}_{\|u_k - y_0\|_{\mathcal{G}}}(\psi) < r + \eta\right\}$$

holds and eventually from (1) we obtain

$$\begin{aligned} &\frac{1}{n^\alpha} \left|\left\{k \leq n : \mathcal{S}_{\|u_k - y_0\|_{\mathcal{G}}}(\psi) < r + \eta\right\}\right| \geq \frac{|P_n|}{n^\alpha} \geq 1 - \gamma \\ i.e., &\frac{1}{n^\alpha} \left|\left\{k \leq n : \mathcal{S}_{\|u_k - y_0\|_{\mathcal{G}}}(\psi) \geq r + \eta\right\}\right| < 1 - (1 - \gamma) = \gamma. \end{aligned}$$

Thus we have,

$$\left\{n \in \mathbb{N} : \tfrac{1}{n^\alpha} \left|\left\{k \leq n : \mathcal{S}_{\|u_k - y_0\|_{\mathcal{G}}}(\psi) \geq r + \eta\right\}\right| \geq \gamma\right\} \subseteq P \in \mathcal{I}.$$

This demonstrates that $y_0 \in \mathcal{I}^{r,\alpha}_{st}(\mathcal{G}) - LIMu$ and subsequently

$$\mathcal{I}^{r,\alpha}_{st}(\mathcal{G}) - LIMu = (u_0 + \bar{N}(r, \psi))$$

holds. Since, $diam((u_0 + \bar{N}(r, \psi))) = 2r$, so in general upper bound $2r$ of the diameter of the set $\mathcal{I}^{r,\alpha}_{st}(\mathcal{G}) - LIMu$ cannot be decreased anymore. □

Theorem 3.7. *Let $u = (u_k), w = (w_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, for a given $r \geq 0$ the following holds:*
(i) If $u_k \xrightarrow{\mathcal{I}^{r,\alpha}_{st}(\mathcal{G})} u_0$, then $\mu u_k \xrightarrow{\mathcal{I}^{r,\alpha}_{st}(\mathcal{G})} \mu u_0$, for any $\mu \in \mathbb{R}$.
(ii) If $u_k \xrightarrow{\mathcal{I}^{r,\alpha}_{st}(\mathcal{G})} u_0$ and $w_k \xrightarrow{\mathcal{I}^{r,\alpha}_{st}(\mathcal{G})} w_0$, then $u_k + w_k \xrightarrow{\mathcal{I}^{r,\alpha}_{st}(\mathcal{G})} u_0 + w_0$.

Proof. (i) If $\mu = 0$, then there is nothing to prove. So let us assume that $\mu \neq 0$. Then, since $u_k \xrightarrow{\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})} u_0$, so for each $\psi \in (0,1]$ and $\eta, \gamma > 0$, we get

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\left\{k \leq n : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \geq \frac{r+\eta}{|\mu|}\right\}\right| \geq \gamma\right\} \in \mathcal{I}. \tag{2}$$

Now

$$\begin{aligned}\frac{1}{n^\alpha}\left|\{k \leq n : \mathcal{S}_{\|\mu u_k - \mu u_0\|_{\mathcal{G}}}(\psi) \geq r+\eta\}\right| &= \frac{1}{n^\alpha}\left|\{k \leq n : |\mu| \cdot \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \geq r+\eta\}\right| \\ &\leq \frac{1}{n^\alpha}\left|\left\{k \leq n : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \geq \frac{r+\eta}{|\mu|}\right\}\right|.\end{aligned}$$

From the above inequation, the inclusion

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\left\{k \leq n : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \geq \frac{r+\eta}{|\mu|}\right\}\right| < \gamma\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\left\{k \leq n : \mathcal{S}_{\|\mu u_k - \mu u_0\|_{\mathcal{G}}}(\psi) \geq r+\eta\right\}\right| < \gamma\right\}$$

holds and consequently, from (2) we have,

$$\left\{n \in \mathbb{N} : \tfrac{1}{n^\alpha}\left|\{k \leq n : \mathcal{S}_{\|\mu u_k - \mu u_0\|_{\mathcal{G}}}(\psi) \geq r+\eta\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

Hence, $\mu u_k \xrightarrow{\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})} \mu u_0$.

(ii) Since $u_k \xrightarrow{\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})} u_0$ and $w_k \xrightarrow{\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})} w_0$ holds, so for each $\psi \in (0,1]$ and $\eta, \gamma > 0$, we get $P, Q \in \mathcal{I}$, where

$$P = \left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\left\{k \leq n : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \geq \frac{r+\eta}{2}\right\}\right| \geq \frac{\gamma}{2}\right\}$$

and

$$Q = \left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\left\{k \leq n : \mathcal{S}_{\|w_k - w_0\|_{\mathcal{G}}}(\psi) \geq \frac{r+\eta}{2}\right\}\right| \geq \frac{\gamma}{2}\right\}.$$

Let $M = (\mathbb{N} \setminus P) \cap (\mathbb{N} \setminus Q)$. Then, $M \in \mathcal{F}(\mathcal{I})$ and so $M \neq \emptyset$. Let us consider an element $m \in M$.

Then we have,

$$\tfrac{1}{m^\alpha}\left|\left\{k \leq m : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \geq \tfrac{r+\eta}{2}\right\}\right| < \tfrac{\gamma}{2}$$

and

$$\tfrac{1}{m^\alpha}\left|\left\{k \leq m : \mathcal{S}_{\|w_k - w_0\|_{\mathcal{G}}}(\psi) \geq \tfrac{r+\eta}{2}\right\}\right| < \tfrac{\gamma}{2}.$$

Now as the inclusion

$$\left\{k \leq m : \mathcal{S}_{\|(u_k + w_k) - (u_0 + y_0)\|_{\mathcal{G}}}(\psi) \geq r+\eta\right\} \subseteq \left\{k \leq m : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \geq \frac{r+\eta}{2}\right\} \cup \left\{k \leq m : \mathcal{S}_{\|w_k - w_0\|_{\mathcal{G}}}(\psi) \geq \frac{r+\eta}{2}\right\}$$

supplies, so we have to obtain

$$\begin{aligned}&\frac{1}{m^\alpha}\left|\left\{k \leq m : \mathcal{S}_{\|(u_k + w_k) - (u_0 + w_0)\|_{\mathcal{G}}}(\psi) \geq r+\eta\right\}\right| \\ &\leq \frac{1}{m^\alpha}\left|\left\{k \leq m : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \geq \frac{r+\eta}{2}\right\}\right| \\ &\qquad + \frac{1}{m^\alpha}\left|\left\{k \leq m : \mathcal{S}_{\|w_k - w_0\|_{\mathcal{G}}}(\psi) \geq \frac{r+\eta}{2}\right\}\right| \\ &< \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma.\end{aligned}$$

According to the above inequality, we can conclude that

$$\left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left|\left\{k \leq n : \mathcal{S}_{\|(u_k+w_k)-(u_0+y_0)\|_{\mathcal{G}}}(\psi) \geq r+\eta\right\}\right| \geq \gamma\right\} \subseteq \mathbb{N} \setminus M.$$

Since $P, Q \in \mathcal{I}$, so $\mathbb{N} \setminus M = P \cup Q \in \mathcal{I}$ and so,

$$\left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left|\left\{k \leq n : \mathcal{S}_{\|(u_k+w_k)-(u_0+y_0)\|_{\mathcal{G}}}(\psi) \geq r+\eta\right\}\right| \geq \gamma\right\} \in \mathcal{I}$$

and the proof ends. □

Theorem 3.8. *Let $0 < \alpha \leq \beta \leq 1$, then $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G}) \subseteq \mathcal{I}_{st}^{r,\beta}(\mathcal{G})$ holds for a fixed $r \geq 0$.*

Proof. Let $u = (u_k)$ be a sequence in the $(Y, \|\cdot\|_{\mathcal{G}})$ so that $u_k \xrightarrow{\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})} u_0$. Then, the result follows from the subsequent inclusion

$$\left\{n \in \mathbb{N} : \frac{1}{n^{\beta}} \left|\left\{k \leq n : \mathcal{S}_{\|u_k-u_0\|_{\mathcal{G}}}(\psi) \geq r+\eta\right\}\right| \geq \gamma\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left|\left\{k \leq n : \mathcal{S}_{\|u_k-u_0\|_{\mathcal{G}}}(\psi) \geq r+\eta\right\}\right| \geq \gamma\right\}.$$

□

Definition 3.9. A sequence $u = (u_k)$ in the GNLS $(Y, \|\cdot\|_{\mathcal{G}})$ is called to be gradually $\mathcal{I}-$statistically bounded (in short $\mathcal{I}_{st}(\mathcal{G})-$bounded) provided that for each $\psi \in (0, 1]$, there is an $M > 0$ so that

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{k \leq n : \mathcal{S}_{\|u_k\|_{\mathcal{G}}}(\psi) \geq M\right\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

Definition 3.10. A sequence $u = (u_k)$ in the GNLS $(Y, \|\cdot\|_{\mathcal{G}})$ is called to be gradually $\mathcal{I}-$statistically bounded of order $\alpha(0 < \alpha \leq 1)$ (in short $\mathcal{I}_{st}^{\alpha}(\mathcal{G})-$ bounded) provided that there is an $M > 0$ so that

$$\left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left|\{k \leq n : \mathcal{S}_{\|u_k\|_{\mathcal{G}}}(\psi) \geq M\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

Theorem 3.11. *Every $\mathcal{I}_{st}^{\alpha}(\mathcal{G})-$bounded sequence is also $\mathcal{I}_{st}(\mathcal{G})-$bounded for $0 < \alpha \leq 1$.*

Proof. The proof follows directly from the definition, so omitted. □

Theorem 3.12. *A sequence $u = (u_k)$ in the GNLS $(Y, \|\cdot\|_{\mathcal{G}})$ is $\mathcal{I}_{st}^{\alpha}(\mathcal{G})-$ bounded iff there exists some $r \geq 0$ such that $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G}) - LIMu \neq \emptyset$.*

Proof. Firstly, let us presume that $u = (u_k)$ be a $\mathcal{I}_{st}^{\alpha}(\mathcal{G})-$bounded sequence in Y. Suppose

$$P = \left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left|\{k \leq n : \mathcal{S}_{\|u_k\|_{\mathcal{G}}}(\psi) \geq M\}\right| \geq \gamma\right\} \in \mathcal{I}$$

for some $M > 0$. Suppose $r' = \sup\{\mathcal{S}_{\|u_k\|_{\mathcal{G}}}(\psi) : k \leq m, m \in \mathbb{N} \setminus P\}$. Then, the set $\mathcal{I}_{st}^{r',\alpha}(\mathcal{G}) - LIMu$ includes the origin of X so that $\mathcal{I}_{st}^{r',\alpha}(\mathcal{G}) - LIMu \neq \emptyset$.

Conversely, presume that there is a $r \geq 0$ so that $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G}) - LIMu \neq \emptyset$. Let $u_0 \in \mathcal{I}_{st}^{r,\alpha}(\mathcal{G}) - LIMu$. Then, for every $\eta, \gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left|\left\{k \leq n : \mathcal{S}_{\|u_k-u_0\|_{\mathcal{G}}}(\psi) \geq r+\eta\right\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

Fix $\eta = \mathcal{S}_{\|u_0\|_{\mathcal{G}}}(\psi)$ and let $M = r + \mathcal{S}_{\|u_0\|_{\mathcal{G}}}(\psi)$. Then, for each $\gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left|\left\{k \leq n : \mathcal{S}_{\|u_k-u_0\|_{\mathcal{G}}}(\psi) \geq M\right\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

Hence, $u = (u_k)$ is $\mathcal{I}-$statistically bounded of order $\alpha(0 < \alpha \leq 1)$. □

Theorem 3.13. *Let $u = (u_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, the set $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G}) - LIMu$ is convex.*

Proof. Let $y_0, y_1 \in \mathcal{I}_{st}^{r,\alpha}(\mathcal{G}) - LIMu$ and $\eta > 0$ be given. Suppose $P_0 = \{k \in \mathbb{N} : \mathcal{S}_{\|u_k - y_0\|_{\mathcal{G}}}(\psi) \geq r + \eta\}$ and $P_1 = \{k \in \mathbb{N} : \mathcal{S}_{\|u_k - y_1\|_{\mathcal{G}}}(\psi) \geq r + \eta\}$. According to Theorem 3.6, for any $\gamma > 0$, we have the following:

$$\left\{n \in \mathbb{N} : \tfrac{1}{n^\alpha} \left|\{k \leq n : k \in P_0 \cup P_1\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

Select $0 < \gamma_1 < 1$ so that $0 < 1 - \gamma_1 < \gamma$.

Take

$$P = \left\{n \in \mathbb{N} : \tfrac{1}{n^\alpha} \left|\{k \leq n : k \in P_0 \cup P_1\}\right| \geq 1 - \gamma_1\right\}.$$

Then, $P \in \mathcal{I}$ and for every $n \notin P$ we have

$$\frac{1}{n^\alpha} \left|\{k \leq n : k \in P_0 \cup P_1\}\right| < 1 - \gamma_1$$
$$\Rightarrow \frac{1}{n^\alpha} \left|\{k \leq n : k \notin P_0 \cup P_1\}\right| \geq 1 - (1 - \gamma_1) = \gamma_1.$$

Therefore, the set $\{k \in \mathbb{N} : k \notin P_0 \cup P_1\}$ is nonempty. Let $k_0 \in (\mathbb{N} \setminus P_0) \cap (\mathbb{N} \setminus P_1)$.

Then, for any $0 \leq \lambda \leq 1$,

$$\begin{aligned}\mathcal{S}_{\|u_{k_0} - (1-\lambda)y_0 - \lambda y_1\|_{\mathcal{G}}}(\psi) &\leq (1-\lambda)\mathcal{S}_{\|u_{k_0} - y_0\|_{\mathcal{G}}}(\psi) + \lambda \mathcal{S}_{\|u_{k_0} - y_1\|_{\mathcal{G}}}(\psi) \\ &< (1-\lambda)(r + \eta) + \lambda(r + \eta) = r + \eta.\end{aligned}$$

Let $Q = \{k \in \mathbb{N} : \mathcal{S}_{\|u_{k_0} - ((1-\lambda)y_0 + \lambda y_1)\|_{\mathcal{G}}}(\psi) \geq r + \eta\}$.

Then, obviously $(\mathbb{N} \setminus P_0) \cap (\mathbb{N} \setminus P_1) \subseteq \mathbb{N} \setminus Q$.

So, for $n \notin P$,

$$\gamma_1 \leq \tfrac{1}{n^\alpha} \left|\{k \leq n : k \notin P_0 \cup P_1\}\right| \leq \tfrac{1}{n^\alpha} \left|\{k \leq n : k \notin Q\}\right|.$$

In other words,

$$\tfrac{1}{n^\alpha} \left|\{k \leq n : k \in Q\}\right| < 1 - \gamma_1 < \gamma.$$

This gives that

$$\mathbb{N} \setminus P \subseteq \left\{n \in \mathbb{N} : \tfrac{1}{n^\alpha} \left|\{k \leq n : k \in Q\}\right| < \gamma\right\}.$$

By taking complements on both sides of the above inclusion and utilizing the fact that $P \in \mathcal{I}$, we obtain

$$\left\{n \in \mathbb{N} : \tfrac{1}{n^\alpha} \left|\{k \leq n : k \in Q\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

This concludes the proof. □

Theorem 3.14. *Let $u = (u_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, the set $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G}) - LIMu$ is gradually closed.*

Proof. When $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G}) - LIMu = \emptyset$, then there is nothing to prove. So let us admit that

$$\mathcal{I}_{st}^{r,\alpha}(\mathcal{G}) - LIMu \neq \emptyset.$$

Consider a sequence (w_k) in $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G}) - LIMu$ such that $(w_k) \xrightarrow{\mathcal{G}} y_0$. Select $\eta, \gamma > 0$. Then, for any $\psi \in (0, 1]$, there is a $i_{\frac{\eta}{2}} \in \mathbb{N}$ so that

$$\mathcal{S}_{\|w_k - y_0\|_{\mathcal{G}}}(\psi) < \tfrac{\eta}{2} \text{ for all } k > i_{\frac{\eta}{2}}.$$

Let $k_0 > i_{\frac{\eta}{2}}$. Then, $y_{k_0} \in \mathcal{I}_{st}^{r,\alpha}(\mathcal{G}) - LIMu$ and ultimately

$$P = \left\{n \in \mathbb{N} : \tfrac{1}{n^\alpha} \left|\left\{k \leq n : \mathcal{S}_{\|u_k - y_{k_0}\|_{\mathcal{G}}}(\psi) \geq r + \tfrac{\eta}{2}\right\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

Assume M indicate the set $\mathbb{N} \setminus P$. Then, $\emptyset \neq M \in \mathcal{F}(\mathcal{I})$. Take $n \in M$. Then, we get

$$\frac{1}{n^{\alpha}} \left| \left\{ k \leq n : \mathcal{S}_{\|u_k - y_{k_0}\|_{\mathcal{G}}}(\psi) \geq r + \frac{\eta}{2} \right\} \right| < \gamma$$
$$\Rightarrow \frac{1}{n^{\alpha}} \left| \left\{ k \leq n : \mathcal{S}_{\|u_k - y_{k_0}\|_{\mathcal{G}}}(\psi) < r + \frac{\eta}{2} \right\} \right| \geq 1 - \gamma.$$

Let $Q_n = \left\{ k \leq n : \mathcal{S}_{\|u_k - y_{k_0}\|_{\mathcal{G}}}(\psi) < r + \frac{\eta}{2} \right\}$. Then, for $k \in Q_n$,

$$\mathcal{S}_{\|u_k - y_0\|_{\mathcal{G}}}(\psi) \leq \mathcal{S}_{\|u_k - y_{k_0}\|_{\mathcal{G}}}(\psi) + \mathcal{S}_{\|y_{k_0} - y_0\|_{\mathcal{G}}}(\psi) < r + \frac{\eta}{2} + \frac{\eta}{2} = r + \eta.$$

Hence, $Q_n \subseteq \left\{ k \leq n : \mathcal{S}_{\|u_k - y_0\|_{\mathcal{G}}}(\psi) < r + \eta \right\}$ which gives that

$$1 - \gamma \leq \frac{|Q_n|}{n^{\alpha}} \leq \frac{1}{n^{\alpha}} \left| \left\{ k \leq n : \mathcal{S}_{\|u_k - y_0\|_{\mathcal{G}}}(\psi) < r + \eta \right\} \right|.$$

Therefore,

$$\frac{1}{n^{\alpha}} \left| \left\{ k \leq n : \mathcal{S}_{\|u_k - y_0\|_{\mathcal{G}}}(\psi) \geq r + \eta \right\} \right| < 1 - (1 - \gamma) = \gamma.$$

This means that

$$\left\{ n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left| \left\{ k \leq n : \mathcal{S}_{\|u_k - y_0\|_{\mathcal{G}}}(\psi) \geq r + \eta \right\} \right| \geq \gamma \right\} \subseteq P \in \mathcal{I}.$$

So, $y_0 \in \mathcal{I}_{st}^{r,\alpha}(\mathcal{G}) - LIMu$ and this conludes the proof. □

Theorem 3.15. *A sequence $u = (u_k)$ in the GNLS $(Y, \|\cdot\|_{\mathcal{G}})$ is $\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})-$ convergent to $u_0 \in Y$ with roughness degree $r \geq 0$ iff there is a sequence $w = (w_k)$ in X which is $\mathcal{I}_{st}^{\alpha}(\mathcal{G})-$convergent to u_0 and $\mathcal{S}_{\|u_k - w_k\|_{\mathcal{G}}}(\psi) \leq r$ for all $k \in \mathbb{N}$.*

Proof. Let $w = (w_k)$ be a sequence in X such that $\mathcal{S}_{\|u_k - w_k\|_{\mathcal{G}}}(\psi) \leq r$ for all $k \in \mathbb{N}$ and $(w_k) \xrightarrow{\mathcal{I}_{st}^{\alpha}(\mathcal{G})} u_0$.

Then, by definition for every $\eta > 0$ and $\gamma > 0$,

$$P = \left\{ n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left| \left\{ k \leq n : \mathcal{S}_{\|w_k - u_0\|_{\mathcal{G}}}(\psi) \geq \eta \right\} \right| \geq \gamma \right\} \in \mathcal{I}.$$

Let $n \notin P$. Then, we have

$$\frac{1}{n^{\alpha}} \left| \left\{ k \leq n : \mathcal{S}_{\|w_k - u_0\|_{\mathcal{G}}}(\psi) \geq \eta \right\} \right| < \gamma$$

which further gives

$$\frac{1}{n^{\alpha}} \left| \left\{ k \leq n : \mathcal{S}_{\|w_k - u_0\|_{\mathcal{G}}}(\psi) < \eta \right\} \right| \geq 1 - \gamma. \tag{3}$$

Let for $n \in \mathbb{N}$, P_n indicate the set $\left\{ k \leq n : \mathcal{S}_{\|w_k - u_0\|_{\mathcal{G}}}(\psi) < \eta \right\}$.

Then, for $k \in P_n$,

$$\mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) \leq \mathcal{S}_{\|u_k - w_k\|_{\mathcal{G}}}(\psi) + \mathcal{S}_{\|w_k - u_0\|_{\mathcal{G}}}(\psi) < r + \eta.$$

This demonstrates that the inclusion

$$P_n \subseteq \left\{ k \leq n : \mathcal{S}_{\|u_k - u_0\|_{\mathcal{G}}}(\psi) < r + \eta \right\}$$

supplies and eventually from (3) we obtain

$$\frac{1}{n^\alpha}\left|\left\{k \le n : \mathcal{S}_{\|u_k-u_0\|_\mathcal{G}}(\psi) < r+\eta\right\}\right| \ge \frac{|P_n|}{n^\alpha} \ge 1-\gamma$$
$$i.e., \frac{1}{n^\alpha}\left|\left\{k \le n : \mathcal{S}_{\|u_k-u_0\|_\mathcal{G}}(\psi) \ge r+\eta\right\}\right| < 1-(1-\gamma) = \gamma.$$

Thus,

$$\left\{n \in \mathbb{N} : \tfrac{1}{n^\alpha}\left|\left\{k \le n : \mathcal{S}_{\|u_k-u_0\|_\mathcal{G}}(\psi) \ge r+\eta\right\}\right| \ge \gamma\right\} \subseteq P \in \mathcal{I}.$$

As a result, $u_k \xrightarrow{\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})} u_0$.

For the converse part, let us admit that $u_k \xrightarrow{\mathcal{I}_{st}^{r,\alpha}(\mathcal{G})} u_0$.

Then, by definition for every $\eta, \gamma > 0$,

$$Q = \left\{n \in \mathbb{N} : \tfrac{1}{n^\alpha}\left|\left\{k \le n : \mathcal{S}_{\|u_k-u_0\|_\mathcal{G}}(\psi) \ge r+\eta\right\}\right| \ge \gamma\right\} \in \mathcal{I}.$$

Let $n \notin Q$. Then, we obtain

$$\tfrac{1}{n^\alpha}\left|\left\{k \le n : \mathcal{S}_{\|u_k-u_0\|_\mathcal{G}}(\psi) \ge r+\eta\right\}\right| < \gamma$$

which further gives

$$\frac{1}{n^\alpha}\left|\left\{k \le n : \mathcal{S}_{\|u_k-u_0\|_\mathcal{G}}(\psi) < r+\eta\right\}\right| \ge 1-\gamma. \tag{4}$$

Presume for $n \in \mathbb{N}$, Q_n indicate the set $\left\{k \le n : \mathcal{S}_{\|u_k-u_0\|_\mathcal{G}}(\psi) < r+\eta\right\}$.

Determine a sequence $w = (w_k)$ by

$$(w_k) = \begin{cases} u_0, & if\ \mathcal{S}_{\|u_k-u_0\|_\mathcal{G}}(\psi) \le r \\ u_k + r\frac{u_0-u_k}{\mathcal{S}_{\|u_k-u_0\|_\mathcal{G}}(\psi)}, & otherwise. \end{cases}$$

At that time, it can be easily observed that $\mathcal{S}_{\|u_k-w_k\|_\mathcal{G}}(\psi) \le r$ for all $k \in \mathbb{N}$. In addition, $\mathcal{S}_{\|w_k-u_0\|_\mathcal{G}}(\psi) = 0$, whenever $\mathcal{S}_{\|u_k-u_0\|_\mathcal{G}}(\psi) \le r$ and $\mathcal{S}_{\|w_k-u_0\|_\mathcal{G}}(\psi) < \eta$, whenever $r < \mathcal{S}_{\|u_k-u_0\|_\mathcal{G}}(\psi) < r+\eta$.

The above fact demonstrates that the following inclusion

$$Q_n \subseteq \left\{k \le n : \mathcal{S}_{\|w_k-u_0\|_\mathcal{G}}(\psi) < \eta\right\}$$

supplies and subsequently from (4),

$$\frac{1}{n^\alpha}\left|\left\{k \le n : \mathcal{S}_{\|w_k-u_0\|_\mathcal{G}}(\psi) < \eta\right\}\right| \ge \frac{|Q_n|}{n^\alpha} \ge 1-\gamma$$
$$i.e., \frac{1}{n^\alpha}\left|\left\{k \le n : \mathcal{S}_{\|w_k-u_0\|_\mathcal{G}}(\psi) \ge \eta\right\}\right| < \gamma.$$

So,

$$\left\{n \in \mathbb{N} : \tfrac{1}{n^\alpha}\left|\left\{k \le n : \mathcal{S}_{\|w_k-u_0\|_\mathcal{G}}(\psi) \ge \eta\right\}\right| \ge \gamma\right\} \subseteq Q \in \mathcal{I}.$$

As a result, $(w_k) \xrightarrow{\mathcal{I}_{st}^{\alpha}(\mathcal{G})} u_0$. This concludes the proof. □

Definition 3.16. A point τ is named to be $\mathcal{I}-$statistically cluster point of order α of a sequence $u = (u_k)$ in the GNLS $(Y, \|\cdot\|_\mathcal{G})$ if

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\left\{k \le n : \mathcal{S}_{\|u_k-\tau\|_\mathcal{G}}(\psi) \ge \eta\right\}\right| < \gamma\right\} \notin \mathcal{I}.$$

For any sequence $u = (u_k)$, the set of all $\mathcal{I}-$statistically cluster point of order α is denoted by $C_u\left(\mathcal{I}^{\alpha}_{st}(\mathcal{G})\right)$.

Theorem 3.17. *Let $u = (u_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, for an arbitrary $\tau \in C_u\left(\mathcal{I}^{\alpha}_{st}(\mathcal{G})\right)$ and $r > 0$, $\mathcal{S}_{\|u_0-\tau\|_{\mathcal{G}}}(\psi) < r$ for all $u_0 \in \mathcal{I}^{r,\alpha}_{st}(\mathcal{G}) - LIMu$.*

Proof. If possible let us presume that for $\tau \in C_x\left(\mathcal{I}^{\alpha}_{st}(\mathcal{G})\right)$ there is a $u_0 \in \mathcal{I}^{r,\alpha}_{st}(\mathcal{G}) - LIMu$ supplying $\mathcal{S}_{\|u_0-\tau\|_{\mathcal{G}}}(\psi) \geq r$. Fix $\eta = \frac{\mathcal{S}_{\|u_0-\tau\|_{\mathcal{G}}}(\psi)-r}{2}$. Then, we get for any $\gamma > 0$,

$$P = \left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}}\left|\left\{k \leq n : \mathcal{S}_{\|u_k-\tau\|_{\mathcal{G}}}(\psi) \geq \eta\right\}\right| < \gamma\right\} \notin \mathcal{I}.$$

Assume

$$Q = \left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}}\left|\left\{k \leq n : \mathcal{S}_{\|u_k-u_0\|_{\mathcal{G}}}(\psi) \geq r + \eta\right\}\right| \geq \gamma\right\}.$$

Let $m \in P$ be so that for $k \leq m, \mathcal{S}_{\|u_k-\tau\|_{\mathcal{G}}}(\psi) < \eta$. Then, we have

$$\mathcal{S}_{\|u_k-u_0\|_{\mathcal{G}}}(\psi) \geq \mathcal{S}_{\|u_0-\tau\|_{\mathcal{G}}}(\psi) - \mathcal{S}_{\|u_k-\tau\|_{\mathcal{G}}}(\psi) > r + \eta.$$

This gives that $P \subseteq Q$ and as a result, $Q \notin \mathcal{I}$, which contradicts to the fact that $u_0 \in \mathcal{I}^{r,\alpha}_{st}(\mathcal{G}) - LIMu$. This concludes the proof. □

4. Conclusion

In this paper, we have extended the concept of $\mathcal{I}$-statistical convergence to rough $\mathcal{I}$-statistical convergence in a GNLS. The theorems we have established, namely Theorem 3.6, Theorem 3.13, and Theorem 3.14, provide insights into various properties of the limit set $\mathcal{I}^{r,\alpha}_{st}(\mathcal{G}) - LIMu$ of a sequence $u = (u_k)$. Additionally, we have presented a theorem, namely Theorem 3.15, which provides an equivalent condition for the $\mathcal{I}^{\alpha}_{st}(\mathcal{G})$-convergence of a sequence. These results can be valuable for future research in exploring rough $\mathcal{I}$-statistically Cauchy sequences in a GNLS.

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