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Commutative comultiplications on the localizations of a wedge sum of spheres and Moore spaces

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Abstract. Let $\mathcal{P}$ be a collection of prime numbers and let $M(G, n)$ be the Moore space of type (G, n), where G is a finitely generated abelian group and n is a positive integer. This paper focuses on the homotopy commutative comultiplication structures on the localization $X_{\mathcal{P}}$ of a wedge $X := \mathbb{S}^m \vee M(G, n)$ of the m-spheres and Moore spaces for $2 \leq m < n$. A list of examples is provided for examination of the phenomena of commutative comultiplications on $X_{\mathcal{P}}$ up to homotopy.

1. Introduction

1.1. *Co-Hopf spaces and Moore spaces*

A co-Hopf space [2] with a homotopy comultiplication and a Hopf space with a homotopy multiplication are the pivotal object classes in the pointed homotopy category and they are Eckmann-Hilton duals [11, 29] with one another in classical homotopy theory. In general terms, a co-Hopf space is composed of many distinctive homotopy comultiplications as well as a number of different properties. In addition, calculation of the cardinality of the set of homotopy (or algebraic) comultiplications is complicated, involving a complex process; see [4–6, 22] and [26] regarding the wedge sum of spheres.

Studies on homotopy comultiplications and same n-types based on various co-Hopf spaces with standard homotopy comultiplications have been reported by several authors to date; see [2] regarding co-Hopf spaces, [18–21] regarding the same n-types of suspension spaces, [14] regarding the local cohomology spectral sequence from the theoretical equivariant homotopy point of view, and [8–10, 24, 27] for digital Hopf spaces and digital Pontryagin algebras.

In classical homotopy theory, a Moore space $M(G, n)$ is the homology analogue of the Eilenberg-MacLane space $K(G, n)$, where G is an abelian group and n is a positive integer. In general, calculation of the cardinal number of the set (or group) $\mathcal{C}(X)$ of homotopy classes of all homotopy comultiplications on a co-Hopf space X is not easy. Pertaining to a special case, as demonstrated in [3], a set-theoretic bijection exists between the set (or group) of homotopy comultiplications $\mathcal{C}(M(G, 2))$ and the group $\mathrm{Ext}(G, G \otimes G)$, and if $n \geq 3$, then $\mathcal{C}(M(G, n))$ is the set consisting of a single class of homotopy as the standard comultiplication up to homotopy; see also [12] regarding a wedge sum of two Moore spaces.

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1.2. Localizations

The localization of a space or a topological (equivariant) spectrum at a collection of prime numbers is similar to the localization of a commutative ring or a module at a collection of prime numbers. The fact that any nilpotent CW-complex could be localized at a collection of prime numbers up to homotopy as a topological counterpart of the localization of algebraic objects is well known. In algebraic topology, according to the localization of a nilpotent CW-space, introduced by D. Sullivan [30], any nilpotent CW-space could be localized up to homotopy at a prime number; see [7] and [16] for the relevant and standard references.

1.3. Goal and organization

The aim of our research is to develop the homotopy comultiplications and commutative comultiplications of the localization at a collection $\mathcal{P}$ of prime numbers of a wedge of the spheres and Moore spaces. In particular, we focus on development and examination of the set of all homotopy commutative comultiplications on the localization $X_{\mathcal{P}}$ of the wedge sum $X := \mathbb{S}^m \vee M(G, n)$ of the spheres and Moore spaces, where $2 \leq m < n$, and G is a finitely generated abelian group.

A description of the fundamental concepts of homotopy comultiplications, Milnor's formula, and the Hopf-Whitney classification is provided in Section 2. A description of the localization counterparts of the Hilton-Milnor formulas along with development of the pivotal concepts regarding the forms of comultiplications on the localization $X_{\mathcal{P}}$ of the wedge product $X := \mathbb{S}^m \vee M(G, n)$ of spheres and Moore spaces of type (G, n) up to homotopy is provided in Section 3. The homotopy commutative comultiplications, as well as a method for calculating the number of possible homotopy commutative comultiplications on the localizations of the wedge sum of the CW-spaces are investigated in Section 4. Examples for use in examining the phenomena of homotopy commutative comultiplications on $X_{\mathcal{P}}$ for a collection $\mathcal{P}$ of prime numbers are provided in Section 5.

1.4. Convention

Most of the spaces described in this paper are subject to the object classes in the homotopy category of pointed connected spaces and homotopy classes of continuous maps that preserves the base point. The notations '$\cong$' and '$\simeq$' for a group isomorphism and a pointed homotopy relation, respectively, will mainly be utilized. For our notational convenience, the homotopy class $\langle f \rangle$ of a homotopy set (or group or abelian group) $[(X, x), (Y, y)]$ consisting of homotopy classes of base point preserving continuous maps from (X, x) to (Y, y) is replaced primarily by $f : (X, x) \to (Y, y)$.

2. Milnor-Hopf-Whitney theorems

A *co-Hopf space* is a pair (X, φ) consisting of a pointed space $X := (X, x_0)$ and a base point preserving continuous map $\varphi : (X, x_0) \to (X, x_0) \vee (X, x_0)$ such that

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & X \vee X \\ & {\scriptstyle 1_X}\searrow & \downarrow{\scriptstyle \pi^1} \\ & & X \end{array}$$

and

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & X \vee X \\ & {\scriptstyle 1_X}\searrow & \downarrow{\scriptstyle \pi^2} \\ & & X \end{array}$$

are homotopy commutative diagrams; that is,

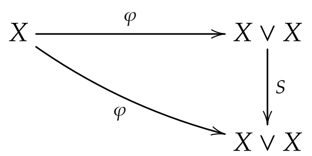
$$\pi^1 \circ \varphi \simeq 1_X \simeq \pi^2 \circ \varphi,$$

where

- x_0 is the base point of X;
- 1_X is the identity map of X; and
- $\pi^1, \pi^2 : X \vee X \to X$ are the first and second projections, respectively.

The base point preserving continuous map $\varphi : X \to X \vee X$ is said to be a *homotopy comultiplication* (or simply *comultiplication*) on X.

Definition 2.1. A homotopy comultiplication $\varphi : X \to X \vee X$ on X is said to be *homotopy commutative* if the triangle



commutes up to homotopy, where $S : X \vee X \to X \vee X$ is the switching map sending (x, x_0) to (x_0, x) and (x_0, x) to (x, x_0) for all $x \in X$, where x_0 is the base point of X.

Note that a commutative comultiplication on a co-Hopf space can be regarded as an Eckmann-Hilton dual notion of a commutative multiplication on a Hopf space, which is a counterpart of homotopy and a generalization of the commutative law of a group in group theory.

We refer to the original result reported by P. J. Hilton [15] with respect to the basic Whitehead products and their heights; see [5] and [6] for additional details. Furthermore, a generalization of Hilton's work was developed by J. W. Milnor [28] as follows.

Theorem 2.2. *Let A and B be connected CW-complexes. Then, there exists a homotopy equivalence*

$$\Omega\Sigma(A \vee B) \simeq \Omega\Sigma A \times \Omega\Sigma\Big(\bigvee_{i\geq 0} A^{\wedge i} \wedge B\Big), \tag{1}$$

where $A^{\wedge i} = \underbrace{A \wedge A \wedge \cdots \wedge A}_{i-\text{times}}$ *is the i-fold smash product.*

Proof. See [28] for additional details. □

We end this section with a description of the Hopf-Whitney classification theorem as follows.

Theorem 2.3. *Let X be an n-dimensional CW-complex and Y an $(n-1)$-connected n-simple space. Then there is a one-to-one correspondence*

$$[X, Y] \approx H^n(X; \pi_n(Y))$$

as sets.

Proof. See [32, page 244] for additional details. □

3. Localized homotopy comultiplications

Let $\mathcal{P}$ be a collection of prime numbers, or the empty collection which is denoted by ϕ in this case; that is, $\mathcal{P} = \phi$. A group G is called a *$\mathcal{P}$-local group* if the self-map $p : G \to G$ defined by

$$p(g) = \underbrace{g * g * \cdots * g}_{p-\text{times}}$$

is a one-to-one correspondence between G and itself for all $p \in \mathcal{P}^c$, the complement of $\mathcal{P}$, where $*$ is the binary operation of G. From the topological point of view, a nilpotent CW-space X is called a *$\mathcal{P}$-local space* if the nth homotopy group $\pi_n(X, x_0)$ is $\mathcal{P}$-local for all $n \geq 1$; see [7] and [30] regarding the localization of a nilpotent CW-space.

Development of the structure of all homotopy comultiplications of the localization $X_{\mathcal{P}}$ of the wedge sum $X := \mathbb{S}^m \vee M(G, n)$ of the m-spheres and Moore spaces up to homotopy is described in this section, where $2 \leq m < n$, and G is a finitely generated abelian group.

3.1. Localizations of Hilton-Milnor formulas

Examination of the localized counterparts of the Hilton-Milnor formulas described in Section 2 is as follows.

Proposition 3.1. *Let $X_{\mathcal{P}}$ be the localized wedge sum of spheres $X := \mathbb{S}^{m_1} \vee \mathbb{S}^{m_2} \vee \cdots \vee \mathbb{S}^{m_k}$ with $2 \leq m_i$, $i = 1, 2, \ldots, k$ at a collection $\mathcal{P}$ of prime numbers. Then, there is an isomorphism*

$$\pi_n(X_{\mathcal{P}}) \cong \bigoplus_{j=1}^{\infty} \pi_n(\mathbb{S}_{\mathcal{P}}^{h_{w_j}}) \tag{2}$$

of homotopy groups, where h_{w_j} is the height of the basic (generalized) Whitehead product w_j for all $j \geq 1$. In general, if M is a nilpotent CW-complex with a suspension structure, then there is an isomorphism

$$[M, X_{\mathcal{P}}] \cong \bigoplus_{j=1}^{\infty} [M, \mathbb{S}_{\mathcal{P}}^{h_{w_j}}] \tag{3}$$

of abelian groups.

Proof. See [23, Theorem 2] regarding the formulas (2) and (3), and [1] regarding generalized Whitehead products. □

3.2. Comultiplications on the localizations

Note that a finitely generated abelian group G can be decomposed as a free $\mathbb{Z}$-module of finite rank r and a torsion subgroup T of G; that is,

$$G \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r-\text{times}} \oplus \underbrace{\mathbb{Z}_{p_1{}^{s_1}} \oplus \mathbb{Z}_{p_2{}^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k{}^{s_k}}}_{\text{a finite subgroup } T \text{ of } G}; \tag{4}$$

see [17, page 78] for additional details, where p_i is a prime number and s_i is a positive integer for $i = 1, 2, \ldots, k$.
Notation. The following notations will be used throughout this report.

- G is a finitely generated abelian group decomposed as in (4).
- $X := \mathbb{S}^m \vee M(G, n)$ with $2 \leq m < n$.
- $\alpha : \mathbb{S}^m \to X$ is the first inclusion map.

- $\beta : M(G,n) \to X$ is the second inclusion map.
- $j^1, j^2 : \mathbb{S}^m \to \mathbb{S}^m \vee \mathbb{S}^m$ are the first and second inclusion maps, respectively.
- $\iota^1, \iota^2 : X \to X \vee X$ are the first and second inclusion maps, respectively.
- $\pi^1, \pi^2 : X \vee X \to X$ are the first and second projection maps, respectively.
- $\mathcal{P}$ is a collection of prime numbers.
- $X_{\mathcal{P}}$ is the topological localization of a nilpotent CW-space X at $\mathcal{P}$.
- $f_{\mathcal{P}} : X_{\mathcal{P}} \to Y_{\mathcal{P}}$ is the $\mathcal{P}$-localization of a map $f : X \to Y$ between connected nilpotent CW-spaces X and Y.
- $S : X \vee X \to X \vee X$ is the switching map sending (x, x_0) to (x_0, x) and (x_0, x) to (x, x_0) for all $x \in X$, where x_0 is the base point of X, or its $\mathcal{P}$-localization.
- $C(X) \subseteq [X; X \vee X]$ is the set of all homotopy classes of homotopy commutative comultiplications on $X := \mathbb{S}^m \vee M(G,n)$, or its localizations.

Development of the homotopy classes for consideration of all homotopy comultiplications on the $\mathcal{P}$-localization $X_{\mathcal{P}} := \mathbb{S}^m_{\mathcal{P}} \vee M(G,n)_{\mathcal{P}}$ of X at a collection $\mathcal{P}$ of prime numbers is as follows.

Theorem 3.2. *Let $X_{\mathcal{P}}$ be the $\mathcal{P}$-localization of $X := \mathbb{S}^m \vee M(G,n)$ with $2 \leq m < n$, where G is a finitely generated abelian group decomposed as in (4). Then, each comultiplication*

$$\varphi = \varphi_Q : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$$

can be expressed as follows:

$$\begin{cases} \varphi \circ \alpha_{\mathcal{P}} & \simeq \iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}} + \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \\ \varphi \circ \beta_{\mathcal{P}} & \simeq \iota^1_{\mathcal{P}} \circ \beta_{\mathcal{P}} + \iota^2_{\mathcal{P}} \circ \beta_{\mathcal{P}} + Q. \end{cases} \tag{5}$$

Here,

- *The additions originate from the homotopy additions in $\pi_m(X_{\mathcal{P}} \vee X_{\mathcal{P}})$ and $[M(G,n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}]$, respectively;*
- *$Q = (Q_1, Q_2, \ldots, Q_t, \ldots, Q_r, Q_T) : M(G,n)_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$;*
- *$Q_t = (\alpha_{\mathcal{P}} \vee \alpha_{\mathcal{P}})_\sharp(\sum_{j=3}^{\infty} w_j \circ v_j)$ for $t = 1, 2, \ldots, r$, where w_j is the jth generalized Whitehead product consisting of at least one homotopy element of the first inclusion $j^1_{\mathcal{P}} : \mathbb{S}^m_{\mathcal{P}} \to \mathbb{S}^m_{\mathcal{P}} \vee \mathbb{S}^m_{\mathcal{P}}$ and at least one homotopy element of the second inclusion $j^2_{\mathcal{P}} : \mathbb{S}^m_{\mathcal{P}} \to \mathbb{S}^m_{\mathcal{P}} \vee \mathbb{S}^m_{\mathcal{P}}$ as a factor, localized at $\mathcal{P}$, and v_j is any homotopy class in the homotopy group $[M(\mathbb{Z},n)_{\mathcal{P}}, \mathbb{S}^{h_{w_j}}_{\mathcal{P}}]$ for $j = 3, 4, 5, \ldots$; and*
- *Q_T is the homotopy class in the homotopy group $[M(T_{\mathcal{P}},n), X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ indicated by $(\alpha_{\mathcal{P}} \vee \alpha_{\mathcal{P}})_\sharp(\sum_{j=3}^{\infty} w_j \circ x_j)$, so that x_j is any homotopy element in $[M(T_{\mathcal{P}},n), \mathbb{S}^{h_{w_j}}_{\mathcal{P}}]$ for $j = 3, 4, 5, \ldots$, where $T_{\mathcal{P}}$ is the localization of the torsion subgroup T of G at the collection $\mathcal{P}$ of prime numbers.*

Proof. See [25, Theorem 3.10] for additional details. □

Definition 3.3. The homotopy class $\langle Q \rangle = \langle (Q_1, Q_2, \ldots, Q_t, \ldots, Q_r, Q_T) \rangle$ of the homotopy group $[M(G,n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ in (5) is called a *homotopy perturbation* of the comultiplication $\varphi : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$.

Examination of the homotopy perturbations is required in order to determine the pivotal properties of homotopy comultiplications on a wedge sum of localizations of CW-spaces. Examples of homotopy comultiplications with various homotopy perturbations on the localization $X_{\mathcal{P}} = \mathbb{S}^m_{\mathcal{P}} \vee M(G,n)_{\mathcal{P}}$ of a 1-connected CW-space as a wedge sum of spheres and Moore spaces are provided in Section 4.

According to Theorem 3.2, if a prime number $p \in \mathcal{P}$ is not equal to any of the p_i's for $i = 1, 2, \dots, k$, then the homotopy perturbation $\langle Q \rangle$ of the homotopy comultiplication

$$\varphi = \varphi_Q : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$$

concludes with the homotopy class $\langle Q \rangle = \langle (Q_1, Q_2, \dots, Q_t, \dots, Q_r) \rangle$ of $[M(G,n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ because, in this particular case, $T_{\mathcal{P}}$ is a trivial group.

4. Homotopy commutative comultiplications

This section includes an examination of the formulations of all possible homotopy commutative comultiplications on the topological localization $X_{\mathcal{P}}$ of the wedge sum $X := \mathbb{S}^m \vee M(G,n)$, $2 \le m < n$ at a collection $\mathcal{P}$ of prime numbers.

We examine the conditions for making the homotopy comultiplication $\varphi : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ homotopy commutative as follows.

Theorem 4.1. *Let $\varphi : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ be the homotopy comultiplication in Theorem 3.2. Then, φ is homotopy commutative if and only if*

$$S \circ Q \simeq Q,$$

where $Q : M(G,n)_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ is a homotopy perturbation, and $S : X_{\mathcal{P}} \vee X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ is the switching map.

Proof. According to the fundamental decomposition properties of the wedge sum on the homotopy groups, the homotopy comultiplication $\varphi : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ is homotopy commutative if and only if

$$S \circ \varphi \circ \alpha_{\mathcal{P}} \simeq \varphi \circ \alpha_{\mathcal{P}} \tag{6}$$

and

$$S \circ \varphi \circ \beta_{\mathcal{P}} \simeq \varphi \circ \beta_{\mathcal{P}}, \tag{7}$$

where $\alpha_{\mathcal{P}} : \mathbb{S}^m_{\mathcal{P}} \to X_{\mathcal{P}}$ is the $\mathcal{P}$-localization of the first inclusion map $\alpha : \mathbb{S}^m \to X$ sending x to (x, y_0), and $\beta_{\mathcal{P}} : M(G,n)_{\mathcal{P}} \to X_{\mathcal{P}}$ is the $\mathcal{P}$-localization of the second inclusion map sending y to (x_0, y) for all $x \in \mathbb{S}^m$ and $y \in M(G,n)$, where x_0 and y_0 are base points of $\mathbb{S}^m$ and $M(G,n)$, respectively. Since $[\mathbb{S}^m_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ is abelian for $2 \le m < n$, we have

$$\begin{aligned} S \circ \varphi \circ \alpha_{\mathcal{P}} &\simeq S(\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}} + \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}) \\ &\simeq \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}} + \iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}} \\ &\simeq \iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}} + \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}} \\ &\simeq \varphi \circ \alpha_{\mathcal{P}} \end{aligned}$$

which provides the proof of (6). In a similar manner, we also obtain

$$\begin{aligned} S \circ \varphi \circ \beta_{\mathcal{P}} &\simeq S(\iota^1_{\mathcal{P}} \circ \beta_{\mathcal{P}} + \iota^2_{\mathcal{P}} \circ \beta_{\mathcal{P}} + Q) \\ &\simeq \iota^2_{\mathcal{P}} \circ \beta_{\mathcal{P}} + \iota^1_{\mathcal{P}} \circ \beta_{\mathcal{P}} + S \circ Q \\ &\simeq \iota^1_{\mathcal{P}} \circ \beta_{\mathcal{P}} + \iota^2_{\mathcal{P}} \circ \beta_{\mathcal{P}} + S \circ Q \end{aligned} \tag{8}$$

in the homotopy group $[M(G,n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}]$, which is also abelian. Therefore, (7) holds if and only if the formula (8) is equal to

$$\varphi \circ \beta_{\mathcal{P}} \simeq \iota^1_{\mathcal{P}} \circ \beta_{\mathcal{P}} + \iota^2_{\mathcal{P}} \circ \beta_{\mathcal{P}} + Q,$$

as required. □

We now provide examples of homotopy comultiplications on the localization $X_{\mathcal{P}} = \mathbb{S}^m_{\mathcal{P}} \vee M(G,n)_{\mathcal{P}}$ of a 1-connected CW-space as a wedge sum of the m-spheres and Moore spaces for $2 \leq m < n$ as follows.

Example 4.2. Let G be a finitely generated abelian group decomposed as in (4), and let $\kappa_1 : M(G,n)_{\mathcal{P}} \to \mathbb{S}^{2m-1}_{\mathcal{P}}$ be the homotopy class of a continuous map that preserves the base point. An element Q_1 of $[M(G,n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ is defined as the composition

$$M(G,n)_{\mathcal{P}} \xrightarrow{\kappa_1} \mathbb{S}^{2m-1}_{\mathcal{P}} \xrightarrow{[j^1,\, j^2]_{\mathcal{P}}} \mathbb{S}^m_{\mathcal{P}} \vee \mathbb{S}^m_{\mathcal{P}} \xrightarrow{\alpha_{\mathcal{P}} \vee \alpha_{\mathcal{P}}} X_{\mathcal{P}} \vee X_{\mathcal{P}}$$

of base point preserving continuous maps; that is,

$$Q_1 \simeq (\alpha_{\mathcal{P}} \vee \alpha_{\mathcal{P}}) \circ [j^1, j^2]_{\mathcal{P}} \circ \kappa_1.$$

It can be demonstrated that the map $\varphi_{Q_1} : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ indicated by

$$\begin{cases} \varphi \circ \alpha_{\mathcal{P}} & \simeq \iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}} + \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \\ \varphi \circ \beta_{\mathcal{P}} & \simeq \iota^1_{\mathcal{P}} \circ \beta_{\mathcal{P}} + \iota^2_{\mathcal{P}} \circ \beta_{\mathcal{P}} + Q_1 \end{cases}$$

is a homotopy comultiplication with a homotopy perturbation Q_1. Indeed, we can observe that

$$\pi^1_{\mathcal{P}} \circ Q_1 \simeq c_{x_0} \simeq \pi^2_{\mathcal{P}} \circ Q_1,$$

where $c_{x_0} : M(G,n)_{\mathcal{P}} \to X_{\mathcal{P}}$ is a constant map at x_0 in $X_{\mathcal{P}}$.

Lemma 4.3. *Let $\varphi_{Q_1} : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ be the homotopy comultiplication in Example 4.2. Then, φ_{Q_1} is a homotopy commutative comultiplication if and only if* (1) *m is even or* (2) *m is odd and $\kappa_1 \simeq (-1)\kappa_1$.*

Proof. It can be observed that

$$Q_1 \simeq (\alpha_{\mathcal{P}} \vee \alpha_{\mathcal{P}}) \circ [j^1, j^2]_{\mathcal{P}} \circ \kappa_1 \simeq [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}] \circ \kappa_1$$

and

$$\begin{aligned} S \circ Q_1 & \simeq S_\sharp[\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}] \circ \kappa_1 \\ & \simeq [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}] \circ \kappa_1 \\ & \simeq (-1)^{m^2}[\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}] \circ \kappa_1 \end{aligned}$$

according to the anticommutative property of the Whitehead products, where $S_\sharp : [M(G,n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}] \to [M(G,n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ is a homomorphism between homotopy groups induced by the switching map $S : X_{\mathcal{P}} \vee X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$. If m is even, then we have

$$S \circ Q_1 \simeq Q_1.$$

If m is odd, we obtain

$$\begin{aligned} S \circ Q_1 & \simeq (-1)^{m^2}[\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}] \circ \kappa_1 \\ & \simeq (-[\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]) \circ \kappa_1 \\ & \simeq [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}](-1) \circ \kappa_1. \end{aligned}$$

Because $[\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]$ is a basic Whitehead product, φ_{Q_1} is a homotopy commutative comultiplication if and only if

$$\kappa_1 \simeq (-1)\kappa_1,$$

as required. □

Example 4.4. Let G be a finitely generated abelian group decomposed as in (4), and let $\kappa_2 : M(G,n)_{\mathcal{P}} \to \mathbb{S}^{3m-2}_{\mathcal{P}}$ be the homotopy class of a continuous map that preserves the base point. A homotopy element Q_2 of $[M(G,n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ is defined as the composition

$$M(G,n)_{\mathcal{P}} \xrightarrow{\kappa_2} \mathbb{S}^{3m-2}_{\mathcal{P}} \xrightarrow{[j^1,[j^1,j^2]]_{\mathcal{P}}} \mathbb{S}^{m}_{\mathcal{P}} \vee \mathbb{S}^{m}_{\mathcal{P}} \xrightarrow{\alpha_{\mathcal{P}} \vee \alpha_{\mathcal{P}}} X_{\mathcal{P}} \vee X_{\mathcal{P}}$$

of base point preserving continuous maps; that is,

$$Q_2 \simeq (\alpha_{\mathcal{P}} \vee \alpha_{\mathcal{P}}) \circ [j^1,[j^1,j^2]]_{\mathcal{P}} \circ \kappa_2.$$

It can be demonstrated that the map $\varphi_{Q_2} : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ indicated by

$$\begin{cases} \varphi \circ \alpha_{\mathcal{P}} & \simeq \iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}} + \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \\ \varphi \circ \beta_{\mathcal{P}} & \simeq \iota^1_{\mathcal{P}} \circ \beta_{\mathcal{P}} + \iota^2_{\mathcal{P}} \circ \beta_{\mathcal{P}} + Q_2 \end{cases}$$

is a homotopy comultiplication with a homotopy perturbation Q_2. Indeed, we can observe that

$$\pi^1_{\mathcal{P}} \circ Q_2 \simeq c_{x_0} \simeq \pi^2_{\mathcal{P}} \circ Q_2,$$

where $c_{x_0} : M(G,n)_{\mathcal{P}} \to X_{\mathcal{P}}$ is a constant map at x_0 in $X_{\mathcal{P}}$.

Lemma 4.5. *Let $\varphi_{Q_2} : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ be the homotopy comultiplication in Example 4.4. Then, φ_{Q_2} is a homotopy commutative comultiplication if and only if $\kappa_2 \simeq c_{s_0}$, where c_{s_0} is the constant map at s_0 in $\mathbb{S}^{3m-2}_{\mathcal{P}}$.*

Proof. It should be noted that

$$Q_2 \simeq (\alpha_{\mathcal{P}} \vee \alpha_{\mathcal{P}}) \circ [j^1,[j^1,j^2]]_{\mathcal{P}} \circ \kappa_2 \simeq [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2$$

and

$$\begin{aligned} S \circ Q_2 & \simeq S_\sharp[\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2 \\ & \simeq [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2 \\ & \simeq [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, (-1)^{m^2}[\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2 \end{aligned}$$

according to the anticommutative property of the Whitehead products, where $S_\sharp : [M(G,n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}] \to [M(G,n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ is a homomorphism between homotopy groups induced by the switching map $S : X_{\mathcal{P}} \vee X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$. If m is even, then we have

$$S \circ Q_2 \simeq [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2,$$

and thus

$$\begin{aligned} c_{x_0} & \simeq Q_2 - S \circ Q_2 \\ & \simeq [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2 - [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2, \end{aligned}$$

where $c_{x_0} : M(G,n)_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ is the constant map at (x_0, x_0) in $X_{\mathcal{P}} \vee X_{\mathcal{P}}$. Because all of the Whitehead products are basic Whitehead products, we can see that the homotopy class $\kappa_2 : M(G,n)_{\mathcal{P}} \to \mathbb{S}^{3m-2}_{\mathcal{P}}$ can be regarded as inessential, and similarly for the case of an odd number, as required. □

Example 4.6. Let G be a finitely generated abelian group decomposed as in (4), and let $\kappa_3 : M(G,n)_{\mathcal{P}} \to \mathbb{S}^{3m-2}_{\mathcal{P}}$ be the homotopy class of a continuous map that preserves the base point. An element Q_3 of $[M(G,n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ is defined as the composition

$$M(G,n)_{\mathcal{P}} \xrightarrow{\kappa_3} \mathbb{S}^{3m-2}_{\mathcal{P}} \xrightarrow{[j^2,[j^1,j^2]]_{\mathcal{P}}} \mathbb{S}^{m}_{\mathcal{P}} \vee \mathbb{S}^{m}_{\mathcal{P}} \xrightarrow{\alpha_{\mathcal{P}} \vee \alpha_{\mathcal{P}}} X_{\mathcal{P}} \vee X_{\mathcal{P}}$$

of base point preserving continuous maps; that is,

$$Q_3 \simeq (\alpha_{\mathcal{P}} \vee \alpha_{\mathcal{P}}) \circ [j^2,[j^1,j^2]]_{\mathcal{P}} \circ \kappa_3.$$

It can be demonstrated that the map $\varphi_{Q_3} : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ indicated by

$$\begin{cases} \varphi \circ \alpha_{\mathcal{P}} & \simeq \iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}} + \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \\ \varphi \circ \beta_{\mathcal{P}} & \simeq \iota^1_{\mathcal{P}} \circ \beta_{\mathcal{P}} + \iota^2_{\mathcal{P}} \circ \beta_{\mathcal{P}} + Q_3 \end{cases}$$

is a homotopy comultiplication with a homotopy perturbation Q_3. Indeed, we can see that in this case

$$\pi^1_{\mathcal{P}} \circ Q_3 \simeq c_{x_0} \simeq \pi^2_{\mathcal{P}} \circ Q_3,$$

where $c_{x_0} : M(G, n)_{\mathcal{P}} \to X_{\mathcal{P}}$ is a constant map at x_0 in $X_{\mathcal{P}}$.

Lemma 4.7. *Let $\varphi_{Q_3} : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ be the homotopy comultiplication in Example 4.6. Then, φ_{Q_3} is a homotopy commutative comultiplication if and only if $\kappa_3 \simeq c_{s_0}$, where c_{s_0} is the constant map at s_0 in $\mathbb{S}^{3m-2}_{\mathcal{P}}$.*

Proof. It should be noted that

$$Q_3 \simeq (\alpha_{\mathcal{P}} \vee \alpha_{\mathcal{P}}) \circ [j^2, [j^1, j^2]]_{\mathcal{P}} \circ \kappa_3 \simeq [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3$$

and

$$\begin{aligned} S \circ Q_3 & \simeq S_\sharp [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3 \\ & \simeq [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3 \\ & \simeq [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, (-1)^{m^2} [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3, \end{aligned}$$

where $S_\sharp : [M(G, n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}] \to [M(G, n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ is a homomorphism between homotopy groups induced by the switching map $S : X_{\mathcal{P}} \vee X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$. If m is even, then the homotopy comultiplication $\varphi_{Q_3} : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ is homotopy commutative if and only if

$$[\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3 \simeq [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3;$$

that is,

$$[\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3 - [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3 \simeq c_{x_0},$$

where $c_{x_0} : M(G, n)_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ is the constant map at (x_0, x_0) in $X_{\mathcal{P}} \vee X_{\mathcal{P}}$. Because all of the Whitehead products are basic Whitehead products, we can see that the homotopy class $\kappa_3 : M(G, n)_{\mathcal{P}} \to \mathbb{S}^{3m-2}_{\mathcal{P}}$ can be regarded as inessential.

If m is an odd number, then the homotopy comultiplication $\varphi_{Q_3} : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ is homotopy commutative if and only if

$$[\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3 \simeq (-[\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]]) \circ \kappa_3;$$

that is,

$$[\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3 + [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3 \simeq c_{x_0}, \tag{9}$$

where $c_{x_0} : M(G, n)_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ is the constant map at (x_0, x_0) in $X_{\mathcal{P}} \vee X_{\mathcal{P}}$. Because all of the Whitehead products in (9) are basic Whitehead products, we see that the homotopy class $\kappa_3 : M(G, n)_{\mathcal{P}} \to \mathbb{S}^{3m-2}_{\mathcal{P}}$ can be regarded as inessential, as required. □

To assess the number of possible homotopy commutative comultiplications on the localization $X_{\mathcal{P}}$ of a wedge $X := \mathbb{S}^m \vee M(G, n)$ of the m-sphere and Moore space with $2 \leq m < n$, a homotopy perturbation Q of a homotopy comultiplication $\varphi_Q : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ on $X_{\mathcal{P}}$ can be constructed as follows.

Example 4.8. Let

- $Q_1 \simeq (\alpha_{\mathcal{P}} \vee \alpha_{\mathcal{P}}) \circ [j^1, j^2]_{\mathcal{P}} \circ \kappa_1$;
- $Q_2 \simeq (\alpha_{\mathcal{P}} \vee \alpha_{\mathcal{P}}) \circ [j^1, [j^1, j^2]]_{\mathcal{P}} \circ \kappa_2$; and

- $Q_3 \simeq (\alpha_{\mathcal{P}} \vee \alpha_{\mathcal{P}}) \circ [j^2, [j^1, j^2]]_{\mathcal{P}} \circ \kappa_3$

be the homotopy classes in Examples 4.2, 4.4, and 4.6, respectively. A homotopy class $\varphi_Q : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ is defined by

$$\begin{cases} \varphi \circ \alpha_{\mathcal{P}} & \simeq \iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}} + \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \\ \varphi \circ \beta_{\mathcal{P}} & \simeq \iota^1_{\mathcal{P}} \circ \beta_{\mathcal{P}} + \iota^2_{\mathcal{P}} \circ \beta_{\mathcal{P}} + Q, \end{cases}$$

where

$$Q \simeq Q_1 + Q_2 + Q_3.$$

It can be demonstrated that $\varphi_Q : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ is a homotopy comultiplication with a homotopy perturbation $Q \simeq Q_1 + Q_2 + Q_3$; that is,

$$\begin{aligned} Q \ &\simeq [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}] \circ \kappa_1 + [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2 \\ &\quad + [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3 \end{aligned} \tag{10}$$

in an abelian group $[M(G, n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}]$.

Theorem 4.9. *Let $\varphi_Q : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ be the homotopy comultiplication in Example 4.8. Then, φ_Q is a homotopy commutative comultiplication if and only if* (1) *m is an even number, and $\kappa_2 \simeq \kappa_3$ or* (2) *m is odd, $\kappa_1 \simeq (-1)\kappa_1$, and $\kappa_2 \simeq -\kappa_3$.*

Proof. Let $S_\sharp : [M(G, n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}] \to [M(G, n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ be the homomorphism between homotopy groups induced by the switching map $S : X_{\mathcal{P}} \vee X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$. It should be noted that the homotopy group $[M(G, n), X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ is abelian, and that all of the Whitehead products from $Q_1, Q_2, Q_3, S_\sharp(Q_1), S_\sharp(Q_2)$, and $S_\sharp(Q_3)$ in Lemmas 4.3, 4.5, and 4.7 are basic Whitehead products. We can also see that $\varphi_Q : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ is a homotopy comultiplication with a homotopy perturbation

$$\begin{aligned} Q \ &\simeq [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}] \circ \kappa_1 + [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2 \\ &\quad + [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3 \end{aligned} \tag{11}$$

in an abelian group $[M(G, n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}]$. We now have

$$\begin{aligned} S \circ Q \ &\simeq S_\sharp\big([\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}] \circ \kappa_1 \\ &\quad + [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2 \\ &\quad + [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3\big) \\ &\simeq [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}] \circ \kappa_1 \\ &\quad + [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2 \\ &\quad + [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3 \\ &\simeq (-1)^{m^2}[\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}] \circ \kappa_1 \\ &\quad + [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, (-1)^{m^2}[\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2 \\ &\quad + [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, (-1)^{m^2}[\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3. \end{aligned} \tag{12}$$

It should be noted that the homotopy comultiplication $\varphi_Q : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ is homotopy commutative if and only if

$$Q - S \circ Q \simeq c_{x_0} : M(G, n)_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$$

in the homotopy group $[M(G, n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}]$, which is abelian, where $c_{x_0} : M(G, n)_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ is the constant map at (x_0, x_0) in $X_{\mathcal{P}} \vee X_{\mathcal{P}}$. According to (11) and (12), we obtain

$$\begin{aligned} Q - S \circ Q \ &\simeq [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}] \circ \kappa_1 \\ &\quad + [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2 \\ &\quad + [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3 \\ &\quad - \big((-1)^{m^2}[\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}] \circ \kappa_1 \\ &\quad + [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, (-1)^{m^2}[\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2 \\ &\quad + [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, (-1)^{m^2}[\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3\big). \end{aligned}$$

If m is even, then

$$\begin{aligned} Q - S \circ Q \simeq\ & [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}] \circ \kappa_1 \\ & + [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2 \\ & + [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3 \\ & - [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}] \circ \kappa_1 \\ & - [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2 \\ & - [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3. \end{aligned}$$

Because the homotopy group $[M(G,n)_{\mathcal{P}}, X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ is abelian, we observe that the homotopy comultiplication $\varphi_Q : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ is homotopy commutative if and only if

$$\begin{aligned} Q - S \circ Q & \simeq [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2 \\ & \quad + [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3 \\ & \quad - [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_2 \\ & \quad - [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ \kappa_3 \\ & \simeq [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ (\kappa_2 - \kappa_3) \\ & \quad + [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ (\kappa_3 - \kappa_2) \\ & \simeq c_{x_0} \end{aligned} \tag{13}$$

so that

$$\kappa_2 \simeq \kappa_3$$

because all of the Whitehead products in (13) are basic Whitehead products.

If m is an odd number, then $\varphi_Q : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ is homotopy commutative if and only if

$$\begin{aligned} Q - S \circ Q & \simeq 2[\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}] \circ \kappa_1 \\ & \quad + [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ (\kappa_2 + \kappa_3) \\ & \quad + [\iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, [\iota^1_{\mathcal{P}} \circ \alpha_{\mathcal{P}}, \iota^2_{\mathcal{P}} \circ \alpha_{\mathcal{P}}]] \circ (\kappa_3 + \kappa_2) \\ & \simeq c_{x_0} \end{aligned} \tag{14}$$

so that

$$\kappa_1 \simeq (-1)\kappa_1$$

and

$$\kappa_2 \simeq -\kappa_3$$

because all of the Whitehead products in (14) are basic Whitehead products, as required. □

Let $C(Y)$ be the set of all homotopy classes of homotopy commutative comultiplications on a co-Hopf space Y, and let $|C(Y)|$ be the cardinality of $C(Y)$.

Corollary 4.10. *Let $X_{\mathcal{P}}$ be the localization of the wedge sum $X := \mathbb{S}^m \vee M(G,n)$, $2 \le m < n \le 4m-4$ at a collection $\mathcal{P}$ of prime numbers, and let $\varphi_Q : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ be the homotopy comultiplication in Example 4.8. If m is an even number, then*

$$|C(X_{\mathcal{P}})| = |[M(G,n)_{\mathcal{P}}, \mathbb{S}^{2m-1}_{\mathcal{P}}]| \times |[M(G,n)_{\mathcal{P}}, \mathbb{S}^{3m-2}_{\mathcal{P}}]|.$$

If m is odd, then

$$|C(X_{\mathcal{P}})| = |\{\kappa_1 \in [M(G,n)_{\mathcal{P}}, \mathbb{S}^{2m-1}_{\mathcal{P}}] \mid \kappa_1 \simeq (-1)\kappa_1\}| \times |[M(G,n)_{\mathcal{P}}, \mathbb{S}^{3m-2}_{\mathcal{P}}]|.$$

Proof. The proof follows from Theorem 4.9 and the range hypothesis. Indeed, if n is even, then the homotopy comultiplication

$$\varphi_Q : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$$

is homotopy commutative for any homotopy class κ_1 in homotopy group $[M(G,n)_{\mathcal{P}}, \mathbb{S}^{2m-1}_{\mathcal{P}}]$, and $\kappa_2 \simeq \kappa_3$ in $[M(G,n)_{\mathcal{P}}, \mathbb{S}^{3m-2}_{\mathcal{P}}]$. If n is odd, then

$$\varphi_Q : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$$

is homotopy commutative for any homotopy class κ_1 satisfying $\kappa_1 \simeq (-1)\kappa_1$ in $[M(G,n)_{\mathcal{P}}, \mathbb{S}^{2m-1}_{\mathcal{P}}]$, and $\kappa_2 \simeq -\kappa_3$ in $[M(G,n)_{\mathcal{P}}, \mathbb{S}^{3m-2}_{\mathcal{P}}]$, as required. □

5. Examples

A list of examples is provided for use in evaluation of the structure of homotopy commutative comultiplications on the topological localization $X_{\mathcal{P}}$, or the rationalization $X_{\mathbb{Q}}$, of a wedge $X := \mathbb{S}^m \vee M(G, n)$, where $2 \leq m < n$.

In Tables:

- $C(X_{\mathcal{P}})$ is the set or the group consisting of all homotopy commutative comultiplications on the topological localization $X_{\mathcal{P}}$ of X at a collection $\mathcal{P}$ of prime numbers.
- The set $C(X_{\mathcal{P}})$ is in one-to-one correspondence with each group, as shown in the last columns in Tables 1, 2, 3, 4, 5, and 6.
- $\{e\}$ is the trivial group.
- $\mathbb{Z}_n$ is the group of integers modulo n.
- $\mathbb{Z}_{\mathcal{P}}$ is the $\mathcal{P}$-localization of the ring of integers.
- ϕ is the empty set, and $C(X_{\mathcal{P}}) = C(X_{\mathbb{Q}})$ in this case.

m	G	n	$\mathcal{P}$	$C(X_{\mathcal{P}})$
2	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5$	3	$\{3, 7\}$	$\{e\}$
2	$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$	3	$\{5, 7\}$	$\{e\}$
2	$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$	3	$\{7, 11\}$	$\{e\}$

Table 1: The finite group cases for $m = 2$

m	G	n	$\mathcal{P}$	$C(X_{\mathcal{P}})$
3	$\mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7$	4	$\{2, 5, 13\}$	$\mathbb{Z}_2$
3	$\mathbb{Z}_2 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{11}$	4	$\{2, 5, 7\}$	$\mathbb{Z}_2$
3	$\mathbb{Z}_2 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{14}$	4	$\{2, 5, 7, 13\}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

Table 2: The finite group cases for $m = 3$

m	G	n	$\mathcal{P}$	$C(X_{\mathcal{P}})$
4	$\mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{15}$	6	$\{2, 5, 7, 17\}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$
4	$\mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{22}$	6	$\{2, 3, 11, 17\}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{11}$
4	$\mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{33}$	6	$\{2, 5, 13, 17\}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5$
$m \geq 2$	all finite groups	$n > m$	ϕ	$\{e\}$

Table 3: The finite group cases for the others

m	G	n	$\mathcal{P}$	$C(X_{\mathcal{P}})$
2	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	3	$\{3, 5\}$	$\mathbb{Z}_{\{3,5\}}$
2	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5$	3	$\{3, 7\}$	$\mathbb{Z}_{\{3,7\}}$
2	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$	4	$\{5, 7\}$	$\{e\}$

Table 4: The infinite group cases for $m = 2$

m	G	n	$\mathcal{P}$	$C(X_{\mathcal{P}})$
3	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7$	4	$\{5, 7\}$	$\{e\}$
3	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7$	5	$\{3, 11\}$	$\mathbb{Z}_{\{3,11\}} \oplus \mathbb{Z}_{\{3,11\}}$
3	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{11}$	5	$\{5, 13\}$	$\mathbb{Z}_{\{5,13\}} \oplus \mathbb{Z}_{\{5,13\}}$

Table 5: The infinite group cases for $m = 3$

m	G	n	$\mathcal{P}$	$C(X_{\mathcal{P}})$
4	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{13}$	6	$\{2, 13\}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{13}$
4	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_{11} \oplus \mathbb{Z}_{13}$	9	$\{2, 17\}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
4	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_{19}$	9	ϕ	$\{e\}$
5	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_{11} \oplus \mathbb{Z}_{19}$	9	ϕ	$\mathbb{Q} \oplus \mathbb{Q}$

Table 6: The infinite group cases for the others

Proof. We first provide the proof of the case of a finite group with $m = 3$: Let $X = \mathbb{S}^3 \vee M(G, 4)$, where $G = \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7$ and $\mathcal{P} = \{2, 5, 13\}$. Then, according to Corollary 4.10, the Hopf-Whitney theorem, and the universal coefficient theorem for cohomology, we have

$$
\begin{aligned}
|C(X_{\mathcal{P}})| &= |\{\kappa_1 \in [M(G,4)_{\mathcal{P}}, \mathbb{S}^5_{\mathcal{P}}] \mid 2\kappa_1 \simeq c_{s_0}\}| \times |[M(A,4)_{\mathcal{P}}, \mathbb{S}^7_{\mathcal{P}}]|\}| && \text{by Corollary 4.10}\\
&= |\{\bar{\kappa}_1 \in H^5(M(\mathbb{Z}_2 \oplus \mathbb{Z}_5, 4); \pi_5(\mathbb{S}^5_{\mathcal{P}})) \mid 2\bar{\kappa}_1 = 0\}| \times 1 && \text{by Hopf-Whitney theorem}\\
&= |\{\bar{\kappa}_1 \in H^5(M(\mathbb{Z}_2 \oplus \mathbb{Z}_5, 4); \mathbb{Z}_{\mathcal{P}}) \mid 2\bar{\kappa}_1 = 0\}|\\
&= |\{\bar{\kappa}_1 \in \mathrm{Ext}(H_4(M(\mathbb{Z}_2 \oplus \mathbb{Z}_5, 4); \mathbb{Z}), \mathbb{Z}_{\mathcal{P}}) \mid 2\bar{\kappa}_1 = 0\}| && \text{by UCT for cohomology}\\
&= |\{\bar{\kappa}_1 \in \mathrm{Ext}(\mathbb{Z}_2 \oplus \mathbb{Z}_5, \mathbb{Z}_{\mathcal{P}}) \mid 2\bar{\kappa}_1 = 0\}|\\
&= |\mathbb{Z}_2|.
\end{aligned}
$$

Here,

- $c_{s_0} : M(G, 4)_{\mathcal{P}} \to \mathbb{S}^5_{\mathcal{P}}$ is the constant map at s_0 in the homotopy group $[M(G, 4)_{\mathcal{P}}, \mathbb{S}^5_{\mathcal{P}}]$ consisting of the $\mathcal{P}$-localizations of the homotopy classes in the cohomotopy group $\pi^5(M(G, 4))$;
- $\bar{\kappa}_1$ is the element of cohomology, extension products, and torsion groups corresponding to the homotopy element $\kappa_1 \in [M(G, 4)_{\mathcal{P}}, \mathbb{S}^5_{\mathcal{P}}]$; and
- 0 is the trivial element of cohomology, extension products, and torsion groups.

Secondly, we provide the proof of the case of a finitely generated infinite abelian group with $m = 4$: Let $X = \mathbb{S}^4 \vee M(G, 6)$, where

$$G = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{13}$$

and $\mathcal{P} = \{2, 13\}$. We then obtain

$$
\begin{aligned}
|C(X_{\mathcal{P}})| &= |[M(G,6)_{\mathcal{P}}, \mathbb{S}^7_{\mathcal{P}}]| \times |[M(G,6)_{\mathcal{P}}, \mathbb{S}^{10}_{\mathcal{P}}]| && \text{by Corollary 4.10}\\
&= |[M(G,6)_{\mathcal{P}}, \mathbb{S}^7_{\mathcal{P}}]| \times 1\\
&= |[M(\mathbb{Z}_{\mathcal{P}} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{13}, 6), \mathbb{S}^7_{\mathcal{P}}]|\\
&= |[M(\mathbb{Z}_{\mathcal{P}}, 6) \vee M(\mathbb{Z}_2, 6) \vee M(\mathbb{Z}_{13}, 6), \mathbb{S}^7_{\mathcal{P}}]|\\
&= |[M(\mathbb{Z}_{\mathcal{P}}, 6), \mathbb{S}^7_{\mathcal{P}}] \oplus [M(\mathbb{Z}_2, 6), \mathbb{S}^7_{\mathcal{P}}] \oplus [M(\mathbb{Z}_{13}, 6), \mathbb{S}^7_{\mathcal{P}}]|\\
&= |\{e\} \oplus H^7(M(\mathbb{Z}_2, 6); \pi_7(\mathbb{S}^7_{\mathcal{P}})) \oplus H^7(M(\mathbb{Z}_{13}, 6); \pi_7(\mathbb{S}^7_{\mathcal{P}}))| && \text{by Hopf-Whitney theorem}\\
&= |\mathrm{Ext}(H_6(M(\mathbb{Z}_2, 6); \mathbb{Z}), \mathbb{Z}_{\mathcal{P}}) \oplus \mathrm{Ext}(H_6(M(\mathbb{Z}_{13}, 6); \mathbb{Z}), \mathbb{Z}_{\mathcal{P}})| && \text{by UCT for cohomology}\\
&= |\mathbb{Z}_2 \oplus \mathbb{Z}_{13}|,
\end{aligned}
$$

where $\{e\}$ is the trivial group as the homotopy group $[M(\mathbb{Z}_{\mathcal{P}}, 6), \mathbb{S}^7_{\mathcal{P}}]$ consisting of the $\mathcal{P}$-localizations of the homotopy classes in the cohomotopy group $\pi^7(M(G, 6))$.

The remaining parts can be proven in a similar manner using the previously described results included in Section 4 and the pivotal theorems in algebraic topology, including the cohomotopy group, the cellular approximation theorem, the universal coefficient theorem in cohomology, the Hopf-Whitney classification theorem, obstruction theory [13, Lemma 17.19], and the homotopy group of spheres [31]. □

6. Conclusions

The investigation of studies on homotopy comultiplications and homotopy multiplications on co-Hopf spaces and Hopf spaces, respectively, has been explored by several authors to date. Indeed, it is well known that a co-Hopf space with a homotopy comultiplication and a Hopf space with a homotopy multiplication are the pivotal object classes in the pointed homotopy category and they are Eckmann-Hilton duals with each other in classical homotopy theory.

In general terms, there exist many distinctive homotopy comultiplications on a co-Hopf space along with a lot of different properties. It is well known that any nilpotent CW-complex could be localized at a collection of prime numbers up to homotopy as a topological (or homotopy) counterpart of the localization of algebraic objects.

In this article, we have described the localization counterparts of the Hilton-Milnor formulas along with the development of the pivotal concepts of the forms of comultiplications on the localizations of CW-spaces. In particular, we have developed the homotopy comultiplications and homotopy commutative comultiplications of the localization of a wedge $X := \mathbb{S}^m \vee M(G, n)$ of the m-spheres and Moore spaces at a collection $\mathcal{P}$ of prime numbers, where $2 \leq m < n$, and G is a finitely generated abelian group. Finally, we have provided the lists of examples for the homotopy commutative comultiplications on $X_{\mathcal{P}}$.

In a subsequent paper, we explore the homotopy associative and commutative comultiplication structures on nilpotent CW-spaces, more general CW-spaces, and their localizations, which are not simply connected spaces at all.

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