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A topology on the product of quasi-local rings

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Abstract. Let $\{R_i\}_{i\in I}$ be a family of quasi-local rings and let $R = \prod_{i\in I} R_i$ be their product. In this paper, we investigate the prime spectrum of R under different topologies, with special attention given to maximal ideals based on the ultrafilters on I. Additionally, we will compute the number of topologies on R.

1. Introduction

All rings in this article are assumed to be commutative and unitary. We also use the following notation: $\beta(I)$ denotes the set of all ultrafilters on I, the cardinality of a set I is a measure that represents the number of elements in I, often denoted as $|I|$, Spec(R) denotes the set of all prime ideals in a ring R, and $U(R)$ denotes the set of all unit elements in a ring R.

Let $\{R_i\}_{i\in I}$ be a family of quasi-local rings, and $R = \prod_{i\in I} R_i$ be their product. Several papers in the literature have addressed the problem of characterizing the prime ideals of R using purely algebraic methods (see [6, 7]). On the other hand, the notion of $\mathcal{F}$-limit is related to a construction proposed by S. Garcia-Ferreira and L. M. Ruza-Montilla in [5], which gives some topological properties of the prime spectrum of $\mathbb{Q}^{\mathbb{N}}$. H. Mouadi and D. Karim applied this notion to find a relation between the elements of $\prod_{i\in I} R_i$ and R_i (see [12]), as well as for defining topologies on other sets (see [11]).

Therefore, our objective is to address the following question:

(**Q**): How many pairwise non-homeomorphic topologies exist in Spec($\prod_{i\in I} R_i$)?

The rest of this paper is organized as follows:

In Section 2, we define and discuss some properties of the $\mathcal{F}$-limit of a family of ideals, along with characterizations of the product of the rings using ultrafilters.

In Section 3, we establish the relationship between $\mathcal{F}$-topologies on Spec($\prod_{i\in I} R_i$) and on $\beta(I)$, where I is an infinite set, and each R_i is a quasi-local ring. Afterward, we will discuss some properties of the $\mathcal{F}$-topology. Finally, at the conclusion of this work, we will provide an answer to question (**Q**).

2. $\mathcal{F}-$ lim of a family of ideals

In our study, we will be working within the framework of Zermelo-Fraenkel set theory with the axiom of choice (ZFC). We will assume the axioms of ZFC unless otherwise specified.

Let us recall the definition of a filter on a set I. A subset $\mathcal{U}$ of the power set of I is called a filter if it satisfies the following conditions:

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1. $\emptyset \notin \mathcal{U}$ and $I \in \mathcal{U}$.
2. If $J_1, J_2 \in \mathcal{U}$, then $J_1 \cap J_2 \in \mathcal{U}$.
3. If $J \in \mathcal{U}$ and $J \subseteq J' \subseteq I$, then $J' \in \mathcal{U}$.

An ultrafilter $\mathcal{U}$ is a special type of filter that satisfies the additional condition: For any subset J of I, either J or its complement $I \setminus J$ belongs to $\mathcal{U}$. If $\mathcal{U}$ consists of all subsets of I containing a specific element $p \in I$, we refer to it as a principal (or fixed) ultrafilter. Otherwise, if $\mathcal{U}$ includes subsets of I that do not contain any fixed element, it is known as a non-principal (or free) ultrafilter. For more information about filters and ultrafilters, refer to [2].

Theorem 2.1. ([2, Corollary 7.4]) *Let I be an infinite set, and let $\mid I \mid = \alpha$. Then $\mid \beta(I) \mid = 2^{2^{\alpha}}$.*

Definition 2.2. Let R be a ring, I an infinite set, $\mathcal{F}$ an ultrafilter on I, and $\{P_i : i \in I\} \subseteq \operatorname{Spec}(R)$. We define the $\mathcal{F}$-limit of $\{P_i : i \in I\}$ as follows:

$$\mathcal{F} - \lim_{i \in I} P_i := \{a \in R : \{i \in I \ : \ a \in P_i\} \in \mathcal{F}\}.$$

It can be noted that:

$$\mathcal{F} - \lim_{i \in I} P_i = \bigcup_{F \in \mathcal{F}} (\bigcap_{i \in F} P_i)$$

Example 2.3.

- Let $\mathcal{F}$ be a nonprincipal ultrafilter on $Spec(\mathbb{Z})$. Then:

  $$(0) = \mathcal{F} - \lim_{p \in Spec(\mathbb{Z})} (p).$$

  For more information, see [12].

- Let $R = \mathbb{Z}^I$ and $\mathcal{U}$ be an ultrafilter on I. In reference [10], R. Levy et al denoted the minimal prime ideal in R by $(\mathcal{U})$. Then,

  $$(\mathcal{U}) = \mathcal{U} - \lim_{i \in I} (0_i).$$

Theorem 2.4. ([5, Theorem 2.1]) *Let R be a ring, I an infinite set, $\mathcal{F}$ an ultrafilter on I, and $\{P_i : i \in I\} \subseteq \operatorname{Spec}(R)$.*

1. *If $A \in \mathcal{F}$, then*

   $$\mathcal{F} - \lim_{i \in I} P_i = \mathcal{F} \mid_A - \lim_{i \in A} P_i.$$

   where $\mathcal{F} \mid_A = \{B \subseteq A \ : \ B \in \mathcal{F}\}$.
2. *Let Γ be an infinite set, and $\sigma : \Delta \to \Gamma$ be a surjective function. For each $j \in \Gamma$, we define $Q_j = P_i$ if $\sigma(i) = j$. Then, we have:*

   $$\mathcal{F} - \lim_{i \in \Delta} P_i = C - \lim_{j \in \Gamma} Q_j,$$

   where $\sigma(\mathcal{F}) = \{\sigma[F] \ : \ F \in \mathcal{F}\} = C$.

Consider a family of rings $\{R_i\}_{i \in I}$ and their product $\prod_{i \in I} R_i$. Elements of this product are commonly approached in two distinct manners. The first, more formal approach involves defining $\prod_{i \in I} R_i$ as the set of all functions $f : I \to \bigcup_{i \in I} R_i$ satisfying $f(i) \in R_i$ for each $i \in I$. On the other hand, an alternative perspective considers elements of $\prod_{i \in I} R_i$ as tuples $\langle a_i \rangle_{i \in I}$, where $a_i \in R_i$ holds true for each i. Within this perspective, the addition and multiplication of tuples are defined based on coordinate-wise addition and multiplication operations.

In the following discussion, we introduce the notation to describe sets of indices where some elements of a ring of products and extend it to ideals inside the ring. Our focus lies in the realm of quasi-local rings, commencing our exploration by defining them, supported by illustrative examples.

Definition 2.5. A commutative ring R is termed a **quasi-local ring** if there exists a unique maximal ideal $\mathfrak{m}$ in R. Symbolically, a ring R is quasi-local if and only if there exists a unique ideal $\mathfrak{m} \subset R$ such that for any proper ideal $\mathfrak{a} \subset R$, either $\mathfrak{a} \subset \mathfrak{m}$ or $\mathfrak{a} = R$.

Example 2.6.

1. The ring $\frac{\mathbb{Z}}{p^n\mathbb{Z}}$ is a quasi-local ring for p prime and $n \geq 1$. Its unique maximal ideal is composed of all multiples of p.
2. Let R be a commutative ring and P a prime ideal in R. The localization of R at the complement of P is denoted by R_P or $S^{-1}R$ where $S = R \setminus P$.
3. A nonzero ring in which every element is either a unit or nilpotent is a quasi-local ring.

For commutative rings, distinguishing between left, right, or two-sided ideals becomes unnecessary: a commutative ring is deemed local only when possessing a singular maximal ideal. Before 1960, several authors stipulated local rings to be both left and right Noetherian, labeling non-Noetherian local rings as quasi-local. This article, however, refrains from imposing such a requirement.

Notation 2.7.

1. Let $\{R_i\}_{i\in I}$ be a nonempty family of commutative rings indexed by a set I, and let $R = \prod_{i\in I} R_i$ be their product. For each $f \in R$, we define $\mathcal{Y}(f)$ as the set $\{i \in I \ \ f(i) \in R_i \setminus U(R_i)\}$.
2. It is important to note that f is a unit of R if and only if $\mathcal{Y}(f)$ is empty.
3. Furthermore, for an ideal J of R, we define $\mathcal{Y}(J)$ as the set of all $\mathcal{Y}(f)$ where $f \in J$.

In the study of product rings with a family of non-zero rings indexed by an infinite set, the following proposition establishes relationships between ideals, filters, ultrafilters, and maximal ideals, providing insights into their properties and interconnections.

Proposition 2.8. ([3, Proposition 2.3]) *Let $R = \prod_{i\in I} R_i$, where I is an infinite set and each R_i is nonzero ring.*

1. *If J is a proper ideal of R, then $\mathcal{Y}(J)$ forms a filter on I. In the case where J is prime, $\mathcal{Y}(J)$ becomes an ultrafilter.*
2. *Assuming that each R_i is quasilocal and $\mathcal{F}$ is a filter on I, we define $J_{\mathcal{F}} = \{f \in R : \mathcal{Y}(f) \in \mathcal{F}\}$. It can be shown that $J_{\mathcal{F}}$ is an ideal of R, and $\mathcal{Y}(J_{\mathcal{F}}) = \mathcal{F}$. Consequently, $J_{\mathcal{F}}$ is the largest ideal of R satisfying $\mathcal{Y}(J_{\mathcal{F}}) = \mathcal{F}$.*
3. *When each R_i is quasilocal, the mapping $M \to \mathcal{Y}(M)$ establishes a bijection between the set of maximal ideals of R and the set of ultrafilters on I. This bijection also holds between the set of maximal ideals of R containing a fixed ideal I and the set of nonprincipal ultrafilters on I. Additionally, each $P \in Spec(R)$ is contained in a unique maximal ideal of R.*

3. The relationship between the $\mathcal{F}$-topology on $Spec(\prod_{i\in I} R_i)$ and $\beta(I)$

Let R be a commutative ring, the concept of the ultrafilter topology on Spec(R) was originally introduced by M. Fontana and K. A. Loper in their work [3]. Building upon this idea, S. Garcia-Ferreira and L. M. Ruza-Montilla defined the ultrafilter topology in the context of $\mathcal{F}$-limits, as described in [5]. In this section, we aim to establish the relationship between the $\mathcal{F}$-topology on $Spec(\prod_{i\in I} R_i)$ and the space $\beta(I)$.

To begin, we define the $\mathcal{F}$-topology on the product of quasi-local rings, inspired by the aforementioned works:

Definition 3.1. Let $R = \prod_{i\in I} R_i$ be a product of quasi-local rings, and let C be a subset of $Spec(R)$. Consider an ultrafilter $\mathcal{H}$ on C. Following the approach in [3], we define $P_{\mathcal{H}} = \{a \in R \ : \ V(a) \cap C \in \mathcal{H}\}$ as a prime ideal. We refer to $P_{\mathcal{H}}$ as an ultrafilter limit point of C.

In this definition, we adapt the notion of ultrafilter limit points from [3] to the setting of $Spec(\prod_{i\in I} R_i)$. By establishing this correspondence, we aim to establish the relationship between the $\mathcal{F}$-topology on $Spec(\prod_{i\in I} R_i)$ and the space $\beta(I)$.

Definition 3.2. Let R be a commutative ring and C be a subset of Spec(R). We say that C is ultrafilter closed if, for any ultrafilter $\mathcal{H}$ on C, we have $P_{\mathcal{H}} \in C$.

The ultrafilter closed subsets are precisely the closed subsets of the ultrafilter topology τ_u on Spec(R).

Lemma 3.3. ([5, Lemma 3.1]) *Let $C \neq \emptyset$ be a subset of* Spec(R) *and $\mathcal{H}$ be an ultrafilter on C. Then there exists a subset $\{P_i : i \in \Delta\} \subseteq C$ and an ultrafilter $\mathcal{F}$ on Δ such that $P_{\mathcal{H}} = \mathcal{F} - \lim_{i\in\Delta} P_i$.*

Lemma 3.3 states that for a nonempty subset C of Spec(R) and an ultrafilter $\mathcal{H}$ on C, there exists a subset $\{P_i : i \in \Delta\} \subseteq C$ and an ultrafilter $\mathcal{F}$ on Δ such that $P_{\mathcal{H}} = \mathcal{F} - \lim_{i\in\Delta} P_i$. In other words, the lemma establishes a connection between ultrafilters on C and ultrafilters on a indexed subset Δ. This result provides a useful tool for understanding the behavior of ultrafilters in the context of Spec(R).

Theorem 3.4. ([5, Theorem 3.2]) *A subset $C \subseteq$ Spec(R) is said to be ultrafilter closed if and only if for every infinite set Δ, every $P_i \in C$ for each $i \in \Delta$, and ultrafilter $\mathcal{F}$ on Δ, we have $\mathcal{F} - \lim_{i\in\Delta} P_i \in C$.*

Definition 3.5. Let R be a commutative ring, and let $\mathcal{F}$ be an ultrafilter on I. We say that $C \subseteq$ Spec(R) is $\mathcal{F}$-closed if for every collection $\{P_i\}_{i\in I}$ in C, we have $\mathcal{F} - \lim_{i\in I} P_i \in C$.

Theorem 3.6. Let R be a commutative ring, and let $\mathcal{F}$ be a nonprincipal ultrafilter on I. The $\mathcal{F}$-closed subsets form the closed sets of a topology on *Spec*(R), which we refer to as the $\mathcal{F}$-topology and denote by $\tau_{\mathcal{F}}$.

Proof. Let C_1 and C_2 be $\mathcal{F}$-closed subsets of *Spec*(R) for some ultrafilters on I. Let $C = C_1 \cup C_2$, and consider a collection $\{P_i\}_{i\in I}$ in C.

Since $I = \{i \in I : P_i \in C_1\} \cup \{i \in I : P_i \in C_2\} \in \mathcal{F}$, by the definition of an ultrafilter, either $\{i \in I : P_i \in C_1\} \in \mathcal{F}$ or $\{i \in I : P_i \in C_2\} \in \mathcal{F}$. Without loss of generality, let's assume that $A = \{i \in I : P_i \in C_1\} \in \mathcal{F}$.

By Theorem 2.4, we have $\mathcal{F}$-$\lim_{i\in I} P_i = \mathcal{F}|A$-$\lim_{i\in I} P_i \in C_1 \subseteq C$.

Thus, we see that the intersection of $\mathcal{F}$-closed subsets is also a $\mathcal{F}$-closed subset. □

Remark 3.7.
- Every ultrafilter closed subset of *Spec*(R) is $\mathcal{F}$-closed.
- Let R be a ring, and let $\mathcal{F}$ be a nonprincipal ultrafilter on a countable set, such as $\mathbb{N}$. In general, $\tau_u \subseteq \tau_{\mathcal{F}}$, but if *Spec*($R$) is countable (as stated in [5, Theorem 4.4]), we have $\tau_u = \tau_{\mathcal{F}}$.

Now, let us consider the ring $R = \prod_{i\in I} R_i$, where I is an infinite set and each R_i is a quasi-local ring. Suppose we have a collection $\{P_a\}_{a\in A}$ in *Spec*($\prod_{i\in I} R_i$). In this context, we can define the $\mathcal{F}$-topology on *Spec*($\prod_{i\in I} R_i$) as follows. For each nonprincipal ultrafilter $\mathcal{F}$ on A (as defined in Definition 2.2), we consider the prime ideal $\mathcal{F}$-$\lim_{a\in A} P_a$.

It is worth noting that there exists a maximal ideal $J_{\mathcal{U}}$ satisfying $\mathcal{F}$-$\lim_{a\in A} P_a \subseteq J_{\mathcal{U}}$, where $\mathcal{U} \in \beta(I)$ of the set I, as shown in Proposition 2.8. This observation allows us to establish a connection between the sets I and A. In fact, we can replace the set A with I, considering the collection $\{P_i\}_{i\in I}$ instead of $\{P_a\}_{a\in A}$.

Moving forward, we introduce $\beta(I)$, denoted as the Stone extension of I. It is a unique (up to homeomorphism) compact space that contains I as a dense subset. In $\beta(I)$, each point corresponds to an ultrafilter on I. The set $I^* = \beta(I) \setminus I$ represents the remainder, with its points corresponding to the nonprincipal ultrafilters on I, while the principal ultrafilters are identified with the points of I. It's important to note that this construction is not unique and can be explored further in references such as [6, 8].

The Stone set $\hat{A}$ is defined as follows:

$$\hat{A} := \{\mathcal{F} \in \beta(I) : A \in \mathcal{F}\}.$$

Certainly, we can make use of Stone sets and leverage the following lemmas to facilitate our computations. It is evident that $\widehat{I} = \beta(I)$ and $\widehat{\emptyset} = \emptyset$.

Lemma 3.8. For all subsets A and B of I, the following equations hold:

1. $\widehat{A \cup B} = \widehat{A} \cup \widehat{B}$.
2. $\widehat{A \cap B} = \widehat{A} \cap \widehat{B}$.
3. $\widehat{I \setminus A} = \beta(I) \setminus \widehat{A}$.

Proof. The elements of the Stone sets occurring in these equations are ultrafilters $\mathcal{F}$ on I. Let's examine each equation:

1. $\widehat{A \cup B} = \widehat{A} \cup \widehat{B}$: The first equation holds since $\mathcal{F} \in \widehat{A \cup B}$ holds if and only if $A \cup B \in \mathcal{F}$. This is equivalent to $A \in \mathcal{F}$ and $B \in \mathcal{F}$, which can be written as $\mathcal{F} \in \widehat{A} \cap \widehat{B}$. Therefore, $\widehat{A \cup B} = \widehat{A} \cup \widehat{B}$.

2. $\widehat{A \cap B} = \widehat{A} \cap \widehat{B}$: The second equation follows similarly. $\mathcal{F} \in \widehat{A \cap B}$ holds if and only if $A \cap B \in \mathcal{F}$. Since every ultrafilter is a prime filter, we have $A \in \mathcal{F}$ and $B \in \mathcal{F}$, which can be written as $\mathcal{F} \in \widehat{A} \cap \widehat{B}$. Hence, $\widehat{A \cap B} = \widehat{A} \cap \widehat{B}$.

3. $\widehat{I \setminus A} = \beta(I) \setminus \widehat{A}$: The third equation is a direct consequence of the definition of an ultrafilter. $\mathcal{F} \in \widehat{I \setminus A}$ holds if and only if $I \setminus A \in \mathcal{F}$, which is equivalent to $A \notin \mathcal{F}$. Therefore, $\widehat{I \setminus A} = \beta(I) \setminus \widehat{A}$.

These equations demonstrate the properties of Stone sets and how they relate to subsets of I through the operations of union, intersection, and complement. □

Lemma 3.9. The family $\mathcal{B} := \{\hat{A} : A \subseteq I\}$, consisting of all Stone sets, is referred to as the Stone base of $\beta(I)$.

The Stone base $\mathcal{B}$ is a collection of subsets of $\beta(I)$ obtained by considering all possible subsets A of I and forming their corresponding Stone sets $\hat{A}$. Each element $\hat{A}$ of the Stone base represents a set of ultrafilters on I that contain the set A. Thus, $\mathcal{B}$ provides a foundational set system for constructing the Stone sets in $\beta(I)$.

The Stone base $\mathcal{B}$ plays a crucial role in the study of Stone spaces and ultrafilters. It captures the essential structure and properties of the Stone sets and enables us to analyze the topology and algebraic properties of the compact space $\beta(I)$ through its subsets.

As a direct consequence of the properties of the Stone base, we obtain the following results:

- The Stone base $\mathcal{B}$ serves as a base for the open sets of $\beta(I)$. Therefore, a subset U of $\beta(I)$ is open if and only if it can be expressed as the union of a family of Stone sets. In other words, there exists a family $\{A_i : i \in I\}$ of subsets of I such that $U = \bigcup_{i\in I} \hat{A}_i$.

- By considering complements and utilizing the closedness of the Stone base under complementation, we find that a subset Y of $\beta(I)$ is closed if and only if it can be expressed as the intersection of a family of Stone sets. In other words, there exists a family $\{A_i : i \in I\}$ of subsets of I such that $Y = \bigcap_{i\in I} \hat{A}_i$.

- In the space $\beta(I)$, for every point p, the family $\{\hat{A} : A \in p\}$ forms a canonical neighborhood base for p. This means that for each A contained in p, the corresponding Stone set $\hat{A}$ is a neighborhood of p, and any neighborhood of p contains a Stone set of the form $\hat{A}$ for some A contained in p.

These results highlight the topological properties of $\beta(I)$ and provide a convenient framework for understanding the open and closed sets in terms of the Stone base. Additionally, the last result emphasizes the local structure of points in $\beta(I)$, where each point has a neighborhood base consisting of Stone sets associated with the subsets of I contained in the point.

Remark 3.10. For a topological space X, a subset U of X is said to be clopen if it is both closed and open, which means that both U and its complement $X \setminus U$ are open sets.

In the context of $\beta(I)$, every Stone set $\widehat{A}$ is clopen, as stated in Lemma 3.8.

In general, a subset of $\beta(I)$ is clopen if and only if it is of the form $\widehat{A}$ for some subset A of I. To see this, assume that $U \subseteq \beta(I)$ is clopen and consider a family $\mathcal{A}$ of subsets of I such that $U = \bigcup_{A\in\mathcal{A}} \widehat{A}$. Since U is closed, it is also compact in $\beta(I)$. Therefore, the open cover $\{\widehat{A} : A \in \mathcal{A}\}$ of U has a finite subcover. Let $A_1, \ldots, A_n$ be subsets in $\mathcal{A}$ such that $U = \widehat{A_1} \cup \ldots \cup \widehat{A_n}$. It follows that U is the Stone set of $A = A_1 \cup \ldots \cup A_n$.

Thus, every clopen subset of $\beta(I)$ can be represented as a Stone set $\widehat{A}$ for some subset A of I.

The previous construction allows us to give the following definition

Definition 3.11. The Stone-Cech compactification $\beta(I)$ of any set I is defined as the set of all ultrafilters on I. One fundamental property of $\beta(I)$ is that for any map $f : I \to I$, there exists a continuous extension $\tilde{f} : \beta(I) \to \beta(I)$. The topology on $\beta(I)$, known as the Stone-Cech compactification topology, is equipped with the discrete topology.

Furthermore, for $A \subseteq I$, the set $cl_{\beta(I)}(A) = \{\mathcal{F} \in \beta(I) \ : \ A \in \mathcal{F}\}$ is a basic clopen subset of $\beta(I)$.

Given a nonprincipal ultrafilter $\mathcal{U}$ on I and a collection $\{\mathcal{F}_i\}_{i\in I}$ in $\beta(I)$, the $\mathcal{U}$-limit of $\mathcal{F}_i$ is defined as:

$$\mathcal{U} - lim\ \mathcal{F}_i = \{A \subseteq I \ : \ \{i \in I \ : \ A \in \mathcal{F}_i\} \in \mathcal{U}\}$$

This set is an ultrafilter on I.

A subset C of $\beta(I)$ is said to be $\mathcal{U}$-closed if for every collection $\{\mathcal{F}_i\}_{i\in I}$ in C, we have $\mathcal{U} - \lim_{i\in I} \mathcal{F}_i \in C$. The $\mathcal{U}$-closed sets define the $\mathcal{F}$-topology on $\beta(I)$, denoted by $\sigma_{\mathcal{U}}$.

Now, let us consider the ring $R = \prod_{i\in I} R_i$, where each R_i is a quasi-local ring. The description of the maximal ideal in R (as discussed in section 2) reveals that the maximal ideal $J_{\mathcal{F}}$ in R has the form:

$$J_{\mathcal{F}} = \{f \in R\ \{i \in I \ : \ f(i) \in R_i \setminus U(R_i)\} \in \mathcal{F}\}$$

where $\mathcal{F} \in \beta(I)$.

Lemma 3.12. ([12, Proposition 2]) *Let $R = \prod_{i\in I} R_i$, where I is an infinite set and each R_i is a quasi-local ring. Let $\mathcal{F}$ be an ultrafilter on I. If $(J_{\mathcal{F}_i})_{i\in I}$ is a collection of maximal ideals of R, then*

$$\mathcal{F}\text{-}\lim_{i\in I} J_{\mathcal{F}_i} = J_{\mathcal{F}\text{-}\lim_{i\in I} \mathcal{F}_i}$$

Definition 3.13. Assume that $\mathcal{U}$ is a nonprincipal ultrafilter on the set I, and let R denote the infinite product of quasi-local rings $R = \prod_{i\in I} R_i$. Given a subset C of $\beta(I)$, we provide the following definition:

$$C_S := \{P \in spec(R) \ : \ \exists \mathcal{F} \in C \quad (P \subseteq J_{\mathcal{F}})\}.$$

if $C \subseteq spec(R)$, then we let

$$C_I := \{\mathcal{F} \in \beta(I) \ : \ \exists P \in C \quad (P \subseteq J_{\mathcal{F}})\}.$$

Theorem 3.14. Let $\mathcal{U}$ be a nonprincipal ultrafilter on I. If $C \subseteq \beta(I)$ is $\sigma_{\mathcal{U}}$-closed, then C_S is a $\tau_{\mathcal{U}}$-closed subset of $Spec(R)$.

Proof. Let $\{P_i\}_{i\in I}$ be a collection of elements of C_S. By definition, for each $i \in I$, there exists $\mathcal{F}_i \in C$ such that $P_i \subseteq J_{\mathcal{F}_i}$. Let $\mathcal{F} = \mathcal{U}\text{-}\lim_{i\in I} \mathcal{F}_i$. Since C is closed under $\mathcal{U}$-limits, we have $\mathcal{F} \in C$.

By Lemma 3.12, we know that $\mathcal{U}\text{-}\lim J_{\mathcal{F}_i} = J_{\mathcal{F}} \in C_S$. Thus, C_S is a $\tau_{\mathcal{U}}$-closed subset of $Spec(R)$. □

Theorem 3.15. Let $\mathcal{U}$ be a nonprincipal ultrafilter on I. If $C \subseteq \mathrm{Spec}(R)$ is a $\tau_{\mathcal{U}}$-closed subset of $\mathrm{Spec}(R)$, then C_I is $\sigma_{\mathcal{U}}$-closed in $\beta(I)$.

Proof. We assume that $\{\mathcal{F}_i\}_{i\in I}$ is a collection of elements of C_I. For each $i \in I$, there exists $P_i \in C$ such that $P_i \subseteq J_{\mathcal{F}_i}$. Since C is $\tau_{\mathcal{U}}$-closed, we have $\mathcal{U}\text{-}\lim_{i\in I} P_i \in C$.

By Lemma 3.12, it follows that $\mathcal{U}\text{-}\lim_{i\in I} P_i \subseteq \mathcal{U}\text{-}\lim_{i\in I} J_{\mathcal{F}_i} = J_{\mathcal{F}}$, where $\mathcal{F} = \mathcal{U}\text{-}\lim_{i\in I} \mathcal{F}_i$. Therefore, $\mathcal{F} \in C_I$, and C_I is a $\sigma_{\mathcal{U}}$-closed subset of $\beta(I)$. □

Theorem 3.16. Let $R = \prod_{i\in I} R_i$, where each R_i is a quasi-local ring, and let $\mathcal{F}$ be a nonprincipal ultrafilter on I. Consider $C \subseteq \beta(I)$. We claim that $C = (C_S)_I$.

Proof. We consider $\mathcal{F} \in (C_S)_I$. Then $I \in C_S$ such that $I \subseteq J_{\mathcal{F}_i}$. By Definition3.13, we know that there exists $\mathcal{U} \in C$ such that $I \subseteq J_{\mathcal{U}}$. Since R is a quasi-local ring, the ultrafilter $\mathcal{F}$ is the unique ultrafilter with this property. Hence, we must have $\mathcal{F} = \mathcal{U}$. Thus, $(C_S)_I \subseteq C$.

Now, let $\mathcal{F} \in C$. Then, $(J_{\mathcal{U}}) \in C_S$, and therefore $\mathcal{F} \in (C_S)_I$. Hence, we conclude that $C \subseteq (C_S)_I$. □

Lemma 3.17. Let $\mathcal{F}$ be a nonprincipal ultrafilter on I. Consider a family $\{C^j : j \in \Gamma\}$ of nonempty subsets of $\beta(I)$. If $\bigcap_{j\in\Gamma} C^j_S \neq \emptyset$, then $\bigcap_{j\in\Gamma} C^j \neq \emptyset$.

Proof. Let $P \in A = \bigcap_{j\in\Gamma} C^j_S$. For every $j \in \Gamma$, there exists $\mathcal{F}_j \in C^j$ such that $P \subseteq J_{\mathcal{F}_j}$. However, due to the uniqueness property of maximal ideals in $R = \prod_{i\in I} R_i$, we have $\mathcal{F}_j = \mathcal{F}$ for all $j \in \Gamma$. Therefore, $\mathcal{F} \in \bigcap_{j\in\Gamma} C^j$. □

Now, our aim is to determine the number of pairwise non-homeomorphic $\mathcal{F}$-topologies on $Spec(\prod_{i\in I} R_i)$, where each R_i is a quasi-local ring. We refer to [5] for the fact that the $\mathcal{F}$-topology is not compact in general, as exemplified by the case of $Spec(\mathbf{Q}^{\mathbf{N}})$ with the $\mathcal{F}$-topology, where $\mathcal{F}$ is any ultrafilter on $\mathbf{N}$.

Our goal is to calculate the number of pairwise non-homeomorphic $\mathcal{F}$-topologies, where $\mathcal{F} \in \beta(I)$.

Let us introduce the following definition:

Definition 3.18. Let $\mathcal{F}$ be a nonprincipal ultrafilter on I. A topological space X is said to be $\mathcal{F}$-compact if, for every collection $(x_i)_{i\in I}$ in X, there exists $x \in X$ such that $x = \mathcal{F} - \lim_{i\in I} x_i$.

The notion of $\mathcal{F}$-compactness plays a crucial role in characterizing the properties of $\mathcal{F}$-topologies.

Remark 3.19. Let $\mathcal{F}$ be a nonprincipal ultrafilter on I. Then the following statements hold:

1. $\tau_{\mathcal{F}}$ is $\mathcal{F}$-compact on $Spec(\prod_{i\in I} R_i)$.
2. $\sigma_{\mathcal{F}}$ is $\mathcal{F}$-compact on $\beta(I)$.
3. If $C \subseteq Spec(\prod_{i\in I} R_i)$ (resp. $\beta(I)$) is $\mathcal{F}$-compact, then it is $\tau_{\mathcal{F}}$-closed (resp. $\sigma_{\mathcal{F}}$-closed).

Definition 3.20. Let $\mathcal{F}, \mathcal{E} \in \beta(I) \setminus I$. The Comfort pre-order on $\beta(I) \setminus I$ is defined as $\mathcal{F} \leq_C \mathcal{E}$ if every $\mathcal{E}$-compact space is $\mathcal{F}$-compact.

Theorem 3.21. *Let $\mathcal{F}$ and $\mathcal{E}$ be nonprincipal ultrafilters on I. The following statements are equivalent:*
(a) $\mathcal{F} \leq_C \mathcal{E}$.
(b) $\tau_{\mathcal{E}} \subseteq \tau_{\mathcal{F}}$ on $Spec(\prod_{i\in I} R_i)$.
(c) $\sigma_{\mathcal{E}} \subseteq \sigma_{\mathcal{F}}$ on $\beta(I)$.
(d) $\sigma_{\mathcal{E}}$ is $\mathcal{F}$-compact.
(e) $\tau_{\mathcal{E}}$ is $\mathcal{F}$-compact on $Spec(\prod_{i\in I} R_i)$.

Proof. It is evident that (a) implies (d) and (e).

(a) ⇒ (b): Let $C \subseteq Spec(\prod_{i\in I} R_i)$ be a $\tau_{\mathcal{E}}$-closed set. Since the topology $\tau_{\mathcal{E}}$ is $\mathcal{E}$-compact, C is $\mathcal{E}$-compact. By assumption, C is $\mathcal{F}$-compact. According to Remark 3.19, we obtain that C is a $\tau_{\mathcal{F}}$-closed set. This shows that $\tau_{\mathcal{E}} \subseteq \tau_{\mathcal{F}}$.

(b) ⇒ (c): Let C be a $\sigma_{\mathcal{E}}$-closed set. By Theorem 3.14, we know that C_S is a $\tau_{\mathcal{E}}$-closed set. From (b), it follows that C_S is a $\tau_{\mathcal{F}}$-closed set. Now, by Theorem 3.15 and Theorem 3.16, we have that $C = (C_S)_I$ is a $\sigma_{\mathcal{E}}$-closed set. Therefore, $\sigma_{\mathcal{E}} \subseteq \sigma_{\mathcal{F}}$.

(c) ⇒ (a): From (c), $cl_{\sigma_{\mathcal{E}}}(I)$ is a $\sigma_{\mathcal{E}}$-closed set containing I. Then $cl_{\sigma_{\mathcal{F}}}(I) \subseteq cl_{\sigma_{\mathcal{E}}}(I)$ because $cl_{\sigma_{\mathcal{F}}}(I)$ is the smallest $\sigma_{\mathcal{F}}$-closed set containing I. By Definition 3.18, we have $\mathcal{F} \leq_C \mathcal{E}$.

(d) ⇒ (a): By definition, $cl_{\sigma_{\mathcal{E}}}(I)$ is a $\sigma_{\mathcal{E}}$-closed subset, and since $\sigma_{\mathcal{E}}$ is $\mathcal{F}$-compact, we have that $cl_{\sigma_{\mathcal{E}}}(I)$ is $\mathcal{F}$-compact. According to Remark 3.19, $cl_{\sigma_{\mathcal{E}}}(I)$ is $\tau_{\mathcal{F}}$-closed. Therefore, $cl_{\sigma_{\mathcal{E}}}(I)$ is a $\tau_{\mathcal{F}}$-closed set that contains I. Using a similar argument to the one given in the previous implication, we have $\mathcal{F} \leq_C \mathcal{E}$.

(e) ⇒ (b): Let $C \subseteq Spec(\prod_{i\in I} R_i)$ be a $\tau_{\mathcal{E}}$-closed subset. Since $\tau_{\mathcal{E}}$ is $\mathcal{F}$-compact, C is $\mathcal{F}$-compact. By Remark 3.19, C is $\tau_{\mathcal{F}}$-closed. Therefore, $\tau_{\mathcal{E}} \subseteq \tau_{\mathcal{F}}$. □

Corollary 3.22. *Let $\{R_i\}_{i\in I}$ be a collection of quasi-local rings, where I is an infinite set such that $\mid I \mid> \omega$ and every uniform ultrafilter on I is decomposable. Then there exist $2^{|I|}$ pairwise non-homeomorphic $\mathcal{F}$-topologies on $Spec(\prod_{i\in I} R_i)$.*

Proof. According to [9, Theorem 2.7], we have $2^{|I|}$ pairwise $\leq_{RK}$-incomparable uniform ultrafilters ($\mathcal{F} := \{A \subseteq I : |A| = |I|\}$) on I. Since each uniform ultrafilter is decomposable, the Rudin-Kiesler pre-order is equivalent to the Comfort pre-order on the set of uniform ultrafilters (a direct consequence of [4, Theorem 3.11 (3)]). By Theorem 3.21, the topologies $\{\tau_{\mathcal{F}} : \mathcal{F} \in I^*\}$ are pairwise non-homeomorphic. Therefore, we have $2^{|I|}$ pairwise non-homeomorphic topologies on $Spec(\prod_{i\in I} R_i)$. □

Example 3.23. As stated in [9], every uniform ultrafilter on I with $| I |= \aleph_n$ for $n < \omega$ is decomposable. Therefore, we have $2^{\aleph_n}$ pairwise non-homeomorphic topologies on $Spec(\prod_{i\in I} R_i)$, as shown in [9].

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