



## On the continuity of the inverse in (strongly) paratopological gyrogroups

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**Abstract.** In this paper, we consider the continuity of the inverse in (strongly) paratopological gyrogroups. The conclusions are established as follows: (1) A compact Hausdorff paratopological gyrogroup  $G$  is a topological gyrogroup. (2) A Hausdorff locally compact strongly paratopological gyrogroup is a strongly topological gyrogroup. (3) If  $G$  is locally compact strongly paratopological gyrocommutative gyrogroup (without any separation restrictions), then  $G$  is a strongly topological gyrocommutative gyrogroup. (4) If a strongly paratopological gyrogroup  $G$  is a dense  $G_\delta$ -set in a regular feebly compact space  $X$ , then  $G$  is a strongly topological gyrogroup. (5) If a Hausdorff strongly paratopological gyrogroup  $G$  is countably compact and topologically periodic, then  $G$  is a strongly topological gyrogroup.

### 1. Introduction

Finding a natural compactness-type condition on a topological semigroup (or paratopological group) that appear to suggest it is a topological group has many precedents in the literature. According to Ellis' theorem in [13], every locally compact Hausdorff semitopological group is a topological group. Romaguera and Sanchis [24] generalized the famous Numakura's theorem [20] and showed that every compact Hausdorff topological semigroup with two-sided cancellation is a topological group. A conclusion drawn from this result in [24] is that every compact  $T_0$  paratopological group is a topological group. It turns out that in the latter situation, the  $T_0$  constraint can be dropped. Ravsky [22] proved that a compact paratopological group is a topological group. Ellis [14], Grant [16], Brand [11], Bouziad [10], Bokalo and Guran [9], Romaguera and Sanchis [24], Kenderov et al.[18], and others have all generalized the latter fact. Reznichenko investigated automatic continuity in paratopological groups in [23], proving that every completely regular pseu-docompact paratopological group  $G$  is a topological group, i.e., the inversion in  $G$  is continuous. This result was extended to regular pseudocompact paratopological groups by Arhangel'skii and Reznichenko in [1, 2].

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2020 *Mathematics Subject Classification.* Primary 22A22; Secondary 20N05, 22A30

*Keywords.* Paratopological gyrogroup, topological gyrogroup, compact, locally compact, feebly compact, pseudocompact

Received: 28 June 2023; Revised: 11 November 2023; Accepted: 17 November 2023

Communicated by Ljubiša D. R. Kočinac

Research supported by the Natural Science Foundation of Guangdong Province under Grant (Nos. 2021A1515010381; 2020A1515110458). The Innovation Project of Department of Education of Guangdong Province (No. 2022KTSCX145), and the Natural Science Project of Jiangmen City (No. 2021030102570004880), Scientific research project of Guangzhou Panyu Polytechnic (No. 2022KJ02).

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The study of gyrogroups began in the early 1980s with the work of Abraham Albert Ungar [28, 29]. He first developed the concept of gyrovectors and then introduced the more general notion of a gyrogroup, which is a set equipped with a binary operation that combines gyrovector addition and scalar multiplication [30–32]. A gyrogroup, broadly defined, is a group-like structure where the associative law does not hold (see Definition 2.1). Since their introduction, gyrogroups have been a topic of interest for researchers in various areas of mathematics, ranging from group theory and topology to geometry and physics [5, 6]. The study of gyrogroups has led to the development of new algebraic structures and insights into the geometry of hyperbolic spaces and special relativity. Other applications of gyrogroups include the development of control algorithms in engineering and the study of dynamical systems [8].

Atiponrat [3] recently developed the idea of topological gyrogroups as a generalization of topological groups. A paratopological gyrogroup is a gyrogroup with a topology such that its binary operation is jointly continuous. If  $G$  is a paratopological gyrogroup and the inverse operation of  $G$  is continuous, then  $G$  is a topological gyrogroup. Specially, Atiponrat [3] discovered that for a topological gyrogroup,  $T_0$  and  $T_3$  are equivalent. It is worth noting that Cai, Lin and He in [12] proved that every Hausdorff first countable topological gyrogroup is metrizable. Atiponrat and Maungchang [4] studied some separation axioms of paratopological gyrogroups. In [17], Jin and Xie proved that every regular (Hausdorff) locally gyroscopic invariant paratopological gyrogroup  $G$  is completely regular (function Hausdorff), and extended the Pontrjagin conditions of (para)topological groups to (para)topological gyrogroups.

**Example 1.1.** Suppose that  $(\mathbb{R}, \tau_s)$  is Sorgenfrey line and  $(G, \tau)$  is a topological gyrogroup. Then  $\mathbb{R} \times G$  with product topology is a paratopological gyrogroup and not a topological gyrogroup.

As a generalization of paratopological groups, it is natural to consider the conditions for a paratopological gyrogroup to turn out to be a topological gyrogroup. In this paper, we try to study whether a paratopological gyrogroup satisfying a natural compactness-type condition and a separation axiom turns out to be a topological gyrogroup. The paper is organized as follows: In Section 2, we mainly introduce the related concepts and conclusions which are required in this article. In Section 3, we study the continuity of the inverse in (strongly) paratopological gyrogroups. The following results are established. (1) A compact Hausdorff paratopological gyrogroup  $G$  is a topological gyrogroup (see Theorem 3.1). (2) If  $G$  is a Hausdorff locally compact strongly paratopological gyrogroup, then  $G$  is a strongly topological gyrogroup (see Theorem 3.5). (3) Let  $G$  be a strongly paratopological gyrogroup, and  $H$  be an invariant subgyrogroup of  $G$ . If  $H$  and  $G/H$  are strongly topological gyrogroups, then so is  $G$  (see Theorem 3.6). (4) If  $G$  is locally compact strongly paratopological gyrocommutative gyrogroup (without any separation restrictions), then  $G$  is a strongly topological gyrocommutative gyrogroup (see Theorem 3.8). In Section 4, we consider feebly compact paratopological gyrogroups. The following results are established. (1) If a strongly paratopological gyrogroup  $G$  is a dense  $G_\delta$ -set in a regular feebly compact space  $X$ , then  $G$  is a strongly topological gyrogroup (see Theorem 4.6). (2) If a strongly paratopological gyrogroup  $G$  is Hausdorff countable compact and topologically periodic, then  $G$  is a strongly topological gyrogroup (see Theorem 4.12).

No separation restrictions on the topological spaces considered in this paper are imposed unless we mention them explicitly. Body of the paper.

## 2. Definitions and preliminaries

**Definition 2.1.** ([32]) Let  $(G, \oplus)$  be a nonempty groupoid. We say that  $(G, \oplus)$  or just  $G$  (when it is clear from the context) is a gyrogroup if the followings hold:

(G1) There is an identity element  $0 \in G$  such that

$$0 \oplus x = x = x \oplus 0 \quad \text{for all } x \in G.$$

(G2) For each  $x \in G$ , there exists an *inverse element*  $\ominus x \in G$  such that

$$\ominus x \oplus x = 0 = x \oplus (\ominus x).$$

(G3) For any  $x, y \in G$ , there exists an *gyroautomorphism*  $\text{gyr}[x, y] \in \text{Aut}(G, \oplus)$  such that

$$x \oplus (y \oplus z) = (x \oplus y) \oplus \text{gyr}[x, y](z)$$

for all  $z \in G$ ;

(G4) For any  $x, y \in G$ ,  $\text{gyr}[x \oplus y, y] = \text{gyr}[x, y]$ .

For a gyrogroup  $G$  and  $x_1, x_2, \dots, x_k \in G$ , the formula  $((x_1 \oplus x_2) \oplus x_3) \oplus \dots \oplus x_{k-1} \oplus x_k$  will be denoted by  $x_1 \oplus x_2 \oplus \dots \oplus x_k$ .

**Definition 2.2.** ([32]) A gyrogroup  $(G, \oplus)$  is gyrocommutative if its binary operation obeys the gyrocommutative law

$$a \oplus b = \text{gyr}[a, b](b \oplus a)$$

for all  $a, b \in G$ .

**Definition 2.3.** ([32]) Let  $(G, \oplus)$  be a gyrogroup with gyrogroup operation (or, addition)  $\oplus$ . The gyrogroup cooperation (or, coaddition)  $\boxplus$  is a second binary operation in  $G$  given by the equation

$$(*) \quad a \boxplus b = a \oplus \text{gyr}[a, \ominus b]b$$

for all  $a, b \in G$ . The groupoid  $(G, \boxplus)$  is called a cogyrogroup, and is said to be the cogyrogroup associated with the gyrogroup  $(G, \oplus)$ .

Replacing  $b$  by  $\ominus b$  in  $(*)$ , along with  $(*)$  we have the identity

$$a \boxminus b = a \ominus \text{gyr}[a, b]b$$

for all  $a, b \in G$ , where we use the obvious notation,  $a \boxminus b = a \boxplus (\ominus b)$ .

**Definition 2.4.** ([27]) Let  $(G, \oplus)$  be a gyrogroup. A nonempty subset  $H$  of  $G$  is called a subgyrogroup, denoted by  $H \leq G$ , if the following statements hold:

- (1) The restriction  $\oplus|_{H \times H}$  is a binary operation on  $H$ , i.e.  $(H, \oplus|_{H \times H})$  is a groupoid;
- (2) For any  $x, y \in H$ , the restriction of  $\text{gyr}[x, y]$  to  $H$ ,  $\text{gyr}[x, y]|_H : H \rightarrow \text{gyr}[x, y](H)$ , is a bijective homomorphism; and
- (3)  $(H, \oplus|_{H \times H})$  is a gyrogroup.

Furthermore, a subgyrogroup  $H$  of  $G$  is said to be an *L-subgyrogroup* [27], denoted by  $H \leq_L G$ , if  $\text{gyr}[a, h](H) = H$  for all  $a \in G$  and  $h \in H$ .

A *semigroup* is a non-void set  $S$  together with a mapping  $(x, y) \rightarrow xy$  of  $S \times S$  to  $S$  such that  $x(yz) = (xy)z$  for all  $x, y, z$  in  $S$ .

**Proposition 2.5.** ([26]) A nonempty subset  $H$  of  $G$  is a subgyrogroup if and only if  $a \in H$  implies  $\ominus a \in H$  and  $a, b \in H$  implies  $a \oplus b \in H$ .

**Proposition 2.6.** ([26]) A nonempty subset  $X$  of a gyrogroup  $G$  is a subgroup if and only if it is a subgyrogroup of  $G$  and the restriction of  $\text{gyr}[a, b]$  to  $X$  equals the identity map on  $X$  for all  $a, b \in X$ .

In this paper,  $\text{gyr}[a, b](V)$  denotes  $\{\text{gyr}[a, b](v) : v \in V\}$ .

**Theorem 2.7.** ([32]) Let  $(G, \oplus)$  be a gyrogroup. Then, for any  $a, b, c \in G$  we have

- (1)  $(a \oplus b) \oplus c = a \oplus (b \oplus \text{gyr}[b, a]c)$ ; Right Gyroassociative Law
- (2)  $\text{gyr}[a, b] = \text{gyr}[a, b \oplus a]$ ; Right Loop Property
- (3)  $(\ominus a) \oplus (a \oplus b) = b$ ;
- (4)  $(a \ominus b) \boxplus b = a$ ;

- (5)  $(a \boxplus b) \oplus b = a$ ;
- (6)  $\text{gyr}[a, b](c) = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c))$ ;
- (7)  $\ominus(a \oplus b) = \text{gyr}[a, b](\ominus b \oplus a)$ ; Gyrosum Inversion
- (8)  $\text{gyr}[a, b](\ominus x) = \ominus \text{gyr}[a, b]x$ ;
- (9)  $\text{gyr}^{-1}[a, b] = \text{gyr}[b, a]$ ; Inversive symmetry
- (10)  $\ominus(a \boxplus b) = (\ominus b) \boxplus (\ominus a)$ ; The Cogyroautomorphic Inverse Theorem
- (11)  $\text{gyr}[\ominus a, \ominus b] = \text{gyr}[a, b]$ ; Even symmetry
- (12)  $\text{gyr}[a, 0] = \text{gyr}[0, b] = I$ .

**Theorem 2.8.** ([32]) *Let  $(G, \oplus)$  be a gyrocommutative gyrogroup. Then, for any  $a, b, c \in G$  we have*

- (1)  $\ominus(a \oplus b) = \ominus a \oplus b$ ; Gyroautomorphic Inverse Property
- (2)  $a \boxplus b = b \boxplus a$ ;
- (3)  $a \boxplus b = a \oplus ((\ominus a \oplus b) \oplus a)$ .

**Definition 2.9.** ([26]) *A subgyrogroup  $N$  of a gyrogroup  $G$  is normal in  $G$ , written  $N \trianglelefteq G$ , if it is the kernel of a gyrogroup homomorphism of  $G$ .*

**Theorem 2.10.** ([26]) *Let  $N$  be a subgyrogroup of a gyrogroup  $G$ . Then  $N$  is a normal subgyrogroup in  $G$  if and only if  $a \oplus (N \oplus b) = (a \oplus b) \oplus N = (a \oplus N) \oplus b$  for all  $a, b \in G$ .*

Since in Topology ‘normal’ refers to a separation property of spaces, we will use the term ‘invariant’ to denote this property of subgyrogroups.

**Definition 2.11.** ([3]) *A triple  $(G, \tau, \oplus)$  is called a topological gyrogroup if and only if*

- (1)  $(G, \tau)$  is a topological space;
- (2)  $(G, \oplus)$  is a gyrogroup;
- (3) The binary operation  $\oplus : G \times G \rightarrow G$  is continuous where  $G \times G$  is endowed with the product topology and the operation of taking the inverse  $\ominus(\cdot) : G \rightarrow G$ , i.e.  $x \rightarrow \ominus x$ , is continuous.

If a triple  $(G, \tau, \oplus)$  satisfies the first two conditions and its binary operation is continuous, we call such triple a *paratopological gyrogroup* [4]. Sometimes we will just say that  $G$  is a topological gyrogroup (paratopological gyrogroup) if the binary operation and the topology are clear from the context.

**Definition 2.12.** ([7]) *Let  $(G, \tau, \oplus)$  be a topological gyrogroup. We say that  $G$  is a strongly topological gyrogroup if there exists a neighborhood base  $\mathcal{U}$  of the identity  $0$  in  $G$  such that, for every  $U \in \mathcal{U}$ ,  $\text{gyr}[x, y](U) = U$  for any  $x, y \in G$ .*

Similarly, we called a paratopological gyrogroup  $(G, \tau, \oplus)$  a *strongly paratopological gyrogroup* if there exists a neighborhood base  $\mathcal{U}$  of the identity  $0$  in  $G$  such that, for every  $U \in \mathcal{U}$ ,  $\text{gyr}[x, y](U) = U$  for any  $x, y \in G$ .

**Proposition 2.13.** ([4]) *Let  $G$  be a paratopological gyrogroup,  $x, y \in G$  and  $A, B \subseteq G$ .*

- (1) *The left translation  $L_x : G \rightarrow G$ , where  $L_x(y) = x \oplus y$  for every  $y \in G$ , is homeomorphism;*
- (2)  *$A$  is closed if and only if  $x \oplus A$  is closed;*
- (3)  *$A$  is open if and only if  $x \oplus A$  and  $B \oplus A$  are open;*

**Proposition 2.14.** *Let  $G$  be a paratopological gyrogroup,  $x, y \in G$  and  $A, B \subseteq G$ .*

- (1)  *$\text{gyr}[x, y] : G \rightarrow G$ , for every  $x, y \in G$ , is homeomorphism;*
- (2)  *$A$  is closed if and only if  $\text{gyr}[x, y](A)$  is closed;*
- (3)  *$A$  is open if and only if  $\text{gyr}[x, y](A)$  is open.*

*Proof.* By definition,  $\text{gyr}[x, y]$  is bijective. Moreover, the gyrator identity provides that  $\text{gyr}[x, y] = L_{\ominus(x \oplus y)} \circ L_x \circ L_y$  which is a homeomorphism by Proposition 2.13. So (2) and (3) are true.  $\square$

**Proposition 2.15.** *Let  $G$  be a paratopological gyrogroup and  $U$  be a neighborhood of the identity  $0$ . Then there is an open neighborhood  $V$  of  $0$  such that  $\overline{V} \oplus \overline{V} \subseteq \overline{U}$ .*

*Proof.* For  $G$  is a paratopological gyrogroup, then  $op_2 : G \times G \rightarrow G$  defined by  $op_2(x, y) = x \oplus y$  is continuous. Because  $0 \oplus 0 = 0$ , and  $U$  is a neighborhood of the identity  $0$ , there exist a neighborhood  $V$  of  $0$  such that  $\overline{V} \oplus \overline{V} = op_2(\overline{V} \times \overline{V}) = op_2(\overline{V} \times \overline{V}) \subseteq op_2(V \times V) = \overline{V} \oplus \overline{V} \subseteq \overline{U}$ .  $\square$

**Lemma 2.16.** *Let  $G$  be a paratopological gyrogroup and  $\mathcal{U}$  be the neighborhood base at  $0$  of  $G$ . Then for  $B = \bigcap \{ \overline{U} : U \in \mathcal{U} \}$ ,  $\text{gyr}[a, b](B) = B$  for any  $a, b \in G$ .*

*Proof.* For  $a, b \in G$ , suppose  $f(x) = \text{gyr}[a, b](x)$  for any  $x \in G$ . By Proposition 2.14,  $f$  is homeomorphism. Since  $f(0) = \text{gyr}[a, b](0) = 0$ , for  $U \in \mathcal{U}$ , there exists  $V \in \mathcal{U}$  such that  $\text{gyr}[a, b](\overline{V}) = f(\overline{V}) \subseteq \overline{f(V)} \subseteq \overline{U}$ . It follows that  $\text{gyr}[a, b](B) \subseteq B$ , for each  $a, b \in G$ .

It is obvious that  $f^{-1}(x) = \text{gyr}[b, a](x)$  by Theorem 2.7 (9), which is continuous. Since  $f^{-1}(0) = \text{gyr}[b, a](0) = 0$ , for  $U_1 \in \mathcal{U}$ , there exists  $V_1 \in \mathcal{U}$  such that  $\text{gyr}[b, a](\overline{V_1}) = f^{-1}(\overline{V_1}) \subseteq \overline{f^{-1}(V_1)} \subseteq \overline{U_1}$ . It follows that  $\text{gyr}[b, a](B) \subseteq B$ , for each  $a, b \in G$ . Thus we have  $B \subseteq \text{gyr}[a, b](B)$ . So we get  $\text{gyr}[a, b](B) = B$  for any  $a, b \in G$ .  $\square$

**Theorem 2.17.** ([17]) *Let  $G$  be a Hausdorff topological gyrogroup and  $\mathcal{U}$  an open base at the neutral element  $0$  of  $G$ . The following conditions hold:*

- (8) *for every  $U \in \mathcal{U}$  and  $x \in G$ , there exists  $V \in \mathcal{U}$  such that  $V \boxplus x \subseteq x \oplus U$  and  $x \oplus V \subseteq x \boxplus U$ ;*
- (9) *for every  $U \in \mathcal{U}$ , there exists  $V \in \mathcal{U}$  such that  $\ominus V \subseteq U$ .*

**Proposition 2.18.** *Let  $(G, \tau, \oplus)$  be a paratopological gyrogroup,  $F$  a compact subset of  $G$ , and  $O$  an open subset of  $G$  such that  $F \subseteq O$ . Then there exists an open neighborhood  $V$  of the identity element  $0$  such that  $F \oplus V \subseteq O$  and  $V \oplus F \subseteq O$ .*

*Proof.* Since  $\oplus : G \times G \rightarrow G$  is continuous in paratopological gyrogroup  $G$ ,  $\oplus^{-1}(O)$  is an open set in  $G \times G$  and  $\{0\} \times F \subseteq \oplus^{-1}(O)$  for  $F \subseteq O$ . Note that  $\{0\} \times F$  is compact in  $G \times G$ , there exist open sets  $V_1, W$  in  $G$  such that  $\{0\} \times F \subseteq V_1 \times W \subseteq \oplus^{-1}(O)$ . Thus  $V_1 \oplus F = \oplus(V_1 \times F) \subseteq \oplus(V_1 \times W) \subseteq O$ .

Similarly, one can find an open set  $V_2$  in  $G$  such that  $F \oplus V_2 \subseteq O$ . Take  $V = V_1 \cap V_2$ . Then we verify that  $F \oplus V \subseteq O$  and  $V \oplus F \subseteq O$ .  $\square$

**Lemma 2.19.** *Let the neighborhood base  $\mathcal{U}$  at  $0$  of  $G$  witness that  $G$  is a strongly paratopological gyrogroup. Then we have  $a \boxplus U \subseteq a \oplus U$  and  $a \boxplus U \subseteq a \ominus U$  for each  $a \in G, U \in \mathcal{U}$ .*

*Proof.* By Definitions 2.3 and 2.12, we can get  $a \boxplus U = \bigcup_{u \in U} a \boxplus u = \bigcup_{u \in U} a \oplus \text{gyr}[a, \ominus u]u = a \oplus \bigcup_{u \in U} \text{gyr}[a, \ominus u]u \subseteq a \oplus U$  and  $a \boxplus U = \bigcup_{u \in U} a \boxplus u = \bigcup_{u \in U} a \ominus \text{gyr}[a, u]u = a \ominus \bigcup_{u \in U} \text{gyr}[a, u]u \subseteq a \ominus U$ , for each  $a \in G, U \in \mathcal{U}$ .  $\square$

**Lemma 2.20.** *Let the neighborhood base  $\mathcal{U}$  at  $0$  of  $G$  witness that  $G$  is a strongly paratopological gyrogroup. Then for each  $U_1, U_2 \in \mathcal{U}$  we have  $U_1 \oplus U_2 \in \mathcal{U}$ .*

*Proof.* By Definition 2.12, we have  $\text{gyr}[a, b](U_1 \oplus U_2) = \text{gyr}[a, b](U_1) \oplus \text{gyr}[a, b](U_2) = U_1 \oplus U_2$ , for each  $a, b \in G, U_1, U_2 \in \mathcal{U}$ , which implies  $U_1 \oplus U_2 \in \mathcal{U}$ .  $\square$

**Lemma 2.21.** *Let the neighborhood base  $\mathcal{U}$  at  $0$  of  $G$  witness that  $G$  is a strongly paratopological gyrogroup. Then we have  $(a \oplus U) \oplus W = a \oplus (U \oplus W)$  for each  $a \in G, U, W \in \mathcal{U}$ .*

*Proof.* By Definition 2.12 and Theorem 2.7 (1), we have  $(a \oplus U) \oplus W = a \oplus (U \oplus \bigcup_{u \in U} \text{gyr}[u, a]W) \subseteq a \oplus (U \oplus W)$  and  $a \oplus (U \oplus W) = (a \oplus U) \oplus \bigcup_{u \in U} \text{gyr}[a, u]W \subseteq (a \oplus U) \oplus W$ , for each  $a \in G, U, W \in \mathcal{U}$ . Thus we get the result.  $\square$

**Lemma 2.22.** *Let the neighborhood base  $\mathcal{U}$  at 0 of  $G$  witness that  $G$  is a strongly paratopological gyrogroup. If  $U \oplus V \subseteq W$ , then  $\ominus V \ominus U \subseteq \ominus W$ , for each  $W, U, V \in \mathcal{U}$ .*

*Proof.* By Theorem 2.7 (7), we have  $\ominus(U \oplus V) = \bigcup_{u \in U, v \in V} \text{gyr}[u, v](\ominus v \ominus u) \subseteq \ominus W$ . We can get  $\text{gyr}[u, v](\ominus v \ominus u) \in \ominus W$  for every  $u \in U, v \in V$ . By Theorem 2.7(8)(9),  $\ominus v \ominus u \in \text{gyr}[v, u](\ominus W) = \ominus \text{gyr}[v, u](W) = \ominus W$ . Thus we get  $\ominus V \ominus U \subseteq \ominus W$ .  $\square$

**Lemma 2.23.** *Let the neighborhood base  $\mathcal{U}$  at 0 of  $G$  witness that  $G$  is a strongly paratopological gyrogroup. Then for each  $W \in \mathcal{U}$ , there exists  $U \in \mathcal{U}$  such that  $U \oplus (U \oplus U) \subseteq W$  and  $(\ominus U \ominus U) \ominus U \subseteq \ominus W$ .*

*Proof.* Since operator  $\oplus$  is continuous in  $G$ , for each  $W \in \mathcal{U}$  we can find neighbourhoods  $U_1, V$  of 0 such that  $U_1 \oplus V \subseteq W$ . And for  $V \in \mathcal{U}$  there exists an open neighbourhood  $V_1$  of 0 such that  $V_1 \oplus V_1 \subseteq V$ . So  $\ominus V \ominus U_1 \subseteq \ominus W$  and  $\ominus V_1 \ominus V_1 \subseteq \ominus V$  by Lemma 2.22. Let  $U = U_1 \cap V_1$ . Thus we can get  $U \oplus (U \oplus U) \subseteq W$  and  $(\ominus U \ominus U) \ominus U \subseteq \ominus W$ .  $\square$

**Lemma 2.24.** *Let the neighborhood base  $\mathcal{U}$  at 0 of  $G$  witness that  $G$  is a strongly paratopological gyrogroup. If  $V \oplus V \subseteq U$  where  $U, V \in \mathcal{U}$ , then  $\overline{\ominus V} \subseteq U$ .*

*Proof.* We show that  $\overline{\ominus V} \subseteq \ominus U$ . Let  $x \in \overline{\ominus V}$ . Then  $(x \oplus V) \cap (\ominus V) \neq \emptyset$ . Therefore there exist  $v_1, v_2 \in V$  such that  $x \oplus v_1 = \ominus v_2$  and  $x = \ominus v_2 \boxplus v_1 \in \ominus V \boxplus V \subseteq \ominus V \ominus V$  by Lemma 2.19. For  $\ominus V \ominus V \subseteq \ominus U$  by Lemma 2.22, we get  $x \in \ominus U$ .  $\square$

Let  $(G, \tau, \oplus)$  be a paratopological gyrogroup and  $H$  a  $L$ -subgyrogroup of  $G$ . It follows from [27, Theorem 20] that  $G/H = \{a \oplus H : a \in G\}$  is a partition of  $G$ . We denote by  $\pi$  the mapping  $a \mapsto a \oplus H$  from  $G$  onto  $G/H$ . Clearly, for each  $a \in G$ , we have  $\pi^{-1}(\pi(a)) = a \oplus H$ . Denote by  $\tau(G)$  the topology of  $G$ . In the left cosets  $G/H$  of the gyrogroup  $G$ , we define a topology  $\tilde{\tau} = \tau(G/H)$  of subsets as follows:

$$\tilde{\tau} = \tau(G/H) = \{O \subseteq G/H : \pi^{-1}(O) \in \tau(G)\}.$$

A continuous mapping  $f : X \rightarrow Y$  is *perfect* if  $f$  is a closed mapping and all fibers  $f^{-1}(y)$  are compact subsets of  $X$ .

**Proposition 2.25.** *Let  $(G, \tau, \oplus)$  be a paratopological gyrogroup and  $H$  a  $L$ -subgyrogroup of  $G$ . Then the natural homomorphism  $\pi$  from a paratopological gyrogroup  $G$  to its quotient topology on  $G/H$  is an open and continuous mapping.*

*Proof.* The continuity of the map  $\pi$  is obvious. If  $U \subseteq G$  is an open set then  $\pi^{-1}(\pi(U)) = U \oplus H$  and hence  $\pi(U)$  is open.  $\square$

**Proposition 2.26.** *Let  $(G, \tau, \oplus)$  be a paratopological gyrogroup and  $H$  an invariant subgyrogroup of  $G$ . If  $H$  is a compact subgyrogroup of  $G$ , then the quotient mapping  $\pi$  of  $G$  onto the quotient space  $G/H$  is perfect.*

*Proof.* Let  $F$  be a closed subset of the gyrogroup  $G$ . Let  $\tilde{x} \in G/H \setminus \pi(F)$ . Consider an arbitrary point  $x \in \pi^{-1}(\tilde{x})$ . Then  $(x \oplus H) \cap F = \emptyset$ . By Proposition 2.18 there exists an open neighborhood  $U$  of the unit such that  $(U \oplus (x \oplus H)) \cap F = \emptyset$ . Then  $\tilde{x} \in \pi(U \oplus x)$  and  $\pi(U \oplus x) = \pi(U) \oplus \pi(x) = \pi(U) \oplus (\pi(x) \oplus \pi(H)) = \pi(U \oplus (x \oplus H)) \cap \pi(F) = \emptyset$  thus the map  $\pi$  is closed. Furthermore, if  $y \in G/H$  and  $\pi(x) = y$  for some  $x \in G$ , we obtain that  $\pi^{-1}(y) = x \oplus H$  is a compact subset of  $G$ . Hence, the fibers of  $\pi$  are compact. Thus  $\pi$  is perfect.  $\square$

### 3. Compact (strongly) paratopological gyrogroups and locally compact (strongly) paratopological gyrogroups

We give the first of our non-trivial claims on the inverse mapping's continuity in paratopological gyrogroups.

**Theorem 3.1.** <sup>1)</sup> A compact Hausdorff paratopological gyrogroup  $G$  is a topological gyrogroup.

*Proof.* Let  $0$  be the neutral element of  $G$ . Since  $G$  is Hausdorff, the set  $M = \{(x, y) \in G \times G : x \oplus y = 0\}$  is closed in  $G \times G$ .

Let  $F$  be any closed subset of  $G$ , and  $P = (G \times F) \cap M$ . Then  $F$  and  $G \times F$  are compact,  $P$  closed in  $G \times F$ , since  $M$  is closed, and, therefore,  $P$  is compact. It is true that  $(x, y) \in P$  if and only if  $y \in F$  and  $x \oplus y = 0$ , that is,  $x = \ominus y$ . It follows that the image of  $P$  under the natural projection of  $G \times G$  onto the first factor  $G$  is precisely  $\ominus F$ . Since  $P$  is compact and the projection mappings are continuous, we conclude that  $\ominus F$  is compact, and therefore, closed in  $G$ . Thus, the inverse operation in  $G$  is continuous. Hence  $G$  is a topological gyrogroup.  $\square$

A topological space  $X$  is called a *locally compact space* if for every  $x \in X$  there exists a neighbourhood  $U$  of the point  $x$  such that  $\bar{U}$  is a compact subspace of  $X$ .

It is natural to extend Theorem 3.1 to locally compact Hausdorff paratopological gyrogroups, We pose the following problem.

**Problem 3.2.** Is a locally compact Hausdorff paratopological gyrogroup  $G$  with a countable base a topological gyrogroup?

Indeed, Theorem 3.1 can be extended to locally compact Hausdorff strongly paratopological gyrogroups with a slightly more involved argument.

**Lemma 3.3.** Let the neighborhood base  $\mathcal{U}_1$  at  $0$  of  $G$  witness that  $G$  is a strongly paratopological gyrogroup and a family  $\mathcal{U} = \{U_n : n \in \omega\} \subseteq \mathcal{U}_1$ , and  $\{x_n : n \in \omega\}$  is a sequence of points in  $G$  such that  $x_n \in U_n$  for each  $n \in \omega$ , and the next conditions are satisfied:

- (1)  $\overline{U_{n+1} \oplus U_{n+1}} \subseteq U_n$  for each  $n \in \omega$ ;
- (2) the sequence  $\{y_k : k \in \mathbb{N}\}$ , where  $y_k = (((x_1 \oplus x_2) \oplus x_3) \oplus \cdots \oplus x_{k-1}) \oplus x_k$ , has an accumulation point  $y$  in  $G$ .

Then there exists  $k \in \omega$  such that  $\ominus x_{k+1} \in U_0$ .

*Proof.* Since  $y \oplus U_1$  is a neighbourhood of  $y$ , there exists  $k \in \mathbb{N}$  such that  $y_k \in y \oplus U_1$ . Put  $z = \text{gyr}[\ominus y_k, y_{k+1}](\ominus y_{k+1} \oplus y)$ . For  $y_{k+1} = y_k \oplus x_{k+1}$ , we can get  $x_{k+1} = \ominus y_k \oplus y_{k+1}$ . Thus

$$\begin{aligned} \ominus x_{k+1} &= \ominus(\ominus y_k \oplus y_{k+1}) \\ &= \text{gyr}[\ominus y_k, y_{k+1}](\ominus y_{k+1} \oplus y_k) \quad \text{by Theorem 2.7 (7)} \\ &\in \text{gyr}[\ominus y_k, y_{k+1}](\ominus y_{k+1} \oplus (y \oplus U_1)) \\ &= \text{gyr}[\ominus y_k, y_{k+1}]((\ominus y_{k+1} \oplus y) \oplus (\text{gyr}[\ominus y_{k+1}, y]U_1)) \quad \text{by Definition 2.1} \\ &\subseteq \text{gyr}[\ominus y_k, y_{k+1}]((\ominus y_{k+1} \oplus y) \oplus U_1) \quad \text{by Definition 2.12} \\ &= \text{gyr}[\ominus y_k, y_{k+1}](\ominus y_{k+1} \oplus y) \oplus \text{gyr}[\ominus y_k, y_{k+1}](U_1) \\ &= \text{gyr}[\ominus y_k, y_{k+1}](\ominus y_{k+1} \oplus y) \oplus U_1 \quad \text{by Definition 2.12} \\ &= z \oplus U_1. \end{aligned}$$

Since by condition (2) the sequence  $\{y_m : m \in \mathbb{N}\}$  has an accumulation point  $y$  in  $G$ , for  $k \in \mathbb{N}$  the sequence  $\{\text{gyr}[\ominus y_k, y_{k+1}](\ominus y_{k+1} \oplus y_m) : m \in \mathbb{N}\}$  has an accumulation point  $\text{gyr}[\ominus y_k, y_{k+1}](\ominus y_{k+1} \oplus y) = z$  in  $G$  by Propositions 2.13 and 2.14.

For each  $m > k + 2$ ,  $y_m = y_{k+1} \oplus x_{k+2} \oplus \cdots \oplus x_m$ .

Since the neighborhood base  $\mathcal{U}_1$  at  $0$  of  $G$  witness that  $G$  is a strongly paratopological gyrogroup and the family  $\mathcal{U} \subseteq \mathcal{U}_1$ , we have

<sup>1)</sup>Through private communication, it was learned that Piyu Li and others obtained this conclusion independently.

$$\begin{aligned}
 y_m &\in ((y_{k+1} \oplus U_{k+2}) \oplus U_{k+3}) \oplus \cdots \oplus U_{m-1} \oplus U_m \\
 &= (((y_{k+1} \oplus U_{k+2}) \oplus U_{k+3}) \oplus \cdots \oplus U_{m-2}) \oplus (U_{m-1} \oplus U_m) \quad \text{by Lemma 2.21} \\
 &= (((y_{k+1} \oplus U_{k+2}) \oplus U_{k+3}) \oplus \cdots \oplus U_{m-3}) \oplus (U_{m-2} \oplus (U_{m-1} \oplus U_m)) \text{ by Lemmas 2.20, 2.21} \\
 &= (((y_{k+1} \oplus U_{k+2}) \oplus U_{k+3}) \oplus \cdots \oplus U_{m-3}) \oplus ((U_{m-2} \oplus U_{m-1}) \oplus U_m) \quad \text{by Lemma 2.21} \\
 &\dots\dots \\
 &= y_{k+1} \oplus (U_{k+2} \oplus \cdots \oplus U_m).
 \end{aligned}$$

It follows from condition (1) of the lemma that,

$$\ominus y_{k+1} \oplus y_m \in U_{k+2} \oplus \cdots \oplus U_m \subseteq U_{k+1}.$$

Therefore,  $\text{gyr}[\ominus y_k, y_{k+1}](\ominus y_{k+1} \oplus y_m) \in \text{gyr}[\ominus y_k, y_{k+1}](U_{k+1}) \subseteq U_{k+1}$ . So we can get  $z \in \overline{U_{k+1}} \subset U_k$ , which implies that

$$\ominus x_{k+1} \in z \oplus U_1 \subseteq U_k \oplus U_1 \subseteq U_0.$$

This finishes the proof.  $\square$

**Lemma 3.4.** *Let the neighborhood base  $\mathcal{U}$  at 0 of  $G$  witness that  $G$  is a strongly paratopological gyrogroup. Then  $G$  is a strongly topological gyrogroup if and only if the inverse operation is continuous at the identity 0.*

*Proof.* The ‘only if’ part is clear. We just need to prove the ‘if’ part. For  $G$  is a strongly paratopological gyrogroup, by Lemma 2.19, we have that  $x \boxplus U \subseteq x \ominus U$ . Thus we have that  $x \oplus U \subseteq x \boxplus U$  by the the following operation:

$$\begin{aligned}
 x \boxplus U &\subseteq x \ominus U && \text{by Lemma 2.19} \\
 \Rightarrow x \boxplus (\ominus U) &= x \boxplus U \subseteq x \ominus U && \text{by } a \boxplus b = a \boxplus (\ominus b) \\
 \Rightarrow \ominus(U \boxplus (\ominus x)) &= x \boxplus (\ominus U) \subseteq x \ominus U && \text{by Theorem 2.7 (10)} \\
 \Rightarrow U \boxplus (\ominus x) &\subseteq \ominus(x \ominus U) \\
 \Rightarrow U \boxplus x &= U \boxplus (\ominus x) \subseteq \ominus(x \ominus U) && \text{by } a \boxplus b = a \boxplus (\ominus b) \\
 \Rightarrow U &\subseteq \ominus(x \ominus U) \oplus x && \text{by Theorem 2.7 (4)} \\
 \Rightarrow U &\subseteq \bigcup_{u \in U} (\ominus(x \ominus u) \oplus x) && \text{by } \ominus(x \ominus U) \oplus x = \bigcup_{u \in U} (\ominus(x \ominus u) \oplus x) \\
 \Rightarrow U &\subseteq \bigcup_{u \in U} \text{gyr}[x, \ominus u]u && \text{by Theorem 2.7 (6)} \\
 \Rightarrow x \oplus U &\subseteq \bigcup_{u \in U} x \oplus \text{gyr}[x, \ominus u]u \\
 \Rightarrow x \oplus U &\subseteq x \boxplus U && \text{by } x \boxplus u = x \oplus \text{gyr}[x, \ominus u]u \quad (*)
 \end{aligned}$$

For each  $U \in \mathcal{U}$ ,  $x \in G$ , it is obvious that  $(x \oplus U) \ominus x$  is a neighborhood of 0. Hence there exists  $V_1 \in \mathcal{U}$  such that  $V_1 \subseteq (x \oplus U) \ominus x$ , which is equivalent to

$$V_1 \boxplus x \subseteq x \oplus U. \quad (**)$$

Take any  $x \in G$ ,  $O \in \mathcal{U}$ . By (\*\*), there is  $U_1 \in \mathcal{U}$  such that  $U_1 \boxplus (\ominus x) \subseteq \ominus x \oplus O$ . For  $U_1$ , there exists  $U_2 \in \mathcal{U}$  such that  $\ominus U_2 \subseteq U_1$ . For  $U_2$ , one can find  $V_1 \in \mathcal{U}$  such that  $x \oplus V_1 \subseteq x \boxplus U_2$  by (\*). Then we have that

$$\begin{aligned}
 \ominus(x \oplus V_1) &\subseteq \ominus(x \boxplus U_2) \\
 &= \ominus U_2 \boxplus (\ominus x) \\
 &\subseteq U_1 \boxplus (\ominus x) \\
 &\subseteq \ominus x \oplus O.
 \end{aligned}$$



Thus we show that the inverse operation is continuous at any  $x \in G$ . This finishes the proof.  $\square$

**Theorem 3.5.** *If  $G$  is a Hausdorff locally compact strongly paratopological gyrogroup, then  $G$  is a strongly topological gyrogroup.*

*Proof.* To prove  $G$  is a topological gyrogroup, it is just to show that the inverse operation is continuous at  $0 \in G$  by Lemma 3.4.

Let the neighborhood base  $\mathcal{U}$  at 0 of  $G$  witness that  $G$  is a strongly paratopological gyrogroup. We shall prove that for each  $U \in \mathcal{U}$  one can find  $V \in \mathcal{U}$  such that  $\ominus V \subseteq U$ . Assume the contrary, there is a  $U \in \mathcal{U}$  such that for each  $V \in \mathcal{U}$ ,  $\ominus V$  is not a subset of  $U$ . Since  $G$  is a Hausdorff locally compact space, so  $G$  is regular. Thus, we can find a  $U_0 \in \mathcal{U}$  such that  $U \supseteq \overline{U_0}$  is compact.  $G$  is a paratopological gyrogroup, so we can define a sequence  $\{U_n : n \in \omega\}$  of  $\mathcal{U}$  such that  $\overline{U_{n+1} \oplus U_{n+1}} \subseteq U_n$  for each  $n \in \omega$  and there is  $x_n \in U_n$  satisfying  $\ominus x_n \notin U_0$  for each  $n \in \mathbb{N}$ . Put  $y_k = (((x_1 \oplus x_2) \oplus x_3) \oplus \cdots \oplus x_{k-1}) \oplus x_k$ , for each  $k \in \mathbb{N}$ . Then from Lemmas 2.19 and 2.21 it easily follows all elements  $y_k$  are in  $U_0$ . In fact,

$$\begin{aligned} y_k &= (((x_1 \oplus x_2) \oplus x_3) \oplus \cdots \oplus x_{k-1}) \oplus x_k \\ &\in (((x_1 \oplus U_2) \oplus U_3) \oplus \cdots \oplus U_{k-1}) \oplus U_k \\ &= x_1 \oplus (U_2 \oplus \cdots \oplus U_k) \quad \text{by Lemmas 2.20, 2.21} \\ &\subseteq x_1 \oplus U_1 \\ &\subseteq U_1 \oplus U_1 \\ &\subseteq U_0. \end{aligned}$$

Since the closure of  $U_0$  is compact, there exists an accumulation point  $y$  for the sequence  $\{y_k : k \in \mathbb{N}\}$  in  $G$ . Thus by Lemma 3.3 there is a  $k \in \omega$  such that  $\ominus x_{k+1} \in U_0$ . This is a contradiction with  $\ominus x_{k+1} \notin U_0$ . Thus we have proved that the inverse operation  $\ominus$  is continuous at 0. This finishes the proof.  $\square$

In the latter case we try to remove the Hausdorff restriction in Theorem 3.5 To demonstrate this, we need the following theorem, which is inspired by Ravsky’s result [21].

**Theorem 3.6.** *Let  $G$  be a strongly paratopological gyrogroup, and  $H$  be an invariant subgyrogroup of  $G$ . If  $H$  and  $G/H$  are strongly topological gyrogroups, then so is  $G$ .*

*Proof.* To prove  $G$  is a topological gyrogroup, it is just to show that the inverse operation is continuous at  $0 \in G$  by Lemma 3.4.

Let the neighborhood base  $\mathcal{U}$  at 0 of  $G$  witness that  $G$  is a strongly paratopological gyrogroup. We shall prove that for each open neighborhood  $U \in \mathcal{U}$  at 0 in  $G$  there exists an open neighborhood  $V_2 \in \mathcal{U}$  such that  $\ominus V_2 \subseteq U$ . For each open neighborhood  $U \in \mathcal{U}$ , there exists an open neighborhood  $U_1 \in \mathcal{U}$  at 0 in  $G$  such that  $U_1 \oplus U_1 \subseteq U$ . For  $U_1$ , there exists an open neighborhood  $V_1 \in \mathcal{U}$  such that  $V_1 \subseteq U_1$ , and  $(\ominus V_1 \oplus V_1) \cap H \subseteq U_1 \cap H \subseteq U_1$ , for  $H$  is a topological gyrogroup. For  $V_1$ , there exists an open neighborhood  $V_2 \in \mathcal{U}$  such that  $V_2 \subseteq V_1$ , and  $\ominus V_2 \subseteq \ominus(V_2 \oplus H) = \pi^{-1}(\ominus\pi(V_2)) \subseteq \pi^{-1}(\pi(V_1)) = V_1 \oplus H$ , for  $\pi$  is an open mapping of  $G$  onto  $G/H$  and  $G/H$  is a topological gyrogroup. If  $x \in \ominus V_2$  then there exist elements  $v \in V_1, h \in H$  such that  $x = v \oplus h$ . Then  $h = \ominus v \oplus x \in (\ominus V_1 \oplus V_2) \cap H \subseteq U_1$ . Therefore  $x \in V_1 \oplus U_1 \subseteq U_1 \oplus U_1 \subseteq U$ . Thus we have proved that the inverse operation  $\ominus$  is continuous. This finishes the proof.  $\square$

**Proposition 3.7.** *Let  $G$  be a paratopological gyrocommutative gyrogroup and  $\mathcal{U}$  be the neighborhood base at 0 of  $G$ . If  $G$  is locally compact, then  $B = \bigcap \{\overline{U} : U \in \mathcal{U}\}$  is a closed invariant subgyrogroup of  $G$ .*

*Proof.* It is obvious that  $B$  is closed. Firstly, we shall prove that  $B$  is a subgyrogroup. By Proposition 2.15 we can get  $B = \bigcap \{\overline{U} \oplus \overline{U} : U \in \mathcal{U}\}$ . Clearly, for every  $U \in \mathcal{U}$ , we have that  $B \oplus B \subseteq \overline{U} \oplus \overline{U}$ . Hence it follows that  $B \oplus B = B$ . Since  $G$  is locally compact,  $B$  is compact. A nonempty subset  $M$  of  $B$  is called a right ideal in  $B$  if  $M \oplus B \subseteq M$ . Since  $B$  is compact, closed and  $B \oplus B = B$ , applying the Kuratowski-Zorn lemma to the

family of all closed right ideals in  $B$  ordered by inverse inclusion, it contains a minimal closed right ideal, denoted by  $H$ . For an arbitrary element  $x \in H$ , we have that  $x \oplus H \subseteq H \oplus B \subseteq H$ . Following Lemma 2.16, it is clear that  $(x \oplus H) \oplus B = x \oplus (H \oplus \bigcup_{h \in H} \text{gyr}[h, x]B) = x \oplus (H \oplus B) \subseteq x \oplus H$ , i.e.,  $x \oplus H$  is a right ideal in  $B$ . Since  $x \oplus H$  is closed in  $B$ ,  $x \oplus H \subseteq H$ , and  $H$  is a minimal right ideal in  $B$ , we conclude that  $x \oplus H = H$  for each  $x \in H$ . In particular,  $x \oplus (x \oplus H) = H$  for any  $x \in H$ , whence it follows that  $x \oplus (x \oplus y) = x$  for some  $y \in H$  and hence  $\ominus x = y \in H$ . In its turn, this implies that  $0 \in H$ ,  $H = B$ , and that  $B$  is a subgyrogroup of  $G$ .

Secondly, we shall prove that  $B$  is an invariant subgyrogroup.

**Claim 1.**  $a \oplus B = B \oplus a$  for each  $a \in G$ .

For  $a \in G$ , suppose  $f_1(x) = \ominus a \oplus (x \oplus a)$  for any  $x \in G$ . So,  $f_1 = L_{\ominus a} \circ R_a$  which is continuous by Proposition 2.13. Since  $f_1(0) = \ominus a \oplus (0 \oplus a) = 0$ , for  $U \in \mathcal{U}$ , there exists  $V \in \mathcal{U}$  such that  $f_1(\overline{V}) = \ominus a \oplus (\overline{V} \oplus a) \subseteq \overline{U}$ . It follows that  $\ominus a \oplus (B \oplus a) \subseteq B$ , for each  $a \in G$ , that is  $B \oplus a \subseteq a \oplus B$ .

On the other hand, for  $a \in G$ , suppose  $f_2(x) = \ominus a \oplus (a \oplus (a \oplus x) \oplus a)$  for any  $x \in G$ . So,  $f_2 = L_{\ominus a} \circ R_{\ominus a} \circ L_a \circ L_a$  which is continuous by Proposition 2.13. Since  $f_2(0) = \ominus a \oplus (a \oplus (a \oplus 0) \oplus a) = 0$ , for  $U_1 \in \mathcal{U}$ , there exists  $V_1 \in \mathcal{U}$  such that  $f_2(\overline{V_1}) = \ominus a \oplus (a \oplus (a \oplus \overline{V_1}) \oplus a) \subseteq \overline{U_1}$ . It follows that  $\ominus a \oplus (a \oplus (a \oplus B) \oplus a) \subseteq B$ , for each  $a \in G$ . Since  $G$  is a gyrocommutative gyrogroup, for each  $h \in B$  we have

$$\begin{aligned} (a \oplus h) \boxplus a &= (a \oplus h) \boxplus (\ominus a) \\ &= \ominus a \boxplus (a \oplus h) && \text{by Theorem 2.8} \\ &= \ominus a \oplus (a \oplus (a \oplus h) \oplus a). && \text{by Theorem 2.8} \end{aligned}$$

So we can get  $(a \oplus B) \boxplus a = \ominus a \oplus (a \oplus (a \oplus B) \oplus a) \subseteq B$ , which means  $a \oplus B \subseteq B \oplus a$ . In conclusion, we get  $a \oplus B = B \oplus a$ .

**Claim 2.**  $(a \oplus B) \oplus b = (a \oplus b) \oplus B$  for each  $a, b \in G$ .

For  $a, b \in G$ , suppose  $f_3(x) = \ominus(a \oplus b) \oplus ((a \oplus x) \oplus b)$  for any  $x \in G$ . So,  $f_3 = L_{\ominus(a \oplus b)} \circ R_b \circ L_a$  which is continuous by Proposition 2.13. Since  $f_3(0) = \ominus(a \oplus b) \oplus ((a \oplus 0) \oplus b) = 0$ , for  $U_2 \in \mathcal{U}$ , there exists  $V_2 \in \mathcal{U}$  such that  $f_3(\overline{V_2}) = \ominus(a \oplus b) \oplus ((a \oplus \overline{V_2}) \oplus b) \subseteq \overline{U_2}$ . It follows that  $\ominus(a \oplus b) \oplus ((a \oplus B) \oplus b) \subseteq B$ , for each  $a, b \in G$ , that is  $(a \oplus B) \oplus b \subseteq (a \oplus b) \oplus B$ .

Also, for  $a, b \in G$ , suppose  $f_4(x) = \ominus a \oplus (\ominus b \oplus ((b \oplus (a \oplus b \oplus x)) \oplus b))$  for any  $x \in G$ . So,  $f_4 = L_{\ominus a} \circ R_{\ominus b} \circ L_{\ominus b} \circ L_b \circ L_{a \oplus b}$  which is continuous by Proposition 2.13. Since  $G$  is a gyrocommutative gyrogroup, for each  $h \in B$  we have

$$\begin{aligned} ((a \oplus b) \oplus h) \boxplus b &= ((a \oplus b) \oplus h) \boxplus (\ominus b) \\ &= (\ominus b) \boxplus ((a \oplus b) \oplus h) && \text{by Theorem 2.8} \\ &= \ominus b \oplus ((b \oplus (a \oplus b \oplus h)) \oplus b). && \text{by Theorem 2.8} \end{aligned}$$

So  $f_4(0) = \ominus a \oplus (\ominus b \oplus ((b \oplus (a \oplus b \oplus 0)) \oplus b)) = \ominus a \oplus (((a \oplus b) \oplus 0) \boxplus b) = 0$ , for  $U_3 \in \mathcal{U}$ , there exists  $V_3 \in \mathcal{U}$  such that  $f_4(\overline{V_3}) = \ominus a \oplus (((a \oplus b) \oplus \overline{V_3}) \boxplus b) \subseteq \overline{U_3}$ . It follows that  $\ominus a \oplus (((a \oplus b) \oplus B) \boxplus b) \subseteq B$ , that is  $(a \oplus b) \oplus B \subseteq (a \oplus B) \oplus b$  for each  $a, b \in G$ . In conclusion, we get  $(a \oplus B) \oplus b = (a \oplus b) \oplus B$  for each  $a, b \in G$ .

**Claim 3.**  $a \oplus (B \oplus b) = (a \oplus b) \oplus B$  for each  $a, b \in G$ .

For  $a, b \in G$ , suppose  $f_5(x) = \ominus(a \oplus b) \oplus (a \oplus (b \oplus x)) = \text{gyr}[a, b](x)$  for any  $x \in G$ . By Lemma 2.16, we can get  $\text{gyr}[a, b](B) = B$ , that is  $\ominus(a \oplus b) \oplus (a \oplus (b \oplus B)) = B$ . Thus,  $\ominus(a \oplus b) \oplus (a \oplus (B \oplus b)) = B$ , by Claim 1. In conclusion, we have  $(a \oplus b) \oplus B = a \oplus (B \oplus b)$ .

From Claims 1, 2 and 3 it follows that  $B$  is an invariant subgyrogroup of  $G$  by Theorem 2.10. Hence we prove that  $B$  is a closed invariant subgyrogroup of  $G$ .  $\square$

**Theorem 3.8.** *If  $G$  is locally compact strongly paratopological gyrocommutative gyrogroup, then  $G$  is a strongly topological gyrocommutative gyrogroup.*

*Proof.* Let the neighborhood base  $\mathcal{U}$  at 0 of  $G$  witness that  $G$  is a strongly paratopological gyrogroup. Since  $B = \bigcap \{\overline{U} : U \in \mathcal{U}\}$  is a closed invariant subgyrogroup of  $G$  by Proposition 3.7, the quotient paratopological gyrogroup  $G/B$  is a  $T_1$ -space. Since  $B$  is compact, the quotient homomorphism  $\pi : G \rightarrow G/B$  is a closed mapping by Proposition 2.26. So we can get the space  $G/B$  is locally compact. We prove that  $G/B$  is

Hausdorff. Suppose for a contradiction that two distinct elements  $a, b \in G/B$  cannot be separated by open neighborhoods. Take  $x, y \in G$  with  $\pi(x) = a$  and  $\pi(y) = b$ . Then  $(x \oplus B) \cap (y \oplus B) = \emptyset$ . Since  $G$  is locally compact, it exists  $V \in \mathcal{U}$  such that  $\overline{V}$  is compact. By our assumption, for distinct elements  $a, b \in G/B$ , and any pair of open sets  $U_1, V_1 \subseteq G/B, a \in U_1, b \in V_1$ , we have  $U_1 \cap V_1 \neq \emptyset$ . So the family

$$\{(x \oplus \overline{U}) \cap (y \oplus \overline{U}) : U \in \mathcal{U}, U \subseteq V\}$$

of closed subsets of the compact space  $x \oplus \overline{V}$  has the finite intersection property, which in its turn implies that  $(x \oplus B) \cap (y \oplus B) \neq \emptyset$ . This contradiction proves that  $G/B$  is Hausdorff. Since  $G/B$  is a locally compact paratopological gyrogroup, it must be a topological gyrogroup by Theorem 3.5. According to Theorem 3.6,  $G$  is also a topological gyrogroup.  $\square$

**Problem 3.9.** Can the condition ‘gyrocommutative’ in Theorem 3.8 be omitted?

#### 4. Pseudocompact strongly paratopological gyrogroups

Theorem 3.5 will be extended to pseudocompact (and regular countably compact) paratopological gyrogroups in the following section. The following lemmas can be used to derive additional necessary conditions for a paratopological gyrogroup to be a topological gyrogroup.

**Lemma 4.1.** Suppose that  $G$  is a paratopological gyrogroup, and  $U$  is any open neighborhood of the neutral element  $0$  in  $G$ . Then  $\overline{M} \subseteq \ominus U \oplus M$  for each subset  $M$  of  $G$ .

*Proof.* If  $x \notin \ominus U \oplus M$ , that is  $(U \oplus x) \cap M = \emptyset$ , which implies there exists an open set  $U \oplus x$  containing  $x$  that has no intersection with  $M$ . So,  $x \notin \overline{M}$ .  $\square$

**Lemma 4.2.** Let the neighborhood base  $\mathcal{U}$  at  $0$  of  $G$  witness that  $G$  is a strongly paratopological gyrogroup and not a topological gyrogroup. Then there exists an open neighbourhood  $U$  of the neutral element  $0$  in  $\mathcal{U}$  such that  $U \cap (\ominus U)$  is nowhere dense in  $G$ , that is, the interior of the closure of  $U \cap (\ominus U)$  is empty.

*Proof.* Since  $(G, \tau, \oplus)$  is not a topological gyrogroup, the inverse operation in  $G$  is discontinuous. Therefore, it is discontinuous at  $0$  by Lemma 3.4, and we can choose an open neighbourhood  $W$  of  $0$  such that  $0 \notin \text{int}(\ominus W)$ . Since operator  $\oplus$  is continuous in  $G$ , we can find an open neighbourhood  $U$  of  $0$  such that  $U \oplus (U \oplus U) \subseteq W$ . That is  $(\ominus U \oplus U) \oplus U \subseteq \ominus W$  by Lemma 2.23. We claim that the set  $U \cap (\ominus U)$  is nowhere dense in  $G$ .

Assume the contrary. Then there exists a non-empty open set  $V$  in  $G$  such that  $V \subseteq \overline{U \cap (\ominus U)}$ . From Lemma 4.1 it follows that  $V \subseteq \overline{U \cap (\ominus U)} \subseteq \ominus U \oplus (U \cap (\ominus U)) \subseteq \ominus U \oplus U$ . Then  $V \oplus U \subseteq (\ominus U \oplus U) \oplus U \subseteq \ominus W$ . Clearly,  $V \cap U \neq \emptyset$ , and since  $V$  is open and the left translation is homeomorphism, the set  $V \oplus U = V \oplus (\ominus U)$  is open in  $G$ . Therefore,  $0 \in V \oplus U \subseteq \text{int}(\ominus W)$ , a contradiction.  $\square$

The next lemma easily follows from Lemma 4.2.

**Lemma 4.3.** Let the neighborhood base  $\mathcal{U}$  at  $0$  of  $G$  witness that  $G$  is a strongly paratopological gyrogroup such that  $0 \in \text{int}(\overline{\ominus U})$ , for each  $U \in \mathcal{U}$ , Then  $G$  is a strongly topological gyrogroup.

**Lemma 4.4.** Suppose that  $G$  is a  $T_1$ -paratopological gyrocommutative gyrogroup which is not a topological gyrogroup. Then, for each compact subset  $F$  of  $G$  such that  $0 \notin F$ , there exist an open neighborhood  $O(F)$  of  $F$  and an open neighborhood  $O(0)$  of  $0$  such that  $O(F) \cap (\ominus O(0)) = \emptyset$ .

*Proof.* For each  $x \in F$ , we select an open neighborhood  $V_x$  of  $0$  such that  $\ominus x \notin V_x \oplus V_x$ . Then  $(\ominus V_x \oplus x) \cap V_x = \emptyset$ . We can get  $V_x \oplus x = \ominus(\ominus V_x \oplus x)$  and  $\ominus(\ominus V_x \oplus x) \cap (\ominus V_x) = \emptyset$  by Theorem 2.8 (1). Since  $\gamma = \{V_x \oplus x : x \in F\}$  is a family of open sets in  $G$  covering the compact subspace  $F$ , there exists a finite subset  $K$  of  $F$  such that  $F \subseteq \bigcup\{V_x \oplus x : x \in K\}$ . Put  $O(0) = \bigcap\{V_x : x \in K\}$  and  $O(F) = \bigcup\{V \oplus x : x \in K\}$ . Then  $O(0)$  is an open neighborhood of  $0$ ,  $O(F)$  is an open neighborhood of  $F$ , and  $O(F) \cap (\ominus O(0)) = \emptyset$ .  $\square$

**Theorem 4.5.** *Suppose that  $f$  is a perfect homomorphism of a  $T_1$ -strongly paratopological gyrocommutative gyrogroup  $G$  onto a strongly topological gyrogroup  $H$ . Then  $G$  is also a strongly topological gyrocommutative gyrogroup.*

*Proof.* Assume that  $G$  is not a strongly topological gyrogroup. and let the neighborhood base  $\mathcal{U}$  at 0 of  $G$  witness that  $G$  is a strongly paratopological gyrogroup. Then, according to Lemma 4.3, there exists an open neighborhood  $U \in \mathcal{U}$  such that 0 is not in  $\text{int}(\ominus U)$ . Put  $F = f^{-1}f(0)$  and  $F_1 = F \setminus U$ . Since  $F_1$  is compact and 0 is not in  $F_1$ , Lemma 4.4 implies that there exist an open neighborhood  $O(F_1)$  of  $F_1$  and an open neighborhood  $O(0)$  of 0 such that  $O(F_1) \cap (\ominus O(0)) = \emptyset$ .

Since  $W = O(F_1) \cup U$  is an open neighborhood of  $F$  and the mapping  $f$  is closed, there exists an open neighborhood  $V$  of  $f(0)$  in  $H$  such that  $f^{-1}(V) \subseteq W$ . We can also assume that  $\ominus V = V$ , since  $H$  is a topological gyrogroup. Then  $\ominus f^{-1}(V) = f^{-1}(\ominus V) = f^{-1}(V) \subseteq W$ . Finally, put  $W_0 = f^{-1}(V) \cap O(0) \cap U$ . Clearly,  $W_0$  is an open neighborhood of 0 contained in  $U$ . We also have  $\ominus W_0 \subseteq \ominus f^{-1}(V) \subseteq W$  and  $\ominus W_0 \subseteq \ominus O(0)$ . Since  $O(F_1) \cap (\ominus O(0)) = \emptyset$ , it follows that  $\ominus W_0 \subseteq U$ . Therefore,  $0 \in W_0 \subseteq \text{int}(\ominus U)$ , a contradiction.  $\square$

Here, we demonstrate that every pseudocompact paratopological gyrogroup is a topological gyrogroup. It is well known that a Tychonoff space  $X$  is pseudocompact if and only if every locally finite family of open sets in  $X$  is finite. To present results in a broad sense, we recall that a topological space  $X$  is called *feebly compact* if every locally finite family of open sets in  $X$  is finite. Therefore, ‘feebly compact’ is equivalent to ‘pseudocompact’ for Tychonoff spaces. This result is improved on A. V. Arhangel’skii and E. A. Reznichenko’s results.

**Theorem 4.6.** *If a strongly paratopological gyrogroup  $G$  is a dense  $G_\delta$ -set in a regular feebly compact space  $X$ , then  $G$  is a strongly topological gyrogroup.*

*Proof.* Let the open neighborhood base  $\mathcal{U}$  at 0 of  $G$  witness that  $G$  is a strongly paratopological gyrogroup. Assume the contrary. Then, by Lemma 4.2, there exists an open neighbourhood  $U$  of the neutral element 0 in  $\mathcal{U}$  such that  $\overline{U \cap (\ominus U)}$  is nowhere dense. Let  $W$  be an open neighborhood of 0 such that  $\overline{W \oplus W} \subseteq U$ . Put  $O = W \setminus \overline{U \cap (\ominus U)}$ . Then, clearly,  $O \subseteq W \subseteq \overline{O}$  and  $\ominus O \cap U = \emptyset$ . First, we fix a sequence  $\{M_n : n \in \omega\}$  of open sets in  $X$  such that  $G = \bigcap_{n=0}^\infty M_n$ . We are going to define a sequence  $\{U_n : n \in \omega\}$  of open subsets of  $X$  and a sequence  $\{x_n : n \in \omega\}$  of elements of  $G$  such that  $x_n \in U_n$ , for each  $n \in \omega$ . Put  $U_0 = O$ , and pick a point  $x_0 \in U_0 \cap G$ .

Assume now that, for some  $n \in \omega$ , an open subset  $U_n$  of  $X$  and a point  $x_n \in U_n \cap G$  are already defined. Since  $0 \in W \subseteq \overline{O}$ , we have  $x_n \in x_n \oplus \overline{O} = x_n \oplus \overline{O}$ . Since  $U_n$  is an open neighbourhood of  $x_n$ , it follows that  $U_n \cap x_n \oplus O \neq \emptyset$ . We take  $x_{n+1}$  to be any point of  $U_n \cap x_n \oplus O$ . Note that  $x_{n+1} \in G$ , since  $x_n \oplus O \subseteq G$ . Using the regularity of  $X$ , we can find an open neighbourhood  $U_{n+1}$  of  $x_{n+1}$  in  $X$  such that the closure of  $U_{n+1}$  is contained in  $U_n \cap M_n$ , and  $U_{n+1} \cap G \subseteq x_n \oplus O$ . The definition of the sets  $U_n$  and points  $x_n$ , for each  $n \in \omega$ , is complete. Note that  $\overline{U_i} \subseteq U_j$  whenever  $j < i$ . We also have  $x_{n+1} \in x_n \oplus O$ , for each  $n \in \omega$ . Put  $F = \bigcap_{n \in \omega} \overline{U_n}$ . Clearly,  $F \subseteq G$ , and  $F \neq \emptyset$  since  $X$  is feebly compact. The set  $F \oplus W$  is an open neighbourhood of  $F$  in  $G$ . Consider the closure  $P$  of  $F \oplus W$  in  $X$ , and let  $H$  be the closure of  $X \setminus P$  in  $X$ . Then  $H$  is a regular closed subset of  $X$ , so that  $H$  is feebly compact.

We claim that  $H \cap F = \emptyset$ . Indeed, assume the contrary, and fix  $x \in H \cap F$ . Since  $F \oplus W$  is an open neighbourhood of  $F$  in  $G$ , from  $x \in F$  it follows that there exists an open neighbourhood  $V$  of  $x$  in  $X$  such that  $V \cap G \subseteq F \oplus W$ . Then the density of  $G$  in  $X$  implies that  $V \subseteq P$ , while  $x \in V \cap H$  implies that  $V \setminus P \neq \emptyset$ , which is a contradiction. Thus,  $H \cap F = \emptyset$ .

Since  $H$  is feebly compact, our definition of  $F$  implies that  $U_k \cap H = \emptyset$ , for some  $k \in \omega$  (we use that  $\overline{U_i} \subseteq U_j$  whenever  $j < i$ ). Then  $U_k \subseteq P$ . Since  $x_k \in U_k \cap G$ , it follows that  $x_k \in \overline{F \oplus W}$ . However,  $F \subseteq U_{k+2} \cap G \subseteq x_{k+1} \oplus O \subseteq x_{k+1} \oplus W$ . Hence,  $x_k \in \overline{F \oplus W} \subseteq \overline{(x_{k+1} \oplus W) \oplus W} = x_{k+1} \oplus \overline{(W \oplus W)} \subseteq x_{k+1} \oplus U$ , by the definition of  $\mathcal{U}$ . Taking into account that  $x_{k+1} \in x_k \oplus O$ , we obtain that  $x_k \in (x_k \oplus O) \oplus U = x_k \oplus (O \oplus U)$ . Hence,  $0 \in O \oplus U$  and  $\ominus O \cap U \neq \emptyset$ , which is again a contradiction. We prove that  $G$  is a strongly topological gyrogroup.  $\square$

Naturally, Tychonoff spaces are mentioned in the following two corollaries of Theorem 4.6.

**Corollary 4.7.** Every pseudocompact strongly paratopological gyrogroup is a strongly topological gyrogroup.

**Corollary 4.8.** Every Čech-complete strongly paratopological gyrogroup is a strongly topological gyrogroup.

Because countably compact spaces are feebly compact, Theorem 4.6 implies the following fact.

**Corollary 4.9.** Every regular countably compact strongly paratopological gyrogroup is a strongly topological gyrogroup.

In [26], the authors gave an explicit description of  $m \cdot a$  for a gyrogroup  $G$ , an element  $a \in G$  and  $m \in \mathbb{Z}$  as following:

$$0 \cdot a = 0, m \cdot a = a \oplus ((m - 1) \cdot a), m \geq 1, m \cdot a = (-m) \cdot (\ominus a), m < 0,$$

$$a \cdot 0 = 0, a \cdot m = (a \cdot (m - 1))a \oplus a, m \geq 1, a \cdot m = (\ominus a) \cdot (-m), m < 0,$$

and  $a \cdot m = m \cdot a$  for all  $m \in \mathbb{Z}$ .

**Theorem 4.10.** ([26]) Let  $a$  be an element of a gyrogroup. For all  $m, k \in \mathbb{Z}$ ,  $(m \cdot a) \oplus (k \cdot a) = (m + k) \cdot a$ .

**Definition 4.11.** A paratopological gyrogroup  $G$  is called topologically periodic if for each  $x \in G$  and every neighborhood  $U$  of the identity there exists an integer  $n$  such that  $n \cdot x \in U$ .

**Theorem 4.12.** If a strongly paratopological gyrogroup  $G$  is Hausdorff countably compact and topologically periodic, then  $G$  is a strongly topological gyrogroup.

*Proof.* Let the neighborhood base  $\mathcal{U}$  at 0 of  $G$  witness that  $G$  is a strongly paratopological gyrogroup,  $U \in \mathcal{U}$  and  $\{V_i | i \in \omega\}$  a family of neighborhoods of the identity 0 such that  $V_i \in \mathcal{U}$ ,  $V_0 = U$  and  $V_{i+1} \oplus V_{i+1} \subseteq V_i$  for each  $i \in \omega$ . By lemma 2.24 we have  $\ominus(\overline{\ominus V_{i+1}}) \subseteq V_i$

We show that  $F = \cap\{\overline{\ominus V_i} | i \in \omega\} \subseteq U$ . Let  $x \in F$ ,  $x \in \overline{\ominus V_i}$  for each  $i \in \omega$ . Then  $\ominus x \in \overline{\ominus V_{i+1}}$  for each  $i \in \omega$ . Choose  $n \in \omega$  such that  $n \cdot x \in V_1$ , then  $(n - 1) \cdot (\ominus x) \in \underbrace{(\ominus(\overline{\ominus V_{i+1}})) \oplus \dots \oplus (\ominus(\overline{\ominus V_{i+1}}))}_{n-1} \subseteq \underbrace{V_i \oplus \dots \oplus V_i}_{n-1}$  for each

$i \in \omega$ . Now choose  $i_0 \in \omega$  such that  $\underbrace{V_{i_0} \oplus \dots \oplus V_{i_0}}_{n-1} \subseteq V_1$ . Then  $(n - 1) \cdot (\ominus x) \in V_1$ . By Theorem 4.10 we can

get  $(n \cdot x) \oplus ((n - 1) \cdot (\ominus x)) = ((-n) \cdot (\ominus x)) \oplus \underbrace{((n - 1) \cdot (\ominus x))}_{n-1} = ((-n) + (n - 1)) \cdot (\ominus x) = (-1) \cdot (\ominus x) = x$ . Then  $x = (n \cdot x) \oplus ((n - 1) \cdot (\ominus x)) \in V_1 \oplus V_1 \subseteq U$ . Therefore  $F \subseteq U$ . Since  $G$  is a countably compact gyrogroup, there exist  $i_1, \dots, i_k$  such that  $\cap\{\overline{\ominus V_{i_j}} | j = 1, \dots, k\} \subseteq U$ . Thus we have proved that the inverse operation  $\ominus$  is continuous at 0. Hence  $G$  is a strongly topological gyrogroup by Lemma 3.4.  $\square$

### Acknowledgments

We would like to express our gratitude to the referees for providing us with a comprehensive list of comments and suggestions for the paper, as well as for their dedicated efforts to enhance its quality.

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