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Fractional Hadamard-type inequalities for refined $(\alpha, h-m)-p$-convex functions and their consequences

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Abstract. In this paper, a new class of convex functions namely refined $(\alpha, h-m)-p$-convex functions is introduced. By utilizing this new definition, several refinements of different kinds of convex functions are derived. Further, Riemann-Liouville fractional integral operator is utilized to establish Hadamard-type inequalities for this class of convex functions. Furthermore, their connection with already known fractional Hadamard inequalities is established.

1. Introduction

Study of real analysis is nothing without a function. The properties of a function like continuity and differentiability have increased its importance in the subjects of mathematical analysis, economics and differential equations etc. A convex function defined a century ago has fascinating geometric and analytical properties due to which it is studied frequently in the subjects of analysis and optimization. Its extended form is the well known Jensen inequality which in particular generates several classical inequalities. In the development of theory of mathematical inequalities its role is significant. Description of convex function in different convenient forms motivates the researchers to extend its concept in new directions. Especially, to extend, generalize and refine the classical inequalities for convex functions, it can be found in literature that researchers have defined different new types of convex functions.

Nowadays, fractional integral inequalities are extensively studied by using various kinds of integral operators via different types of convex functions see, for example, [1, 7, 9, 11, 13, 14, 16, 19, 27, 28, 31, 33, 34] and references therein.

The aim of this paper is to study the refinements of well known Hadamard-type inequalities by introducing a new class of convex functions namely refined $(\alpha, h-m)-p$-convex functions and applying with Riemann-Liouville fractional integral operators. Next, we give preliminary definitions.

Definition 1.1. *[24] A function $\Delta : [a, b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is called convex if for all $u, v \in [a, b]$ and $t \in [0, 1]$, we have*

$$\Delta(tu + (1-t)v) \leq t\Delta(u) + (1-t)\Delta(v). \tag{1}$$

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If the inequality (1) reverses, then Δ is called concave function.

Definition 1.2. *[32] A function $\Delta : [0, b] \to \mathbb{R}$ is called m-convex if for all $u, v \in [0, b], t \in [0, 1]$ and $m \in [0, 1]$, we have*

$$\Delta(tu + m(1 - t)v) \leq t\Delta(u) + m(1 - t)\Delta(v). \tag{2}$$

Definition 1.3. *[18] A function $\Delta : [0, b] \to \mathbb{R}$ is called (α, m)-convex if for all $u, v \in [0, b]$, $(\alpha, m) \in [0, 1]^2$ and $t \in [0, 1]$, we have*

$$\Delta(tu + m(1 - t)v) \leq t^\alpha\Delta(u) + m(1 - t^\alpha)\Delta(v). \tag{3}$$

Definition 1.4. *[9] Let $h : J \to \mathbb{R}$ is a function with $h \not\equiv 0$ and $(0, 1) \subseteq J$. A function Δ is said to be $(h - m)$-convex, if $\Delta, h \geq 0$ and for each $u, v \in [0, b] \subseteq \mathbb{R}$, we have*

$$\Delta(tu + m(1 - t)v) \leq h(t)\Delta(u) + mh(1 - t)\Delta(v), \tag{4}$$

where $m \in [0, 1]$ and $t \in (0, 1)$.

Definition 1.5. *[9] Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. A function $\Delta : [0, b] \to \mathbb{R}$ is said to be $(\alpha, h - m)$-convex, if Δ is non-negative and for all $u, v \in [0, b]$, $(\alpha, m) \in [0, 1]^2$, $t \in (0, 1)$ we have*

$$\Delta(tu + m(1 - t)v) \leq h(t^\alpha)\Delta(u) + mh(1 - t^\alpha)\Delta(v). \tag{5}$$

Definition 1.6. *[15] Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. Let $I \subset (0, \infty)$ be an interval and $p \in \mathbb{R} \setminus \{0\}$. A function $\Delta : I \to \mathbb{R}$ is said to be $(\alpha, h - m) - p$-convex, if*

$$\Delta((tu^p + m(1 - t)v^p))^{\frac{1}{p}}) \leq h(t^\alpha)\Delta(u) + mh(1 - t^\alpha)\Delta(v), \tag{6}$$

holds provided $(tu^p + m(1 - t)v^p))^{\frac{1}{p}} \in I$ for $t \in [0, 1]$ and $(\alpha, m) \in [0, 1]^2$.

Fractional integrals are extensively used in the theory of inequalities. In this paper, we establish some inequalities of Hadamard type using Riemann-Liouville fractional integrals with $(\alpha, h - m) - p$-convex function.
Now a days many fractional integrals of different kinds are being used to get new results related to the inequalities. For references one can see [3, 4, 7, 9, 10, 17, 19, 21, 26, 28].

Definition 1.7. *[12] Let $\Delta \in L_1[a, b]$. Then left-sided and right-sided Riemann-Liouville fractional integrals of a function Δ of order η where $\Re(\eta) > 0$ are defined as follows:*

$$I^\eta_{a^+}\Delta(x) = \frac{1}{\Gamma(\eta)}\int_a^x (x - t)^{\eta-1}\Delta(t)dt, \quad x > a, \tag{7}$$

$$I^\eta_{b^-}\Delta(x) = \frac{1}{\Gamma(\eta)}\int_x^b (t - x)^{\eta-1}\Delta(t)dt, \quad x < b, \tag{8}$$

where

$$\Gamma(\eta) = \int_0^\infty t^{\eta-1}e^{-t}dt.$$

k-analogue of above definition is given as follows:

Definition 1.8. *[20] Let $\Delta \in L_1[a,b]$. Then left-sided and right-sided Riemann-Liouville k-fractional integrals of a function Δ of order η where $\Re(\eta), k > 0$ are defined as follows:*

$$ {}_kI^{\eta}_{a^+}\Delta(x) = \frac{1}{k\Gamma_k(\eta)} \int_a^x (x-t)^{\frac{\eta}{k}-1}\Delta(t)dt, \quad x > a, \tag{9} $$

$$ {}_kI^{\eta}_{b^-}\Delta(x) = \frac{1}{k\Gamma_k(\eta)} \int_x^b (t-x)^{\frac{\eta}{k}-1}\Delta(t)dt, \quad x < b, \tag{10} $$

where

$$ \Gamma_k(\eta) = \int_0^\infty t^{\eta-1}e^{-\frac{t^k}{k}}dt. $$

The Hadamard inequality has a strong connection with the convexity. It interprets the bounds of the integral mean of the convex functions. Many researches took the motivation and established new inequalities using this fact [25, 29, 30]. The well-known Hadamrd inequality is given as follows:
Let Δ be a convex function on $[a,b]$ with $a < b$. Then the following inequality holds:

$$ \Delta\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \Delta(x)dx \le \frac{\Delta(a)+\Delta(b)}{2}. $$

In this article, a new class of convex functions namely refined $(\alpha, h-m)-p$-convex functions is introduced. By utilizing this new definition, several refinements of different kinds of convex functions are derived. Also, we establish Hadamard-type inequalities for Riemann-Louville fractional integral operator using this new class of convex functions along with refinements of some already known inequalities.
The rest of the paper is organized in the following manner. In Section 2, we introduce refined $(\alpha, h-m)-p$-convex function and prove some fractional integral inequalities of Hadamard-type for refined $(\alpha, h-m)-p$-convex functions via (7) and (8). Also, we have given refinements of few fractional versions of Hadamard inequalities proved in [8, 23, 31]. In Section 3, k-fractional versions of Hadamard inequalities for refined $(\alpha, h-m)-p$-convex functions are given.

2. Main Results

We define a new class of convex functions and prove some inequalities for this type of functions using Riemann-Liouville integral operators.

Definition 2.1. *Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. Let $I \subset (0,\infty)$ be an interval and $p \in \mathbb{R}\setminus\{0\}$. A function $\Delta : I \to \mathbb{R}$ is said to be refined $(\alpha, h-m)-p$-convex function, if*

$$ \Delta((tu^p + m(1-t)v^p))^{\frac{1}{p}}) \le h(t^\alpha)h(1-t^\alpha)\left(\Delta(u) + m\Delta(v)\right), \tag{11} $$

holds and $(tu^p + m(1-t)v^p))^{\frac{1}{p}} \in I$ for $t \in [0,1]$ and $(\alpha, m) \in [0,1]^2$. Reverse of inequality (11) defines Δ as refined $(\alpha, h-m)-p$-concave function.

By comparing inequalities (6) and (11), one can find that if $0 < h(t) < 1$, then (11) gives the refinement of $(\alpha, h-m)-p$-convex function in the form of following inequality:

$$ \begin{aligned} &\Delta((tu^p + m(1-t)v^p)^{\frac{1}{p}}) \le h(t^\alpha)h(1-t^\alpha)\left(\Delta(u) + m\Delta(v)\right) \\ &\le h(t^\alpha)\Delta(u) + mh(1-t^\alpha)\Delta(v). \end{aligned} \tag{12} $$

Likewise, inequality (12) also gives the refinements of different classes of convex function defined in [1, 4–6, 9, 15, 18, 22, 24, 32, 34].
Next, we discuss some special cases of Definition 2.1.

Case 2.2. *For $h(t) = t$ in (11), we get definition for refined $(\alpha, m) - p$-convex function stated as follows:*

Definition 2.3. *Let $I \subset (0, \infty)$ be an interval and $p \in \mathbb{R} \setminus \{0\}$. A function $\Delta : I \to \mathbb{R}$ is said to be refined $(\alpha, m) - p$-convex, if*

$$\Delta((tu^p + m(1-t)v^p)^{\frac{1}{p}}) \leq t^\alpha(1-t^\alpha)(\Delta(u) + m\Delta(v)) \tag{13}$$

holds provided $(tu^p + m(1-t)v^p))^{\frac{1}{p}} \in I$ for $t \in [0,1]$ and $(\alpha, m) \in [0,1]^2$.

Case 2.4. *For $\alpha = 1$ in (11), we get definition of refined $(h - m) - p$-convex function stated as follows:*

Definition 2.5. *Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. Let $I \subset (0, \infty)$ be an interval and $p \in \mathbb{R} \setminus \{0\}$. A function $\Delta : I \to \mathbb{R}$ is said to be refined $(h - m) - p$-convex, if*

$$\Delta((tu^p + m(1-t)v^p))^{\frac{1}{p}}) \leq h(t)h(1-t)(\Delta(u) + m\Delta(v)), \tag{14}$$

holds provided $(tu^p + m(1-t)v^p))^{\frac{1}{p}} \in I$ for $t \in [0,1]$ and $m \in [0,1]$.

Case 2.6. *For $m = 1$ in (11), we get the definition of refined $(\alpha, h) - p$-convex function stated as follows:*

Definition 2.7. *Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. Let $I \subset (0, \infty)$ be an interval and $p \in \mathbb{R} \setminus \{0\}$. A function $\Delta : I \to \mathbb{R}$ is said to be refined $(\alpha, h) - p$-convex, if*

$$\Delta((tu^p + (1-t)v^p))^{\frac{1}{p}}) \leq h(t^\alpha)h(1-t^\alpha)(\Delta(u) + \Delta(v)), \tag{15}$$

holds provided $(tu^p + m(1-t)v^p))^{\frac{1}{p}} \in I$ for $t \in [0,1]$ and $\alpha \in [0,1]$.

Case 2.8. *For $\alpha = 1$ and $h(t) = t^s$ in (11), we get definition of refined $(s, m) - p$-convex function stated as follows:*

Definition 2.9. *Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. Let $I \subset (0, \infty)$ be an interval and $p \in \mathbb{R} \setminus \{0\}$. A function $\Delta : I \to \mathbb{R}$ is said to be refined $(s, m) - p$-convex, if*

$$\Delta((tu^p + m(1-t)v^p))^{\frac{1}{p}}) \leq t^s(1-t^s)(\Delta(u) + m\Delta(v)), \tag{16}$$

holds provided $(tu^p + m(1-t)v^p))^{\frac{1}{p}} \in I$ for $t \in [0,1]$ and $m \in [0,1]$.

Case 2.10. *For $\alpha = 1$ and $h(t) = t^{-s}$ in (11), we get definition of refined $(s, m) - p-$Godunova-Levin function stated as follows:*

Definition 2.11. *Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. Let $I \subset (0, \infty)$ be an interval and $p \in \mathbb{R} \setminus \{0\}$. A function $\Delta : I \to \mathbb{R}$ is said to be refined $(s, m) - p-$Godunova-Levin convex, if*

$$\Delta((tu^p + m(1-t)v^p))^{\frac{1}{p}}) \leq \frac{1}{t^s(1-t^s)}(\Delta(u) + m\Delta(v)), \tag{17}$$

holds provided $(tu^p + m(1-t)v^p))^{\frac{1}{p}} \in I$ for $t \in [0,1]$ and $m \in [0,1]$.

Case 2.12. *For $p = -1$ in (11), we get definition of refined $(\alpha, h - m) - HA-$ convex function stated as follows:*

Definition 2.13. *Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. Let $I \subset (0, \infty)$ be an interval and $p \in \mathbb{R} \setminus \{0\}$. A function $\Delta : I \to \mathbb{R}$ is said to be refined $(\alpha, h - m) - HA-$ convex, if*

$$\Delta\left(\frac{uv}{u(1-t)m + vt}\right) \leq h(t^\alpha)h(1-t^\alpha)(\Delta(u) + m\Delta(v)), \tag{18}$$

holds.

Remark 2.14. *(i) If $\alpha = 1$, $h(t) = t^s$ and $m = 1$, $p = 1$ then Definition 2.1 reduces to the definition of s-tgs-convex function stated in [2, Definition 3].*
(ii) If $\alpha = 1$, $m = 1$ and $h(t) = t^{-s}$, $p = 1$ then Definition 2.1 reduces to the definition of Godunova-Levin-Dragomir tgs-convex function stated in [2, Definition 4].
(iii) If $\alpha = 1$, $m = 1$ and $h(t) = t$, $p = 1$ then Definition 2.1 reduces to the definition of tgs-convex function stated in [31, Definition 2.1].
(iv) If $\alpha = 1$, $m = 1$ and $h(t) = 1$, $p = 1$ then Definition 2.1 reduces to the definition of p-function stated in [6].
(v) If $\alpha = 1 = m$, and $p = 1$ then Definition 2.1 reduces to the definition of refined h–convex function stated in [23, Definition 7].

Theorem 2.15. *Let $\Delta : [a, mb] \to \mathbb{R}$ be a positive, refined $(\alpha, h - m) - p$-convex function with $0 \le a < mb$ and $\Delta \in L_1[a, mb]$. Then, for $(\alpha, m) \in (0, 1]^2$ and $p \in \mathbb{R} - \{0\}$, the following fractional integral inequality holds:*

$$\frac{\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right)}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)} \le \frac{\Gamma(\eta+1)}{(mb^p-a^p)^\eta}\left[I^\eta_{a^{p+}}\Delta(mb^p)+m^{\eta+1}I^\eta_{(mb^p)^-}\Delta\left(\frac{a^p}{m}\right)\right] \tag{19}$$
$$\le \eta\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right]\int_0^1 h(t^\alpha)h(1-t^\alpha)t^{\eta-1}dt,$$

with $\eta > 0$.

Proof. The following inequality holds for refined $(\alpha, h - m) - p$-convex function

$$\Delta\left(\left(\frac{u^p+mv^p}{2}\right)^{\frac{1}{p}}\right) \le h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)[\Delta(u)+m\Delta(v)]. \tag{20}$$

Setting $u = (ta^p + m(1-t)b^p)^{\frac{1}{p}}$, $v = \left(\frac{a^p}{m}(1-t)+b^pt\right)^{\frac{1}{p}}$ in (20), we get the following inequality

$$\frac{1}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \le \Delta\left((ta^p+m(1-t)b^p)^{\frac{1}{p}}\right)$$
$$+m\Delta\left(\left(\frac{a^p}{m}(1-t)+b^pt\right)^{\frac{1}{p}}\right).$$

Using Definition (2.1) and integrating the resulting inequality over the interval $[0, 1]$ after multiplying with $t^{\eta-1}$, we get

$$\frac{\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right)}{\eta h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)} \le \int_0^1 \Delta((ta^p+m(1-t)b^p)^{\frac{1}{p}})t^{\eta-1}dt \tag{21}$$
$$+m\int_0^1 \Delta\left(\left(\frac{a^p}{m}(1-t)+b^pt\right)^{\frac{1}{p}}\right)t^{\eta-1}dt$$
$$\le \left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right]\int_0^1 h(t^\alpha)h(1-t^\alpha)t^{\eta-1}dt.$$

Changing the variable, applying Definition 1.7 and multiplying by η, we get inequality (19). □

The extension of inequality (19) is given in the following result.

Theorem 2.16. *Under the assumptions of Theorem 2.15, further if $h(t) \le \frac{1}{\sqrt{2}}$, then the following inequality holds:*

$$
\begin{aligned}
&2\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \le \frac{1}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \\
&\le \frac{\Gamma(\eta+1)}{(mb^p-a^p)^\eta}\left[I^\eta_{a^{p+}}\Delta(mb^p)+m^{\eta+1}I^\eta_{(mb^p)^-}\Delta\left(\frac{a^p}{m}\right)\right] \\
&\le \eta\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right]\int_0^1 h(t^\alpha)h(1-t^\alpha)t^{\eta-1}dt \\
&\le \frac{1}{2}\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right].
\end{aligned} \tag{22}
$$

Proof. It is given $h(t) \le \frac{1}{\sqrt{2}}$. Therefore, we have

$$\int_0^1 h(t^\alpha)h(1-t^\alpha)t^{\eta-1}dt \le \frac{1}{2\eta}$$

and

$$\frac{1}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \ge 2\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right).$$

By using these inequalities along with (19), we obtain (22) as required. □

Remark 2.17. *(i) If $m = 1$, $\alpha = 1$, $p = 1$ $h(t) = t$ in (19), then we get [31, Theorem 3.1].*
(ii) If $\eta = 1$, $\alpha = 1$, $p = 1$ $m = 1$, $h(t) = t$ in (19), then we get [31, Theorem 2.1].
(iii) If $\alpha = 1$, $m = 1$, $p = 1$ in (19), then we get [23, Theorem 8].
(iv) If $\alpha = 1, m = 1, p = 1, \eta = 1$ in (19), then we get [23, Theorem 2].
(v) If $p = 1$ in (19), then we get [8, Theorem 1].
(vi) If $p = 1$ in (22), then we get [8, Theorem 2].

Corollary 2.18. *By using $h(t) = t$ in (19), the following fractional integral inequality for refined $(\alpha, m) - p$-convex function is obtained.*

$$
\begin{aligned}
&\frac{2^{2\alpha}}{(2^\alpha-1)}f\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \le \frac{\Gamma(\eta+1)}{(mb^p-a^p)^\eta}\left[I^\eta_{a^{p+}}\Delta(mb^p)+m^{\eta+1}I^\eta_{(mb^p)^-}\Delta\left(\frac{a^p}{m}\right)\right] \\
&\le \frac{\alpha\eta}{(\eta+\alpha)(\eta+2\alpha)}\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right].
\end{aligned} \tag{23}
$$

Remark 2.19. *(i)If $m = 1$ in (23), then the result for refined $(\alpha - p)$-convex function can be obtained.*
(ii)If $m = 1, p = 1$ in (23), then the result for refined α-convex function can be obtained.

Corollary 2.20. *By using $\alpha = 1$ in (19), the following fractional integral inequality for refined $(h - m) - p$-convex function is obtained.*

$$
\begin{aligned}
&\frac{1}{h^2\left(\frac{1}{2}\right)}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \le \frac{\Gamma(\eta+1)}{(mb^p-a^p)^\eta}\left[I^\eta_{a^{p+}}\Delta(mb^p)+m^{\eta+1}I^\eta_{(mb^p)^-}\Delta\left(\frac{a^p}{m}\right)\right] \\
&\le \eta\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right]\int_0^1 h(t)h(1-t)t^{\eta-1}dt.
\end{aligned} \tag{24}
$$

Corollary 2.21. *By using $\alpha = 1$ in (22), the following fractional integral inequality for refined $(h-m)-p$-convex function is obtained.*

$$\begin{aligned} &2\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{1}{h^2\left(\frac{1}{2}\right)}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \\ &\leq \frac{\Gamma(\eta+1)}{(mb^p-a^p)^\eta}\left[I^\eta_{a^{p+}}\Delta(mb^p)+m^{\eta+1}I^\eta_{(mb^p)^-}\Delta\left(\frac{a^p}{m}\right)\right] \\ &\leq \eta\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right]\int_0^1 h(t)h(1-t)t^{\eta-1}dt \\ &\leq \frac{1}{2}\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right]. \end{aligned} \tag{25}$$

Remark 2.22. *(i) If $\eta, m = 1$ in (25), then we get [8, Corollary 3].*
(ii) If $\eta, m = 1, p = 1$ in (25), then we get [23, Corollary 1].

Corollary 2.23. *By using $\alpha = 1$ and $h(t) = t^s$ in (19), the following fractional integral inequality for refined $(s,m)-p$-convex function is obtained.*

$$\begin{aligned} &2^{2s}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{\Gamma(\eta+1)}{(mb^p-a^p)^\eta}\left[I^\eta_{a^{p+}}\Delta(mb^p)+m^{\eta+1}I^\eta_{(mb^p)^-}\Delta\left(\frac{a^p}{m}\right)\right] \\ &\leq \eta\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right]\mathbb{B}(1+s,s+\eta). \end{aligned} \tag{26}$$

Remark 2.24. *(i) If $p = 1$ in (26), then [8, corollary 4] is obtained.*
(ii) If $m = 1 = p$ in (26), then the result for s-tgs convex function can be obtained.

Theorem 2.25. *Under the assumption of Theorem 2.15, the following fractional integral inequality holds:*

$$\begin{aligned} &\frac{1}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \\ &\leq \frac{2^\eta\Gamma(\eta+1)}{(mb^p-a^p)^\eta}\left[I^\eta_{\left(\frac{a^p+mb^p}{2}\right)^+}\Delta(mb^p)+m^{\eta+1}I^\eta_{\left(\frac{a^p+mb^p}{2m}\right)^-}\Delta\left(\frac{a^p}{m}\right)\right] \\ &\leq \eta\left[\Delta(a)+2m\Delta(b)+\Delta\left(\frac{a}{m^2}\right)\right]\int_0^1 h\left(\left(\frac{t}{2}\right)^\alpha\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{\eta-1}dt, \end{aligned} \tag{27}$$

with $\eta > 0$.

Proof. Let $u = \left(\frac{ta^p}{2}+m(\frac{2-t}{2})b^p\right)^{\frac{1}{p}}$, $v = \left(\frac{a^p}{m}(\frac{2-t}{2})+\frac{tb^p}{2}\right)^{\frac{1}{p}}$ in (20), we get the following inequality

$$\begin{aligned} &\frac{1}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \leq \Delta\left(\left(\frac{ta^p}{2}+m\left(\frac{2-t}{2}\right)b^p\right)^{\frac{1}{p}}\right) \\ &+m\Delta\left(\left(\frac{a^p}{m}\left(\frac{2-t}{2}\right)+\frac{tb^p}{2}\right)^{\frac{1}{p}}\right). \end{aligned}$$

Using Definition (2.1) and integrating the resulting inequality over $[0,1]$ after multiplying with $t^{\eta-1}$, we get

$$\begin{aligned}
&\frac{\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right)}{\eta h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)} \quad (28)\\
&\leq \int_0^1 \Delta\left(\frac{ta^p}{2}+m\left(\frac{2-t}{2}\right)b^p\right)t^{\eta-1}dt+\int_0^1 \Delta\left(\frac{a^p}{m}\left(\frac{2-t}{2}\right)+\frac{tb^p}{2}\right)t^{\eta-1}dt\\
&\leq \left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right]\int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{\eta-1}dt.
\end{aligned}$$

Making change of variable and using Definition 1.7, we get inequality (27). □

The extension of inequality (27) is given in the following result.

Theorem 2.26. *Along with the assumptions of Theorem 2.25, further if $h(t^\alpha) \leq \frac{1}{\sqrt{2}}$, then the following inequality holds:*

$$\begin{aligned}
&2\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{1}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \quad (29)\\
&\leq \frac{2^\eta\Gamma(\eta+1)}{(mb^p-a^p)^\eta}\left[I^\eta_{\left(\frac{a^p+mb^p}{2}\right)^+}\Delta(mb^p)+m^{\eta+1}I^\eta_{\left(\frac{a^p+mb^p}{2m}\right)^-}\Delta\left(\frac{a^p}{m}\right)\right]\\
&\leq \eta\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right]\int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{\eta-1}dt\\
&\leq \frac{1}{2}\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right].
\end{aligned}$$

Proof. It is given $h(t) \leq \frac{1}{\sqrt{2}}$. Therefore we have

$$\int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{\eta-1}dt \leq \frac{1}{2\eta}$$

and

$$\frac{1}{mh\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \geq \frac{2}{m}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right).$$

By using these inequalities along with (27) we will get (29). □

Remark 2.27. *If $p = 1$ in (27), then we get [8, Theorem 3].*
If $p = 1$ in (29), then we get [8, Theorem 4].
If $\alpha = 1$, $\eta = 1$, $m = 1$, $h(t) = t$ in (27), then we get [31, Theorem 2.1].

Corollary 2.28. *By using $h(t) = t$ in (27), the following fractional integral inequality for refined $(\alpha, m) - p$-convex function is obtained.*

$$\begin{aligned}
&\frac{2^{2\alpha}}{(2^\alpha-1)}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \quad (30)\\
&\leq \frac{2^\eta\Gamma(\eta+1)}{(mb^p-a^p)^\eta}\left[I^\eta_{\left(\frac{a^p+mb^p}{2}\right)^+}\Delta(mb^p)+m^{\eta+1}I^\eta_{\left(\frac{a^p+mb^p}{2m}\right)^-}\Delta\left(\frac{a^p}{m}\right)\right]\\
&\leq \frac{\eta(2^\alpha(\eta+2\alpha)-(\eta+\alpha))}{2^{2\alpha}(\eta+\alpha)(\eta+2\alpha)}\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right].
\end{aligned}$$

Remark 2.29. *If $m = 1$ in (30), then the result for refined $(\alpha - p)$-convex function can be obtained. If $m = p = 1$ in (30), then the result for refined α-convex function can be obtained.*

Corollary 2.30. *By using $\alpha = 1$ in (27), the following fractional integral inequality for refined $(h - m) - p$-convex function is obtained.*

$$\begin{aligned}&\frac{1}{h^2\left(\frac{1}{2}\right)}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right)\\&\leq\frac{2^\eta\Gamma(\eta+1)}{(mb^p-a^p)^\eta}\Big[I^\eta_{\left(\frac{a^p+mb^p}{2}\right)^+}\Delta(mb^p)+m^{\eta+1}I^\eta_{\left(\frac{a^p+mb^p}{2m}\right)^-}\Delta\left(\frac{a^p}{m}\right)\Big]\\&\leq\eta\Big[\Delta(a)+2m\Delta(b)+m^2\Delta\Big(\frac{a}{m^2}\Big)\Big]\int_0^1 h\Big(\frac{t}{2}\Big)h\Big(\frac{2-t}{2}\Big)t^{\eta-1}dt.\end{aligned}\tag{31}$$

Remark 2.31. *If $p = 1$ in (31), then [8, corollary 8] can be obtained. If $m = p = 1$ in (31), then the result for refined h-convex function can be obtained.*

Corollary 2.32. *By using $\alpha = 1$ in (29), we get the following inequality*

$$\begin{aligned}&2\Delta\left(\left(\frac{a^p+mb^p}{2}\right)\right)^{\frac{1}{p}}\leq\frac{1}{h^2\left(\frac{1}{2}\right)}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)\right)^{\frac{1}{p}}\\&\leq\frac{2^\eta\Gamma(\eta+1)}{(mb^p-a^p)^\eta}\Big[I^\eta_{\left(\frac{a^p+mb^p}{2}\right)^+}\Delta(mb^p)+m^{\eta+1}I^\eta_{\left(\frac{a^p+mb^p}{2m}\right)^-}\Delta\left(\frac{a^p}{m}\right)\Big]\\&\leq\eta\Big[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\Big]\int_0^1 h\left(\frac{t}{2}\right)h\left(\frac{2-t}{2}\right)t^{\eta-1}dt\\&\leq\frac{1}{2}\Big[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\Big].\end{aligned}\tag{32}$$

Corollary 2.33. *By using $\alpha = 1$ and $h(t) = t^s$ in (27), the following fractional integral inequality for refined $(s, m) - p$-convex function is obtained.*

$$\begin{aligned}&2^{2s}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)\right)^{\frac{1}{p}}\leq\frac{2^\eta\Gamma(\eta+1)}{(mb^p-a^p)^\eta}\Big[I^\eta_{\left(\frac{a^p+mb^p}{2}\right)^+}\Delta(mb^p)+m^{\eta+1}I^\eta_{\left(\frac{a^p+mb^p}{2m}\right)^-}\Delta\left(\frac{a^p}{m}\right)\Big]\\&\leq 2^{\eta-1}\eta\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right]\mathbb{B}\left(s+\eta,1+s\right).\end{aligned}\tag{33}$$

Remark 2.34. *If $m = 1$ in (33), then the result for s-tgs convex function can be obtained.*

Remark 2.35. *From the above theorems, one can also deduce inequalities for refined $(h-m)-p$, refined $(\alpha, h-m)-HA$ convex function and refined $(s, m)-p$-Godunova-Levin function by adjusting parameters as in Definitions (2.7), (2.11), (2.13).*

3. k-fractional versions of Hadamard inequalities for refined $(\alpha, h - m) - p$-convex function

This section presents the k-fractional versions of the results discussed in Section 2.

Theorem 3.1. *Under the assumption of Theorem 2.15, the following k-fractional integral inequality holds:*

$$\begin{aligned}&\frac{\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right)}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)}\leq\frac{\Gamma_k(\eta+k)}{(mb^p-a^p)^{\frac{\eta}{k}}}\Big[{}_kI^\eta_{a^p+}\Delta(mb^p)+m^{\frac{\eta}{k}+1}{}_kI^\eta_{mb^p-}\Delta\left(\frac{a^p}{m}\right)\Big]\\&\leq\frac{\eta}{k}\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right]\int_0^1 h(t^\alpha)h(1-t^\alpha)t^{\frac{\eta}{k}-1}dt,\end{aligned}\tag{34}$$

with $\eta, k > 0$.

Proof. Replacing η by $\frac{\eta}{k}$ and applying Definition (1.8), (19) gives inequality (34). □

Corollary 3.2. *By setting $h(t) = t$ in (34), the following k-fractional integral inequality holds:*

$$\frac{2^{2\alpha}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right)}{(2^\alpha-1)} \leq \frac{\Gamma_k(\eta+k)}{(mb^p-a^p)^{\frac{\eta}{k}}}\left[{}_kI^{\eta}_{a^p+}\Delta(mb^p)+m^{\frac{\eta}{k}+1}{}_kI^{\eta}_{mb^p-}\Delta\left(\frac{a^p}{m}\right)\right]$$
$$\leq \frac{k\alpha\eta}{(\eta+\alpha k)(\eta+2\alpha k)}\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right].$$

Corollary 3.3. *Using $\alpha = 1$, (34) gives the following k-fractional integral inequality for refined $(h-m)-p$-convex function.*

$$\frac{\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right)}{h^2\left(\frac{1}{2}\right)} \leq \frac{\Gamma_k(\eta+k)}{(mb^p-a^p)^{\frac{\eta}{k}}}\left[{}_kI^{\eta}_{a^p+}\Delta(mb^p)+m^{\frac{\eta}{k}+1}{}_kI^{\eta}_{mb^p-}\Delta\left(\frac{a^p}{m}\right)\right] \tag{35}$$
$$\leq \frac{\eta}{k}\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right]\int_0^1 h(t)h(1-t)t^{\frac{\eta}{k}-1}dt.$$

Corollary 3.4. *Using $\alpha = 1$ and $h(t) = t^s$, (34) gives the following k-fractional integral inequality for refined $(s,m)-p$-convex function.*

$$2^{2s}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{\Gamma_k(\eta+k)}{(mb^p-a^p)^{\frac{\eta}{k}}}\left[{}_kI^{\eta}_{a^p+}\Delta(mb^p)+m^{\frac{\eta}{k}+1}{}_kI^{\eta}_{mb^p-}\Delta\left(\frac{a^p}{m}\right)\right] \tag{36}$$
$$\leq \frac{\eta}{k}\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right]\mathbb{B}\left(1+s,s+\frac{\eta}{k}\right).$$

Theorem 3.5. *Under the assumption of Theorem 2.25, for $k > 0$, the following k-fractional integral inequality holds:*

$$\frac{1}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \tag{37}$$
$$\leq \frac{2^{\frac{\eta}{k}}\Gamma_k(\eta+k)}{(mb^p-a^p)^{\frac{\eta}{k}}}\left[{}_kI^{\eta}_{\left(\frac{a^p+mb^p}{2}\right)^+}\Delta(mb^p)+m^{\left[\frac{\eta}{k}+1\right]}I^{\eta}_{\left(\frac{a^p+mb^p}{2m}\right)^-}\Delta\left(\frac{a^p}{m}\right)\right]$$
$$\leq \frac{\eta}{k}\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right]\int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{\frac{\eta}{k}-1}dt,$$

with $\eta > 0$.

Proof. Replacing η by $\frac{\eta}{k}$ and applying Definition (1.8), (27) gives inequality (37). □

Corollary 3.6. *By setting $h(t) = t$ in (37), the following k-fractional integral inequality for refined $(\alpha,m)-p$-convex function is obtained.*

$$\frac{2^{2\alpha}}{(2^\alpha-1)}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \tag{38}$$
$$\leq \frac{2^{\frac{\eta}{k}}\Gamma_k(\eta+k)}{(mb^p-a^p)^{\frac{\eta}{k}}}\left[{}_kI^{\eta}_{\left(\frac{a^p+mb^p}{2}\right)^+}\Delta(mb^p)+m^{\frac{\eta}{k}+1}I^{\eta}_{\left(\frac{a^p+mb^p}{2m}\right)^-}\Delta\left(\frac{a^p}{m}\right)\right]$$
$$\leq \frac{\eta\left(2^\alpha(\eta+2\alpha k)-(\eta+\alpha k)\right)}{2^{2\alpha}(\eta+\alpha k)(\eta+2\alpha k)}\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right].$$

Remark 3.7. *For $m = 1$, (38) gives the result for refined $\alpha-p$-convex function.*
For $m = 1 = p$, (38) gives the result for refined α-convex function.

Corollary 3.8. *Using $\alpha = 1$ in (37), the following k-fractional integral inequality for refined $(h-m)-p$-convex function is obtained.*

$$\frac{1}{h^2\left(\frac{1}{2}\right)}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \tag{39}$$
$$\leq \frac{2^{\frac{\eta}{k}}\Gamma_k(\eta+k)}{(mb^p-a^p)^{\frac{\eta}{k}}}\left[{}_kI^{\eta}_{\left(\frac{a^p+mb^p}{2}\right)^+}\Delta(mb^p)+m^{\frac{\eta}{k}+1}I^{\eta}_{\left(\frac{a^p+mb^p}{2m}\right)^-}\Delta\left(\frac{a^p}{m}\right)\right]$$
$$\leq \frac{\eta}{k}\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right]\int_0^1 h\left(\frac{t}{2}\right)h\left(\frac{2-t}{2}\right)t^{\frac{\eta}{k}-1}dt.$$

Corollary 3.9. *Using $\alpha = 1$ and $h(t) = t^s$ in (37), the following k-fractional integral inequality for refined $(s, m)-p$-convex function is obtained.*

$$2^{2s}\Delta\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{2^{\frac{\eta}{k}}\Gamma_k(\eta+k)}{(mb^p-a^p)^{\frac{\eta}{k}}}\left[{}_kI^{\eta}_{\left(\frac{a^p+mb^p}{2}\right)^+}\Delta(mb^p)+m^{\frac{\eta}{k}+1}I^{\eta}_{\left(\frac{a^p+mb^p}{2m}\right)^-}\Delta\left(\frac{a^p}{m}\right)\right] \tag{40}$$
$$\leq \frac{2^{\frac{\eta}{k}-1}\eta}{k}\left[\Delta(a)+2m\Delta(b)+m^2\Delta\left(\frac{a}{m^2}\right)\right]\mathbb{B}\left(s+\frac{\eta}{k},1+s\right).$$

Remark 3.10. *If $m = 1 = p$ in (40), then the result for k-fractional s-tgs convex function can be obtained.*

Conclusion

In this article, we have established several Hadamard-type inequalities for refined $(\alpha, h-m)-p$-convex functions. The obtained results are the refinements of some Hadamard-type inequalities previously known in the literature. The k-fractional versions of these inequalities are also obtained for refined $(\alpha, h-m)$−p-convex functions. The work can be extended and generalized for other classes of convex functions for example, strongly refined $(\alpha, h-m)-p$-convex function, refined convex functions via strictly monotone functions etc. Also the work can be generalized for other fractional integral operators for example generalized Riemann-Liouville fractional integrals, unified integral operators etc.

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