Filomat 38:15 (2024), 5239–5246
https://doi.org/10.2298/FIL2415239G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Majorization for starlike and convex functions with respect to conjugate points

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**Abstract.** The concept of majorization is now well-known after the beautiful work of MacGregor, and then followed by Campbell in his sequel of papers. In this paper, we establish the sharp majorization results for the starlike and convex functions with respect to the conjugate points. A geometric application to the harmonic functions is shown.

## 1. Introduction

Let $\mathcal{A}$ be the set of all normalized analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the open unit disc $\mathbb{D} := \{z : |z| < 1\}$. The subclass of $\mathcal{A}$ of univalent functions be denoted by $\mathcal{S}$. Recall the following definition in want of our onward results:

**Definition 1.1.** *[13, 16] Let $f$, $g$ and $\omega$ be analytic in $|z| < r$. The function $g$ is majorized by $f$ denoted by $g << f$ in $|z| < r$, if $|g(z)| \leq |f(z)|$ in $|z| < r$. The function $g$ is subordinate to $f$ denoted by $g \prec f$ in $|z| < r$ if $g(z) = f(\omega(z))$, where $|\omega(z)| \leq |z|$ and $\omega(0)$ in $|z| < r$. Further, if $f$ is univalent then $g \prec f$ if and only if $g(\mathbb{D}_r) \subseteq f(\mathbb{D}_r)$, where $\mathbb{D}_r := \{z : |z| < r\}$.*

In 1936, Biernacki [1] introduced Majorization-Subordination theory, proving that if $g'(0) \geq 0$ and $g \prec f$, where $f \in \mathcal{S}$ in $\mathbb{D}$, then $g << f$ in $|z| < 1/4$. In the subsequent years, Goluzin, Tao Shah, Lewandowski and MacGregor examined a variety of related problems (for greater in depth detail see [2]). In 1951, Goluzin [9] proved that if $g'(0) \geq 0$ and $g \prec f$, where $f \in \mathcal{S}$ then $g' << f'$ in $|z| < 0.12$ and hypothesized the majorization radius as $|z| < 3 - \sqrt{8}$. Later, Tao Shah found this to be true in 1958.

In 1967, MacGregor [13] proved sharp majorization for the class of univalent starlike and convex functions. Later, Campbell [2–4] obtained sharp majorization results for locally univalent functions in his sequel of three papers. Since then many authors proved majorization results for classes of meromorphic functions [10, 18], but claim for the sharpness is still open. In 2019, Teng and Deng [19] obtained results for several subclasses of starlike functions defined in view of Ma and Minda [12] classes, but here proving the sharpness of obtained radii was still open. Cho et al. [5] also proved the majorization result of MacGregor for the Ma-Minda class of starlike function, however, it is not proved that the obtained radius is sharp or not. In fact, it is not possible to mimic the method of sharpness used by MacGregor [13]. It is worth to

2020 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C80.
*Keywords*. Majorization, Starlike and Convex Functions, Harmonic functions.
Received: 20 June 2023; Accepted: 03 April 2024
Communicated by Miodrag Mateljević
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mention here that the sharpness of radii in such problems are more interesting and challenging for the general Ma-Minda classes.

Let us now recall an equivalent definition of majorization from [13].

**Definition 1.2.** *[13] Let $f$ and $g$ be analytic in $\mathbb{D}$. A function $g(z)$ is said to be majorized by $f(z)$, denoted by $g << f$, if there exists an analytic function $\Phi(z)$ in $\mathbb{D}$ satisfying $|\Phi(z)| \leq 1$ and $g(z) = \Phi(z)f(z)$ for all $z \in \mathbb{D}$.*

**Theorem 1.3 (MacGregor Theorem [13]).** *Let $g$ be majorized by $f$ in $\mathbb{D}$ and $g(0) = 0$. If $f(z)$ is univalent in $\mathbb{D}$, then $|g'(z)| \leq |f'(z)|$ in $|z| \leq 2 - \sqrt{3}$. The constant $2 - \sqrt{3}$ is sharp.*

In 2004, the class of starlike and convex functions with respect to conjugate points, respectively were unified by Ravichandran [17], which are as follows:

**Definition 1.4.** *[17] Let us consider*

$$\mathcal{S}_c^*(\psi) = \left\{ f \in \mathcal{A} : \frac{2zf'(z)}{f(z) + \overline{f(\bar{z})}} \prec \psi(z) \right\} \quad \textit{and} \quad C_c(\psi) = \left\{ f \in \mathcal{A} : \frac{2(zf'(z))'}{(f(z) + \overline{f(\bar{z})})'} \prec \psi(z) \right\}.$$

For the standard notations and basic results of Ma and Minda classes of starlike and convex functions $\mathcal{S}^*(\psi)$ and $C(\psi)$ respectively, see [12]. For its connection to the classes $\mathcal{S}_c^*(\psi)$ and $C_c(\psi)$, see [17]. For some recent articles, we refer to see [8, 11] and the references therein. In Theorem 2.1, the Schwarz-pick inequality is inevitable. Further, interesting applications of the well known Schwarz's lemma in connection with the Jack's lemma can be found in the work lead by Mateljević et. al [15].

In this article, we prove sharp version of Theorem 1.3 in the context of $\mathcal{S}_c^*(\psi)$ and $C_c(\psi)$, where the technique of sharpness is different. Several well-known special cases will be derived in a corollary. We will also obtain an application of majorization for harmonic functions [6].

## 2. Majorization

Let us consider the function $k_\phi \in \mathcal{A}$ be given by

$$1 + \frac{zk_\phi''(z)}{k_\phi'(z)} = \phi(z),$$

with $k_\phi(0) = k_\phi'(0) - 1 = 0$, where $\phi$ is a Ma-Minda function, see [12]. Now we prove our main result.

**Theorem 2.1.** *Let $\phi$ be convex in $\mathbb{D}$ with $\phi(0) = 1$ and $\Re\phi(z) > 0$. Suppose $\psi$ be the function satisfying*

$$\psi(z) + \frac{z\psi'(z)}{\psi(z)} = \phi(z). \tag{1}$$

*Further let $m(r) := \min_{|z|=r} |\psi(z)|$. Let $g \in \mathcal{A}$. If $g << f$ in $\mathbb{D}$, where $f \in C_c(\phi)$ then*

$$|g'(z)| \leq |f'(z)|$$

*holds in $|z| \leq r_\psi$, where $r_\psi \in (0, 1)$ is the smallest root of*

$$(1 - r^2)m(r) - 2r = 0. \tag{2}$$

*The radius constant $r_\psi$ is sharp when $m(r) = \psi(-r)$.*

*Proof.* Let $g$ be majorized by $f$, where $f \in \mathcal{C}_c(\phi)$. Then from Defintion 1.2, we have

$$g(z) = \Phi(z)f(z),$$

where $\Phi$ is analytic in $\mathbb{D}$ and satisfy $|\Phi(z)| \leq 1$. Thus,

$$g'(z) = \Phi(z)f'(z) + \Phi'(z)f(z). \tag{3}$$

The well-known Schwarz-Pick inequality for the function $\Phi$ yields

$$|\Phi'(z)| \leq \frac{1-|\Phi(z)|^2}{1-|z|^2}.$$

Let us write $|\Phi(z)| := \beta$. Now using growth and distortion theorems [17] for the class $\mathcal{C}_c(\phi)$ in (3) along with the Schwarz-Pick inequality, we get

$$|g'(z)| \leq |\Phi(z)||f'(z)| + |\Phi'(z)||f(z)| \leq \beta k'_\phi(r) + \frac{1-\beta^2}{1-r^2}k_\phi(r) \tag{4}$$

for $|z| = r$. Since $|f'(z)| \leq k'_\phi(r)$. Therefore,

$$\left|\frac{g'(z)}{f'(z)}\right| \leq \beta + \frac{1-\beta^2}{1-r^2}\frac{k_\phi(r)}{k'_\phi(r)}. \tag{5}$$

Now let $p$ be the Carathéodory function and satisfy

$$p(z) + \frac{zp'(z)}{p(z)} \prec \phi(z). \tag{6}$$

Since $\Re\phi(z) > 0$ and $\phi$ is convex in $\mathbb{D}$, [16, Theorem 3.2d, p. 86] implies that the solution $\psi$ of (1) exists and is analytic in $\mathbb{D}$ with $\Re\psi(z) > 0$ and given by:

$$\psi(z) := q(z)\left(\int_0^z \frac{q(t)}{t}dt\right)^{-1},$$

where

$$q(z) = z\exp\int_0^z \frac{\phi(t)-1}{t}dt.$$

Since $\Re\psi(z) > 0$ and $p$ satisfies the subordination (6), therefore [16, Lemma 3.2e, p. 89] implies that $p \prec \psi$ and $\psi$ is the best univalent dominant. Thus, a function $f \in \mathcal{C}(\phi)$ implies $f \in \mathcal{S}^*(\psi)$, where $\psi$ satisfies (1) follows by taking $p(z) = zf'(z)/f(z)$. Further, note that a function with real coefficients in $\mathcal{C}(\psi)$ belongs to $\mathcal{C}_c(\psi)$. In particular, taking $f = k_\phi$ we conclude that

$$\frac{zk'_\phi(z)}{k_\phi(z)} = \psi(z),$$

which gives, using the hypothesis and maximum principle of modulus

$$\left|\frac{k_\phi(z)}{k'_\phi(z)}\right| \leq \frac{r}{m(r)}, \quad \text{for} \quad |z| = r. \tag{7}$$

Using the inequality (7) in (5) gives

$$\left|\frac{g'(z)}{f'(z)}\right| \leq \beta + \frac{1-\beta^2}{1-r^2}\frac{r}{m(r)} =: h(\beta, r). \tag{8}$$

Finally, to establish $h(\beta, r) \leq 1$, it is enough to see that

$$\frac{\partial}{\partial \beta} H(\beta, r) = -r < 0,$$

where $H(\beta, r) = (1-r^2)m(r) - (1+\beta)r$. Hence, $h(\beta, r) \leq 1$ holds in $|z| = r \leq r_\psi$, where $r_\psi$ is the smallest postive root of

$$H(1, r) := (1-r^2)m(r) - 2r = 0.$$

The existence of the root $r_\psi$ is evident.

**Proof of Sharpness:** To show the result is sharp, let $m_r = \psi(-r)$ and $\Phi(z) = (z+\delta)/(1+\delta z)$, where $-1 \leq \delta \leq 1$. Take $f(z) \in C(\phi)$ such that $zf'(z)/f(z) = \psi(-z)$. Let us consider $r_2$ as the second consecutive positive root (if any) of equation (2), otherwise take $r_2 = 1$. We prove that for each $r \in (r_\psi, r_2)$ we can select $\delta$ so that $0 < f'(r) < g'(r)$, which means $g'$ can not be majorized by $f'$ in $|z| > r_\psi$. First note that the function $f$ is a rotation of $k_\phi$ with real coefficients and belongs to $C_c(\phi)$ such that

$$\frac{f(r)}{f'(r)} = \frac{k_\phi(r)}{k'_\phi(r)} = \frac{r}{\psi(-r)}. \tag{9}$$

Since

$$g'(r) = k'_\phi(r)\left(\frac{r+\delta}{1+\delta r} + \frac{1-\delta^2}{(1+\delta r)^2}\frac{k_\phi(r)}{k'_\phi(r)}\right) =: f'(r)K(\delta, r)$$

and $K(1, r) = 1$, it suffices to show that $\partial K(\delta, r)/\partial\delta < 0$ at $\delta = 1$ in order to establish that $K(1-\epsilon, r) > 1$, and hence $g'(r) > f'(r) > 0$. But at $\delta = 1$, we have:

$$\begin{aligned}\frac{\partial K(\delta, r)}{\partial \delta} &= \frac{2}{(1+r)^2}\left(\frac{1-r^2}{2} - \frac{k_\phi(r)}{k'_\phi(r)}\right)\\ &= \frac{2}{(1+r)^2}\left(\frac{1-r^2}{2} - \frac{r}{\psi(-r)}\right)\\ &< 0\end{aligned}$$

using (2), (8), (9) and the fact that $h(1, r) < 0$ for all $r \in (r_\psi, r_2)$. □

**Remark 2.2.** *It would be the right place to mention some minute details about the proof, which are used here.*

(*i*) *It is important to note here that to obtain* (5) *from* (4), *we have to use growth and distortion theorems [17], separately. Otherwise, exercise that using the sharp upper estimate for the quantity* $\max_{|z|=r}\left|f(z)/(zf'(z))\right|$ *in* (4) *leads to an invalid statement, that is, the non-existence of the required radius. However, on the other hand, maximization of* $\max_{|z|=r}\left|f(z)/(zf'(z))\right|$ *works in [7].*

(*ii*) *Observe that a function with real coefficients in* $C(\psi)$ *belongs to* $C_c(\psi)$. *In particular, taking* $f = k_\phi$ *we concluded that*

$$\frac{zk'_\phi(z)}{k_\phi(z)} = \psi(z),$$

*which is much required to get* (7). *Such interconnections between the classes* $C(\psi)$ *and* $C_c(\psi)$ *is found to be very important to establish the sharpness part of our result.*

Proof of the following result is omitted as it directly follows from Theorem 2.1.

**Theorem 2.3.** *Let us assume that*

$$m(r) := \min_{|z|=r}|\psi(z)| = \begin{cases} \psi(-r), & \text{if } \psi'(0) > 0; \\ \psi(r), & \text{if } \psi'(0) < 0, \end{cases}$$

*where $\psi$ is a univalent function with positive real part and $\psi(0) = 1$. Let $g \in \mathcal{A}$. If $g << f$ in $\mathbb{D}$, where $f \in \mathcal{S}^*_c(\psi)$ then*

$$|g'(z)| \leq |f'(z)|$$

*holds in $|z| \leq r_\psi$, where $r_\psi \in (0, 1)$ is the smallest root of the equation*

$$(1 - r^2)m(r) - 2r = 0.$$

*The radius constant $r_\psi$ is sharp.*

Now we have our result for some well-known choices of the function $\psi$ and some recently studied:

**Corollary 2.4.** *Let $g \in \mathcal{A}$. If $g << f$, where $f \in \mathcal{S}^*_c(\psi)$ in $\mathbb{D}$ then*

$$|g'(z)| \leq |f'(z)|$$

*holds in $|z| \leq r_\psi$, where $r_\psi \in (0, 1)$ is the smallest root of*

$$q(r) = 0,$$

*where*

(*i*) *if* $\psi(z) = \frac{1+Dz}{1+Ez}$, *where* $-1 \leq E < D \leq 1$, *then*

$$q(r) = (1 - r^2)((1 - Dr)/(1 - Er)) - 2r.$$

(*ii*) *if* $\psi(z) = \frac{1+(1-2\alpha)z}{1-z}$, *where* $0 \leq \alpha < 1$, *then*

$$q(r) = (1 - r)(1 - (1 - 2\alpha)r) - 2r.$$

(*iii*) *if* $\psi(z) = \left(\frac{1+z}{1-z}\right)^\eta$, *where* $0 < \eta \leq 1$, *then*

$$q(r) = (1 - r^2)((1 - r)/(1 + r))^\eta - 2r.$$

(*iv*) *if* $\psi(z) = \sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}$, *then*

$$q(r) = (1 - r^2)\left(\sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1 + r}{1 - 2(\sqrt{2} - 1)r}}\right) - 2r.$$

(*v*) *if* $\psi(z) = (b(1 + z))^{1/a}$, *where* $a \geq 1$ *and* $b \geq 1/2$, *then*

$$q(r) = (1 - r^2)(b(1 - r))^{1/a} - 2r.$$

(*vi*) *if* $\psi(z) = e^z$, *then*

$$q(r) = (1 - r^2) - 2re^r.$$

(*vii*) *if* $\psi(z) = z + \sqrt{1 + z^2}$, *then*

$$q(r) = (1 - r^2)(\sqrt{1 + r^2} - r) - 2r.$$

(*viii*) *if* $\psi(z) = \frac{2}{1+e^{-z}}$, *then*

$$q(r) = (1 - r^2) - r(1 + e^r).$$

(*ix*) *if* $\psi(z) = 1 + \sin z$, *then*

$$q(r) = (1 - r^2)(1 - \sin r) - 2r.$$

*The radii constants* $r_\psi$ *in the context of the above cases are all sharp.*

From the Proof of Theorem 2.1, we observed that the sharp bounds for the starlike expression $|zf'(z)|/|f(z)|$ or $|f'(z)|/|f(z)|$ was combined with the well-known Schwarz-Pick inequality to yield the geometric property that $|g'(z)| < |f'(z)|$ in largest disk $B(0, r_0)$ for some $r_0 \in (0, 1)$ whenever $g$ is majorized by $f$. It is worth mentioning that using the bounds on the starlike type expressions, geometrical conclusions like image containment, growth theorem, bound on the Taylor coefficient of $z$, etc. were obtained recently in [15] for certain classes of holomorphic functions (for harmonic cases, see [15]) by the use of Well-known Schwarz's Jack's Lemma (see, [15, Page 113]). We here, for instance, mention one of their results:

**Theorem 2.5.** *[15, Page 120, Sec. 4.1, Theorem 5] Set* $I_f(z) := zf'(z)/f(z)$. *Suppose that*

(*i*) $f$ *and* $\psi$ *are analytic in* $\mathbb{D}$, *and*
(*ii*) $\psi$ *is locally injective in* $\mathbb{D}$.
(*iii*) $I_\psi$ *is injective and* $I_\psi(D_r)$ *is starlike domain with respect to* 0 *for* $r$ *close to* 1.

*If* $I_f(\mathbb{D}) \subset I_\psi(\mathbb{D})$, *then* $f(\mathbb{D}) \subset \psi(\mathbb{D})$.

## 3. Applications in Harmonic functions

Further, a basic geometric application of our result can be observed for harmonic functions [6] of the form $h = f + \bar{g}$, where

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

Here, we invoke Theorem 2.3.

**Corollary 3.1.** *Assume that* $\min_{|z|=r} |\psi(z)| = m(r)$. *Let* $h = f + \bar{g}$ *be the harmonic mapping in* $\mathbb{D}$ *such that* $f \in \mathcal{S}_c^*(\psi)$ *and* $g << f$. *Then* $h$ *is sense-preserving and univalent in* $\mathbb{D}_r$, *where* $r = \min\{r_\psi, r_c\}$ *and* $r_\psi$ *is the unique positive root of the equation*

$$(1 - r^2)m(r) - 2r = 0$$

*and* $r_c$ *is the radius of convexity of functions* $f$ *in* $\mathcal{S}_c^*(\psi)$. *If* $r_c \geq r_\psi$, *then the result is sharp for the case* $m(r) = \psi(-r)$.

We now demonstrate Corollary 3.1 with a particular case by taking $\psi(z) = (1 + z)/(1 - z)$.

**Example 3.2.** *Note that the Koebe function* $f(z) = z/(1 - z)^2$ *belongs to the class* $\mathcal{S}_c^*\left(\frac{1+z}{1-z}\right)$. *Consider the function* $h_\delta : \mathbb{D} \to \mathbb{C}$ *given by*

$$h_\delta(z) = f(z) + \overline{g(z)} := \frac{z}{(1 - z)^2} + \overline{\Phi(z)\frac{z}{(1 - z)^2}},$$

*where*

$$\Phi(z) = \frac{z + \delta}{1 + \delta z} : \mathbb{D} \to \mathbb{D}$$

*is analytic and* $-1 \leq \delta \leq 1$. *It is harmonic in* $\mathbb{D}$. *Clearly,* $g$ *is majorized by* $f$. *However, the function* $h_\delta$ *is not univalent in* $\mathbb{D}$ *for all values of* $\delta$. *It is well known that the Koebe function has the radius of convexity* $2 - \sqrt{3}$. *Now an application of Corrollary 3.1 show that* $h_\delta$ *is sense-preserving and univalent in* $|z| < 2 - \sqrt{3}$. *See, the following figures.*

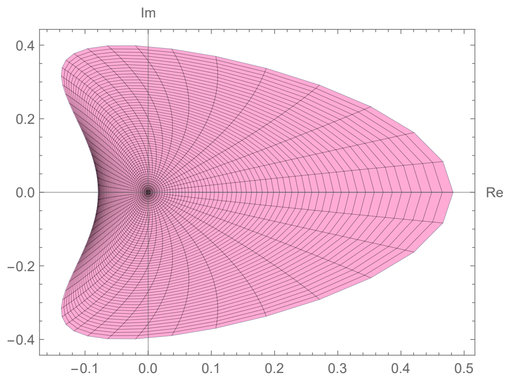


Figure 1: Image of $h_\delta(|z| < 2 - \sqrt{3})$ with $\delta = -0.3$.

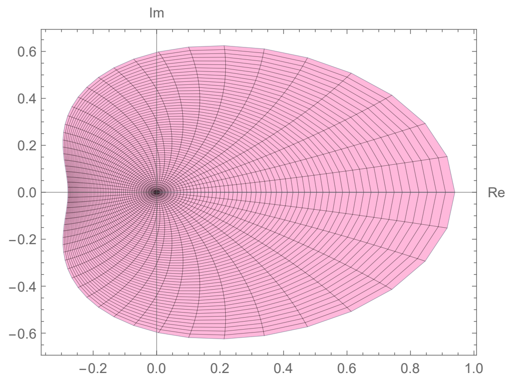


Figure 2: Image of $h_\delta(|z| < 2 - \sqrt{3})$ with $\delta = -0.8$.

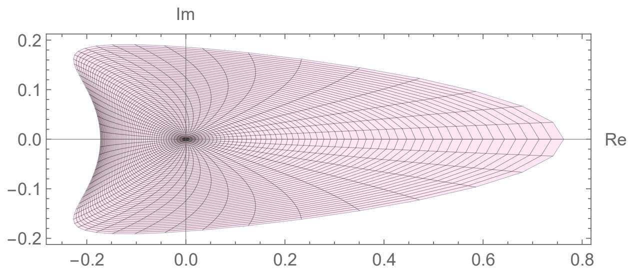


Figure 3: Image of $h_\delta(|z| < 2 - \sqrt{3})$ with $\delta = 0.3$.

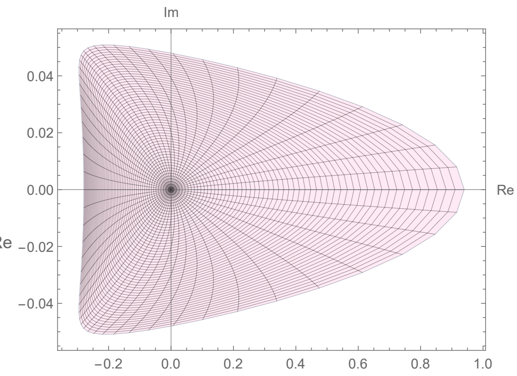


Figure 4: Image of $h_\delta(|z| < 2 - \sqrt{3})$ with $\delta = 0.8$.

**Acknowledgment**. The authors thank the Editor and referees for their careful reading, comments, and suggestions.

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