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A generalization of parametric metric spaces via $\mathcal{B}$-actions

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Abstract. In this study, we introduce a new concept of θ-parametric metric space as a generalization of metric and parametric metric space. We also prove some fixed point theorems for self-mappings defined in the context of such spaces. The utility of our findings is further demonstrated by some examples and some illustrated remarks. The findings presented extend and generalise several existing findings in the literature.

1. Introduction and Preliminaries

Fixed point theory is one of the increasingly prominent fields of nonlinear functional analysis research. It has various applications in a variety of disciplines, including computer sciences (see e.g. [1, 2, 22, 23, 25]), economics, and game theory, among many others. In the setting of numerous abstract metric spaces, the revolutionary Banach contraction principle has been developed, generalized, and refined, and it constitutes an important step in the theoretical progression of metric fixed point theory. (see [5–9, 12–21, 24, 26].

In 2014, Hussain *et al.* [11] established various fixed point results and defined the idea of parametric metric as an intuitive generalization of metric.

Definition 1.1. *[11] Let X be a nonempty set and let $\mathcal{P} : X \times X \times (0, +\infty) \to [0, +\infty)$ be a mapping. We say that P is a parametric metric on X if*

(i): $\mathcal{P}(x, y, \Im) = 0$ *for all* $\Im > 0$ *if and only if* $x = y$ *for all* $x, y \in X$,
(ii): $\mathcal{P}(x, y, \Im) = \mathcal{P}(y, x, \Im)$ *for all* $\Im > 0$,
(iii): $\mathcal{P}(x, y, \Im) \leq \mathcal{P}(x, z, \Im) + \mathcal{P}(z, y, \Im)$ *for all* $x, y \in X$ *and all* $\Im > 0$.

and one says the pair $(X, \mathcal{P})$ is a parametric space.

Example 1.2. *[11] Let X denote the set of all functions $f : (0, +\infty) \to \mathbb{R}$. Define $\mathcal{P} : X \times X \times (0, +\infty) \to [0, +\infty)$ by*

$$\mathcal{P}(f, g, \Im) = |f(\Im) - g(\Im)|$$

for all $f, g \in X$ and all $\Im > 0$. Then $(X, \mathcal{P})$ is a parametric metric space.

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In order to provide a characterisation of well-known fixed point theorems as the Banach and Caristi types, Khojasteh *et al.* [10] developed a novel generalization of metric spaces known as θ-metric spaces in 2013, by altering out the triangular inequality with a broader inequality based on the notion of $\mathcal{B}$-actions.

Definition 1.3. *[10] Let $\theta : [0,+\infty) \times [0,+\infty) \longrightarrow [0,+\infty)$ be a continuous mapping with respect to both variables. Let $Im(\theta) = \{\theta(\zeta,\vartheta) : \zeta \geq 0, \vartheta \geq 0\}$. The mapping θ is called an $\mathcal{B}$-action if and only the following hold :*

($\mathcal{B}$1) $\theta(0,0) = 0$ *and* $\theta(\zeta,\vartheta) = \theta(\vartheta,\zeta)$ *for all* $\zeta,\vartheta \geq 0$,

($\mathcal{B}$2)

$$\theta(\zeta,\vartheta) < \theta(u,v) \Rightarrow \begin{cases} \text{either} \quad \zeta < u, \vartheta \leq v \\ \text{or} \ \ \zeta \leq u, \vartheta < v, \end{cases}$$

($\mathcal{B}$3) for each $r \in Im(\theta)$ and for each $\zeta \in [0,r]$, there exists $\vartheta \in [0,r]$ such that $\theta(\vartheta,\zeta) = r$,

($\mathcal{B}$4) $\theta(\zeta,0) \leq \zeta$ for all $\zeta > 0$.

The set of all $\mathcal{B}$-actions is denoted by Υ.

Example 1.4. *[10] The following functions are examples of $\mathcal{B}$-action :*

1. $\theta_1(\zeta,\vartheta) = \vartheta + \zeta + \vartheta\zeta$,
2. $\theta_2(\zeta,\vartheta) = \frac{\vartheta\zeta}{1+\vartheta\zeta}$,
3. $\theta_3(\zeta,\vartheta) = k(\vartheta + \zeta + \vartheta\zeta)$ *where* $k \in (0,1]$,
4. $\theta_4(\zeta,\vartheta) = \sqrt{\vartheta^2 + \zeta^2}$.

Definition 1.5. *[10] A mapping $\mathcal{E}_\theta : X \times X \longrightarrow [0,+\infty)$ is called a θ-metric on a nonempty set X with respect to $\mathcal{B}$-action $\theta \in \Upsilon$ if d_θ satisfies the following:*

(1) $\mathcal{E}_\theta(x,y) = 0$ *if and only if* $x = y$ *for all* $x,y \in X$,

(2) $\mathcal{E}_\theta(x,y) = \mathcal{E}_\theta(y,x)$ *for all* $x,y \in X$,

(3) $\mathcal{E}_\theta(x,y) \leq \theta(\mathcal{E}_\theta(x,z),\mathcal{E}_\theta(z,y))$ *for all* $x,y,z \in X$.

The pair $(X,\mathcal{E}_\theta)$ is called a θ-metric space.
It is proved in [10] that, in a θ-metric space $(X,\mathcal{E}_\theta)$ every open ball is an open set and each θ-metric d_θ on X generates a Hausdorff first countable topology $\tau_{\mathcal{E}_\theta}$ on X where the set $\{B_{\mathcal{E}_\theta}(x,\frac{1}{n}) : n \in \mathbb{N}\}$ is a local base at x.

2. θ-parametric metric space

First, we introduce the concept of θ-parametric metric space and study some of its properties needed in the following section.

Definition 2.1. *Let X be a non-empty set. A mapping $\mathcal{D}_\theta : X \times X \times (0,+\infty) \to [0,+\infty)$ is called θ-parametric metric space on X if*

($\mathcal{D}_\theta$1) $\mathcal{D}_\theta(x,y,\mathfrak{I}) = 0$ *for all* $\mathfrak{I} > 0$ *if and only if* $x = y$ *for all* $x,y \in X$,

($\mathcal{D}_\theta$2) $\mathcal{D}_\theta(x,y,\mathfrak{I}) = \mathcal{D}_\theta(y,x,\mathfrak{I})$ *for all* $\mathfrak{I} > 0$,

($\mathcal{D}_\theta$3) $\mathcal{D}_\theta(x,y,\mathfrak{I}) \leq \theta(\mathcal{D}_\theta(x,z,\mathfrak{I}),\mathcal{D}_\theta(z,y,\mathfrak{I}))$ *for all* $x,y \in X$ *and all* $\mathfrak{I} > 0$.

Then the pair $(X,\mathcal{D}_\theta)$ is called a θ-parametric metric space.

Remark 2.2. *We mention that every parametric metric space is effectively a θ-parametric metric space with $\theta(u,v) = u+v$, But the converse is not true. We provide the following example to support this claim.*

Example 2.3. *Let $X = \{x, y, z\}$. Define $\theta : [0,+\infty)\times[0,+\infty) \longrightarrow [0,+\infty)$ and $\mathcal{D}_\theta : X\times X\times(0,+\infty) \to [0,+\infty)$ as*

$$\theta(u, v) = u + v + \sqrt{uv}$$

$$\mathcal{D}_\theta(x, x, \mathfrak{I}) = \mathcal{D}_\theta(y, y, \mathfrak{I}) = \mathcal{D}_\theta(z, z, \mathfrak{I}) = 0,$$

$$\mathcal{D}_\theta(x, y, \mathfrak{I}) = \frac{1}{\mathfrak{I}}\ , \quad \mathcal{D}_\theta(x, z, \mathfrak{I}) = \frac{3}{\mathfrak{I}}\ , \quad \mathcal{D}_\theta(z, y, \mathfrak{I}) = \frac{4+\sqrt{2}}{\mathfrak{I}}$$

$$\mathcal{D}_\theta(x, y, \mathfrak{I}) = \mathcal{D}_\theta(y, x, \mathfrak{I})\ , \quad \mathcal{D}_\theta(x, z, \mathfrak{I}) = \mathcal{D}_\theta(z, x, \mathfrak{I})\ , \quad \mathcal{D}_\theta(z, y, \mathfrak{I}) = \mathcal{D}_\theta(y, z, \mathfrak{I})\ .$$

The pair $(X, \mathcal{D}_\theta)$ is θ-parametric metric space.

Note that, $\mathcal{D}_\theta(x, y, \mathfrak{I})+\mathcal{D}_\theta(x, z, \mathfrak{I}) < \mathcal{D}_\theta(z, y, \mathfrak{I})$, thus condition (iii) fails and $\mathcal{D}_\theta$ does not define a parametric metric.

Remark 2.4. *Every θ-parametric metric space $(X, \mathcal{D}_\theta)$ with respect to $\theta(u, v) = k(u + v)$, $k \in (0, 1]$, is a parametric metric space.*
The converse, however, is false in general, for $\theta(u, v) = k(u + v)$, $k \in (0, 1)$, there exits parametric metric space $(X, \mathcal{D})$ which is not θ-parametric metric space. To see this, let $X = \{1, 2, 3\}$ and $\mathcal{D} : X \times X \times (0,+\infty) \to [0,+\infty)$ defined by

$$\mathcal{D}(1, 1, \mathfrak{I}) = \mathcal{D}(2, 2, \mathfrak{I}) = \mathcal{D}(3, 3, \mathfrak{I}) = 0$$

$$\mathcal{D}(1, 2, \mathfrak{I}) = 3\mathfrak{I}\ ,\ \mathcal{D}(1, 3, \mathfrak{I}) = 3\mathfrak{I}\ ,\ \mathcal{D}(2, 3, \mathfrak{I}) = 6\mathfrak{I}\ ,\ k = \frac{1}{3}.$$

It is easy to see that $\mathcal{D}$ is a parametric metric space. Note that $\mathcal{D}(2, 3, \mathfrak{I}) > \theta(\mathcal{D}(2, 1, \mathfrak{I}), \mathcal{D}(1, 3, \mathfrak{I}))$, that is, $\mathcal{D}$ is not a θ-parametric metric space.

Definition 2.5. *Let $(X, \mathcal{D}_\theta)$ be a θ-parametric metric space. The open ball $B_{\mathcal{D}_\theta}(x, r)$ with center $x \in X$ and $r \in Im(\theta)$ is defined as follows :*

$$B_{\mathcal{D}_\theta}(x, r) = \{y \in X : \mathcal{D}_\theta(x, y, \mathfrak{I}) < r\} \text{ for all } \mathfrak{I} > 0$$

Definition 2.6. *Let $(X, \mathcal{D}_\theta)$ be a θ-parametric metric space. A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$, if for every $\hbar > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mathcal{D}_\theta(x_n, x, \mathfrak{I}) < \hbar$ for all $n \geq n_0$ and all $\mathfrak{I} > 0$. that is, $\lim_{n\to+\infty} \mathcal{D}_\theta(x_n, x, \mathfrak{I}) = 0$.*

Definition 2.7. *Let $(X, \mathcal{D}_\theta)$ be θ-parametric metric space and let $\{x_n\}$ be a sequence in X :*

(1) A sequence $\{x_n\}$ is called a Cauchy sequence if, $\lim_{n,m\to+\infty} \mathcal{D}_\theta(x_n, x_m, \mathfrak{I}) = 0$ for all $t > 0$.
(2) A θ-parametric metric space $(X, \mathcal{D}_\theta)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$.

Proposition 2.8. *Let $(X, \mathcal{D}_\theta)$ be a θ-parametric metric space, If the sequence $\{x_k\}$ in X converges to a point x. Then x is unique.*

Proof. Suppose that $\{x_k\}$ converges to x and y. Then, there exists $A > 0$ such that $\mathcal{D}_\theta(x_k, x, \mathfrak{I}) < \frac{1}{k}$ and $\mathcal{D}_\theta(x_k, y, \mathfrak{I}) < \frac{1}{k}$ for each $k \in \mathbb{N}$, all $\mathfrak{I} > 0$.
By the third condition of θ-parametric metric, we have

$$0 \leq \mathcal{D}_\theta(x, y, \mathfrak{I}) \leq \theta(\mathcal{D}_\theta(x_k, x, \mathfrak{I}), \mathcal{D}_\theta(x_k, y, \mathfrak{I})) < \theta(\frac{1}{k}, \frac{1}{k})$$

Taking limit as $k \to +\infty$ in the above inequality and using the continuity of θ with respect to each variable, we have that $\theta(\frac{1}{k}, \frac{1}{k}) \to 0$ and so $\mathcal{D}_\theta(x, y, \mathfrak{I}) = 0$, which implies that $x = y$. □

Lemma 2.9. *Let $(X, \mathcal{D}_\theta)$ be a θ-parametric metric space, then the function $\mathcal{D}_\theta$ is continuous in its two arguments. In other words, if there exist sequences $\{x_k\}$ and $\{y_k\}$ such that $\lim_{k\to+\infty} x_k = x$ and $\lim_{k\to+\infty} y_k = y$, Then*

$$\lim_{k\to+\infty} \mathcal{D}_\theta(x_k, y_k, \mathfrak{I}) = \mathcal{D}_\theta(x, y, \mathfrak{I}) \qquad \text{for all } \mathfrak{I} > 0.$$

Definition 2.10. *Let $(X, \mathcal{D}_\theta)$ be a θ-parametric metric space and let $\mathcal{G} : X \longrightarrow X$ be a self-mapping. $\mathcal{G}$ is said to be a continuous mapping at $x \in X$, if for any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to +\infty$, $\mathcal{G}x_n \to \mathcal{G}x$ as $n \to +\infty$.*

3. Some Fixed Point Results

In this section, we initiate several fixed point results in the setting of θ-parametric metric space. First, we establish the Banach contraction principle in such settings.

Theorem 3.1. *Let $(X, \mathcal{D}_\theta)$ be a complete θ-parametric metric space and $\mathcal{G} : X \to X$ a mapping that satisfies*

$$\mathcal{D}_\theta(\mathcal{G}x, \mathcal{G}y, \mathfrak{I}) \leq \mathcal{K}\mathcal{D}_\theta(x, y, \mathfrak{I}), \tag{1}$$

for all $x, y \in X$, with $\mathcal{K} \in [0, 1)$. Then $\mathcal{G}$ has a unique fixed point $u \in X$.

Proof. Let $x_0 \in X$ be arbitrary and define an iterative sequence $\{x_n\}$ in X by letting $x_{n+1} = \mathcal{G}x_n$ for all $n \geq 0$. Trivially, we note that whenever there exists an index m such that $x_m = x_{m+1}$, then the equalities $x_m = x_{m+1} = \mathcal{G}x_m$ leads to the occurrence that x_m is a fixed point of $\mathcal{G}$. Therefore, to continue our proof we assume That $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$.
We will show that $\lim_{n\to+\infty} \mathcal{D}_\theta(x_{n+1}, x_n, \mathfrak{I}) = 0$.

It follows from (1) that

$$\begin{aligned}\mathcal{D}_\theta(x_{n+1}, x_n, \mathfrak{I}) &= \mathcal{D}_\theta(\mathcal{G}x_n, \mathcal{G}x_{n-1}, \mathfrak{I}) \\ &\leq \mathcal{K}\mathcal{D}_\theta(x_n, x_{n-1}, \mathfrak{I}).\end{aligned}$$

for all $n \in \mathbb{N}$ and all $\mathfrak{I} > 0$. Therefore

$$\begin{aligned}\mathcal{D}_\theta(x_{n+1}, x_n, \mathfrak{I}) &\leq \mathcal{K}\mathcal{D}_\theta(x_n, x_{n-1}, \mathfrak{I}) \\ &\leq \mathcal{K}^2\mathcal{D}_\theta(x_{n-1}, x_{n-2}, \mathfrak{I}) \\ &\;. \\ &\;. \\ &\;. \\ &\leq \mathcal{K}^n\mathcal{D}_\theta(x_1, x_0, \mathfrak{I}) \longrightarrow 0 \;\; as \;\; (n \longrightarrow +\infty).\end{aligned}$$

Hence,

$$\mathcal{D}_\theta(x_{n+1}, x_n, \mathfrak{I}) \longrightarrow 0 \;\; as \;\; (n \longrightarrow +\infty). \tag{2}$$

Now, we will show that the sequence $\{x_n\}$ is bounded. By contradiction, assume that the sequence $\{x_n\}$ is unbounded. That is, there exists subsequence $\{n(i)\}$ with $n(1) = 1$ and for all $i \in \mathbb{N}$, $n(i + 1)$ is the lowest index for which

$$\mathcal{D}_\theta(x_{n(i+1)}, x_{n(i)}, \mathfrak{I}) > \hbar, \tag{3}$$

is not fulfilled, and

$$\mathcal{D}_\theta(x_{m(i)}, x_{n(i)}, \mathfrak{I}) \leq \hbar, \tag{4}$$

is satisfied for all $m \in \{n(i), n(i) + 1, ..., n(i + 1) - 1\}$. Now, Using the triangular inequality, we obtain

$$\begin{aligned}1 &< \mathcal{D}_\theta(x_{n(i+1)}, x_{n(i)}, \mathfrak{I}) \\ &\leq \theta(\mathcal{D}_\theta(x_{n(i+1)}, x_{n(i+1)-1}, \mathfrak{I}), \mathcal{D}_\theta(x_{n(i+1)-1}, x_{n_i}, \mathfrak{I})) \\ &\leq \theta(\mathcal{D}_\theta(x_{n(i+1)}, x_{n(i+1)-1}, \mathfrak{I}), 1).\end{aligned}$$

Taking the limit as $i \to +\infty$ and using $(\mathcal{B}2)$, we get $\lim_{i\to+\infty} \mathcal{D}_\theta(x_{n(i+1)}, x_{n(i)}, \mathfrak{I}) = 1^+$. And, also

$$\begin{aligned}1 &< \mathcal{D}_\theta(x_{n(i+1)}, x_{n(i)}, \mathfrak{I}) \\ &\leq \mathcal{D}_\theta(x_{n(i+1)-1}, x_{n(i)-1}, \mathfrak{I}) \\ &\leq \theta(\mathcal{D}_\theta(x_{n(i+1)}, x_{n(i)}, \mathfrak{I}), \mathcal{D}_\theta(x_{n(i)}, x_{n(i)-1}, \mathfrak{I})) \\ &\leq \theta(1, \mathcal{D}_\theta(x_{n(i)}, x_{n(i)-1}, \mathfrak{I})).\end{aligned}$$

Consequently, $\lim_{i\to+\infty} \mathcal{D}_\theta(x_{n(i+1)-1}, x_{n(i)-1}, \mathfrak{I}) = 1^+$. Taking into account that $\mathcal{D}_\theta(x_{n(i+1)}, x_{n(i)}, \mathfrak{I}) \leq \mathcal{K}\mathcal{D}_\theta(x_{n(i+1)-1}, x_{n(i)-1}, \mathfrak{I})$, we arrive to the contradiction $1 \leq \mathcal{K}1$. Hence, the sequence $\{x_n\}$ is bounded. Next, wi will show that $\{x_n\}$ is a Cauchy sequence. Let $n, l \in \mathbb{N}$ with $l > n$, we have

$$\begin{aligned}\mathcal{D}_\theta(x_l, x_n, \mathfrak{I}) &\leq \mathcal{K}\mathcal{D}_\theta(\mathcal{G}x_{l-1}, \mathcal{G}x_{n-1}, \mathfrak{I})\\ &\leq \mathcal{K}\mathcal{D}_\theta(\mathcal{G}x_{l-2}, \mathcal{G}x_{n-2}, \mathfrak{I})\\ &.\\ &.\\ &.\\ &\leq \mathcal{K}^n\mathcal{D}_\theta(\mathcal{G}x_{l-n}, \mathcal{G}x_0, \mathfrak{I}).\end{aligned}$$

As $\{x_n\}$ is bounded, then $\lim_{n,l\to+\infty} \mathcal{D}_\theta(x_l, x_n, \mathfrak{I}) = 0$, which means that $\{x_n\}$ is a Cauchy sequence. Therefore, there exists $x \in X$ such that $x_n \to x$, we have

$$\mathcal{D}_\theta(x_{n+1}, \mathcal{G}x_n, \mathfrak{I}) \leq \mathcal{K}\mathcal{D}_\theta(x_n, x, \mathfrak{I}) \to 0 \ \textit{as} \ (n \longrightarrow +\infty).$$

Thus, $x_{n+1} \to \mathcal{G}x$ and $\mathcal{G}x = x$. □

Now, we prove Boyd-Wang type theorem in the frame of θ-parametric metric space. First denote by Ψ the set of all functions $\psi : [0, +\infty) \longrightarrow [0, +\infty)$ with the following properties

(ψ1) : ψ is upper semi-continuous from the right i.e. for all sequence $\{\mathfrak{I}_n\}$ in $[0, +\infty)$ such that $\mathfrak{I}_n \to t$ as $n \to +\infty$, we have $\lim_{n\to+\infty} \sup \psi(\mathfrak{I}_n) \leq \psi(\mathfrak{I})$;

(ψ2) : $\psi(\mathfrak{I}) < \mathfrak{I}$ for each $\mathfrak{I} > 0$.

Theorem 3.2. *Let $(X, \mathcal{D}_\theta)$ be a complete θ-parametric metric space and $\mathcal{G} : X \to X$ a mapping that satisfies*

$$\mathcal{D}_\theta(\mathcal{G}x, \mathcal{G}y, \mathfrak{I}) \leq \psi(\mathcal{D}_\theta(x, y, \mathfrak{I})) \tag{5}$$

where $\mathfrak{I} > 0$ and $\psi \in \Psi$. Assume that there exists an element $x \in X$ such that $\mathcal{D}_\theta(x, \mathcal{G}x, \mathfrak{I}) < \infty$. Then $\mathcal{G}$ has a unique fixed point $u \in X$. Moreover, for each $x \in X$, $\lim_{n\to+\infty} \mathcal{G}^n x = u$.

Proof. Define the sequence $\delta_n(\mathfrak{I}) = \mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^{n+1}x, \mathfrak{I})$ for an element $x \in X$. We shall prove that $\delta_n(\mathfrak{I}) \to 0$ as $n \to 0$. We have,

$$\begin{aligned}\delta_n(\mathfrak{I}) = \mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^{n+1}x, \mathfrak{I}) &= \mathcal{D}_\theta(\mathcal{G}\mathcal{G}^{n-1}x, \mathcal{G}\mathcal{G}^n x, \mathfrak{I})\\ &\leq \psi(\mathcal{D}_\theta(\mathcal{G}^{n-1}x, \mathcal{G}^n x, \mathfrak{I}))\\ &< \mathcal{D}_\theta(\mathcal{G}^{n-1}x, \mathcal{G}^n x, \mathfrak{I}) = \delta_{n-1}(\mathfrak{I}).\end{aligned}$$

That is, $\{\mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^{n+1}x, \mathfrak{I})\}$ is a decreasing sequence of nonnegative reals. Therefore, there exist $\eta(\mathfrak{I}) \geq 0$ such that $\lim_{n\to+\infty} \mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^{n+1}x, \mathfrak{I}) = \eta(t)$ for all $\mathfrak{I} > 0$. we show that $\eta(\mathfrak{I}) = 0$ for all $\mathfrak{I} > 0$. Indeed, we assume that there exists $\mathfrak{I}_0 > 0$ such that $\eta(\mathfrak{I}_0) > 0$. Using $\delta_n(\mathfrak{I}) \leq \psi(\delta_{n-1}(\mathfrak{I}))$ together with the upper semicontinuity from the right of ψ, and letting $n \to +\infty$, we obtain

$$\begin{aligned}\eta(\mathfrak{I}_0) &\leq \lim_{n\to+\infty} \sup \psi(\delta_{n-1}(\mathfrak{I}_0))\\ &\leq \psi(\eta(\mathfrak{I}_0)).\end{aligned}$$

This contradicts the assumption (ψ2) on ψ. Hence, we obtain

$$\lim_{n\to+\infty} \mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^{n+1}x, \mathfrak{I}) = 0 \text{ for all } \mathfrak{I} > 0. \tag{6}$$

Now, we shall prove that $\{\mathcal{G}^n x\}$ is a Cauchy sequence. Reasoning by contradiction, we assume that $\{\mathcal{G}^n x\}$ is not Cauchy sequence. Then there exist $\hbar > 0$, $\Im_0 > 0$ and for each $n \in \mathbb{N}$ there is $m = m(n) > n$ such that

$$\mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^m x, \Im_0) \geq \hbar. \tag{7}$$

Where $m(n)$ is chosen as the smallest integer such that (7) is satisfied, which means

$$\mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^{m-1} x, \Im_0) < \hbar. \tag{8}$$

On account of (7), (8) and the triangular inequality, we derive that

$$\begin{aligned}\hbar &\leq \mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^m x, \Im_0)\\ &\leq \theta(\mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^{m-1} x, \Im_0), \mathcal{D}_\theta(\mathcal{G}^{m-1} x, \mathcal{G}^m x, \Im_0))\\ &\leq \theta(\hbar, \mathcal{D}_\theta(\mathcal{G}^{m-1} x, \mathcal{G}^m x, \Im_0)).\end{aligned}$$

Taking the limit as $m \to +\infty$ in the previous inequality and applying (6) and $(\mathcal{B}4)$, we get

$$\hbar \leq \lim_{m\to+\infty} \mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^m x, \Im_0) \leq \theta(\hbar, 0) \leq \hbar.$$

Consequently,

$$\lim_{m\to+\infty} \mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^m x, \Im_0) = \hbar. \tag{9}$$

Again, by using the triangular inequality and (5), we also derive that

$$\begin{aligned}\hbar &\leq \mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^m x, \Im_0)\\ &\leq \theta(\mathcal{G}_\theta(\mathcal{G}^n x, \mathcal{G}^{n+1} x, \Im_0), \mathcal{D}_\theta(\mathcal{G}^{n+1} x, \mathcal{G}^m x, \Im_0))\\ &\leq \theta(\mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^{n+1} x, \Im_0), \theta(\mathcal{D}_\theta(\mathcal{G}^{n+1} x, \mathcal{G}^{m+1} x, \Im_0), \mathcal{D}_\theta(\mathcal{G}^{m+1} x, \mathcal{G}^m x, \Im_0)))\\ &\leq \theta(\mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^{n+1} x, \Im_0), \theta(\psi(_{\mathcal{D}}\theta(\mathcal{G}^n x, \mathcal{G}^m x, \Im_0)), \mathcal{D}_\theta(\mathcal{G}^{m+1} x,^{\mathcal{G}} mx, \Im_0))).\end{aligned}$$

Then, letting $n \to +\infty$, and using (6) and $(\mathcal{B}4)$, we derive that

$$\begin{aligned}\hbar &\leq \theta(0, \theta(\lim_{n\to+\infty} \sup \psi(\mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^m x, \Im_0)), 0))\\ &\leq \theta(\lim_{n\to+\infty} \sup \psi(\mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^m x, \Im_0)), 0)\\ &\leq \lim_{n\to+\infty} \sup \psi(\mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^m x, \Im_0)).\end{aligned} \tag{10}$$

As ψ is continuous, we get

$$\hbar \leq \lim_{m\to+\infty} \psi(\mathcal{D}_\theta(\mathcal{G}^n x, \mathcal{G}^m x, \Im_0)) \leq \psi(\hbar). \tag{11}$$

A contradiction with the assumption (ψ2). Hence, $\{\mathcal{G}^n x\}$ is Cauchy sequence. Owing to the fact that $(X, \mathcal{D}_\theta)$ is a complete, there exists $u \in X$ such that

$$\lim_{n\to+\infty} \mathcal{G}^n x = u. \tag{12}$$

Since $\mathcal{G}$ is continuous, we derive that

$$\mathcal{G}u = \mathcal{G}(\lim_{n\to+\infty} \mathcal{G}^n x) = \lim_{n\to+\infty} \mathcal{G}\mathcal{G}^n x = \lim_{n\to+\infty} \mathcal{G}^{n+1} x = u$$

That is, $\mathcal{G}$ has a fixed point.

Now we show that u is a unique fixed point of $\mathcal{G}$. The proof of this claim is obtained by contradiction, let v be another fixed point of $\mathcal{G}$. Hence,

$$\mathcal{D}_\theta(u, v, \Im) = \mathcal{D}_\theta(\mathcal{G}u, \mathcal{G}v, \Im) \leq \psi(\mathcal{D}_\theta(u, v, \Im)) < \mathcal{D}_\theta(u, v, \Im).$$

Which is a contradiction. Thus, $u = v$. □

Theorem 3.3. *Let $(X, \mathcal{D}_\theta)$ be a complete θ-parametric metric space and let $\mathcal{G} : X \to X$ be a continuous self-mapping satisfying*

$$\mathcal{D}_\theta(\mathcal{G}x, \mathcal{G}y, t) \leq \alpha\Lambda_\theta(x, y, \mathfrak{I}) \tag{13}$$

where

$$\Lambda_\theta(x, y, \mathfrak{I}) = \max\{\mathcal{D}_\theta(x, y, \mathfrak{I}), \mathcal{D}_\theta(x, \mathcal{G}x, \mathfrak{I}), \mathcal{D}_\theta(y, \mathcal{G}y, \mathfrak{I}), \frac{\mathcal{D}_\theta(x, \mathcal{G}x, \mathfrak{I})\mathcal{D}_\theta(y, \mathcal{G}y, \mathfrak{I})}{\mathcal{D}_\theta(x, y, \mathfrak{I})}\}$$

for all $x, y \in X$, $x \neq y$ and all $\mathfrak{I} > 0$, with $\alpha \in [0, 1[$. Then $\mathcal{G}$ has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary and define an iterative sequence $\{x_n\}$ in X by letting $x_{n+1} = \mathcal{G}x_n$ for all $n \geq 0$. Trivially, we note that whenever there exists an index m such that $x_m = x_{m+1}$, then the equalities $x_m = x_{m+1} = \mathcal{G}x_m$ leads to the occurrence that x_m is a fixed point of $\mathcal{G}$. Therefore, to continue our proof we assume That $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$.
We will show that $\lim_{n\to+\infty} \mathcal{D}_\theta(x_{n+1}, x_n, \mathfrak{I}) = 0$

It follows from (13) that

$$\begin{aligned}\mathcal{D}_\theta(x_{n+1}, x_n, \mathfrak{I}) &= \mathcal{D}_\theta(\mathcal{G}x_n, \mathcal{G}x_{n-1}, \mathfrak{I})\\ &\leq \alpha\Lambda_\theta(x_n, x_{n-1}, \mathfrak{I})\end{aligned}$$

Where

$$\begin{aligned}\Lambda_\theta(x_n, x_{n-1}, \mathfrak{I}) &= \{\mathcal{D}_\theta(x_n, x_{n-1}, t), \mathcal{D}_\theta(x_n, \mathcal{G}x_n, \mathfrak{I}), \mathcal{D}_\theta(x_{n-1}, \mathcal{G}x_{n-1}, \mathfrak{I}),\\ &\qquad \frac{\mathcal{D}_\theta(x_n, \mathcal{G}x_n, \mathfrak{I})\mathcal{D}_\theta(x_{n-1}, \mathcal{G}x_{n-1}, \mathfrak{I})}{\mathcal{D}_\theta(x_n, x_{n-1}, \mathfrak{I})}\}\\ &= \max\{\mathcal{D}_\theta(x_n, x_{n-1}, \mathfrak{I}), \mathcal{D}_\theta(x_n, x_{n+1}, \mathfrak{I}), \mathcal{D}_\theta(x_{n-1}, x_n, \mathfrak{I}),\\ &\qquad \frac{\mathcal{D}_\theta(x_n, x_{n+1}, \mathfrak{I})\mathcal{D}_\theta(x_{n-1}, x_n, \mathfrak{I})}{\mathcal{D}_\theta(x_n, x_{n-1}, \mathfrak{I})}\}\\ &= \max\{\mathcal{D}_\theta(x_n, x_{n-1}, \mathfrak{I}), \mathcal{D}_\theta(x_n, x_{n+1}, \mathfrak{I})\}.\end{aligned}$$

Thus, we get two cases :

$$\max\{\mathcal{D}_\theta(x_n, x_{n-1}, \mathfrak{I}), \mathcal{D}_\theta(x_n, x_{n+1}, \mathfrak{I})\} = \mathcal{D}_\theta(x_n, x_{n+1}, \mathfrak{I}),$$

or

$$\max\{\mathcal{D}_\theta(x_n, x_{n-1}, \mathfrak{I}), \mathcal{D}_\theta(x_n, x_{n+1}, \mathfrak{I})\} = \mathcal{D}_\theta(x_n, x_{n-1}, t).$$

Now if $\max\{\mathcal{D}_\theta(x_n, x_{n-1}, \mathfrak{I}), \mathcal{D}_\theta(x_n, x_{n+1}, \mathfrak{I})\} = \mathcal{D}_\theta(x_n, x_{n+1}, \mathfrak{I})$, the consequence of the above inequality, also rewritable as

$$\mathcal{D}_\theta(x_{n+1}, x_n, \mathfrak{I}) \leq \alpha\mathcal{D}_\theta(x_n, x_{n+1}, \mathfrak{I}) < \mathcal{D}_\theta(x_n, x_{n+1}, \mathfrak{I}).$$

This leads to a contradiction. Hence

$$\mathcal{D}_\theta(x_{n+1}, x_n, \mathfrak{I}) \leq \alpha\mathcal{D}_\theta(x_n, x_{n-1}, \mathfrak{I}), \tag{14}$$

for all $n \in \mathbb{N}$ and all $\mathfrak{I} > 0$. Therefore

$$\begin{aligned}\mathcal{D}_\theta(x_{n+1}, x_n, \mathfrak{I}) &\leq \alpha\mathcal{D}_\theta(x_n, x_{n-1}, \mathfrak{I})\\ &\leq \alpha^2\mathcal{D}_\theta(x_{n-1}, x_{n-2}, \mathfrak{I})\\ &\quad .\\ &\quad .\\ &\quad .\\ &\leq \alpha^n\mathcal{D}_\theta(x_1, x_0, \mathfrak{I}) \longrightarrow 0 \;\; as \;\; (n \longrightarrow +\infty).\end{aligned}$$

Thus we have

$$\mathcal{D}_\theta(x_{n+1}, x_n, \Im) \longrightarrow 0 \ \ as \ \ (n \longrightarrow +\infty). \tag{15}$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. Reasoning by contradiction, we assume that is not a Cauchy sequence. Then there exists $\Im_0 > 0$ and $\hbar > 0$ and two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \quad and \quad \mathcal{D}_\theta(x_{m_i}, x_{n_i}, \Im_0) \geq \hbar \tag{16}$$

and

$$\mathcal{D}_\theta(x_{m_i}, x_{n_i-1}, \Im_0) < \hbar \tag{17}$$

Now, Using (14) and the triangular inequality, we derive that

$$\begin{aligned}\hbar &\leq \mathcal{D}_\theta(x_{m_i}, x_{n_i}, \Im_0)\\ &\leq \theta(\mathcal{D}_\theta(x_{m_i}, x_{n_i-1}, \Im_0), \mathcal{D}_\theta(x_{n_i-1}, x_{n_i}, \Im_0))\\ &\leq \theta(\theta(\mathcal{D}_\theta(x_{m_i}, x_{m_i-1}, \Im_0), \mathcal{D}_\theta(x_{m_i-1}, x_{n_i-1}, \Im_0)), \mathcal{D}_\theta(x_{n_i-1}, x_{n_i}, \Im_0))\end{aligned}$$

Taking the limit as $i \to +\infty$ and then using (13) and $(\mathcal{B}4)$, we get

$$\hbar \leq \lim_{i\to+\infty} \mathcal{D}_\theta(x_{m_i-1}, x_{n_i-1}, \Im_0). \tag{18}$$

We also have

$$\begin{aligned}\mathcal{D}_\theta(x_{m_i-1}, x_{n_i-1}, \Im_0) &\leq \theta(\mathcal{D}_\theta(x_{m_i-1}, x_{m_i}, \Im_0), \mathcal{D}_\theta(x_{m_i}, x_{n_i-1}, \Im_0))\\ &\leq \theta(\mathcal{D}_\theta(x_{m_i-1}, x_{m_i}, \Im_0), \hbar)\end{aligned}$$

Taking the limit as $i \to +\infty$ on both sides of the above inequality and then using (13) and $(\mathcal{B}4)$, we get

$$\lim_{i\to+\infty} \mathcal{D}_\theta(x_{m_i-1}, x_{n_i-1}, \Im_0) \leq \hbar \tag{19}$$

Therefore, from (16) and (17), we deduce

$$\lim_{i\to+\infty} \mathcal{D}_\theta(x_{m_i-1}, x_{n_i-1}, \Im_0) = \hbar \tag{20}$$

Applying the condition (9), we obtain

$$\begin{aligned}\mathcal{D}_\theta(x_{m_i}, x_{n_i}, \Im_0) &= \mathcal{D}_\theta(\mathcal{G}x_{m_i-1}, \mathcal{G}x_{n_i-1}, \Im_0)\\ &\leq \alpha\Lambda_\theta(x_{m_i-1}, x_{n_i-1}, \Im_0)\end{aligned} \tag{21}$$

Where

$$\begin{aligned}\Lambda_\theta(x_{m_i-1}, x_{n_i-1}, \Im_0) &= \max\{\mathcal{D}_\theta(x_{m_i-1}, x_{n_i-1}, \Im_0), \mathcal{D}_\theta(x_{m_i-1}, \mathcal{G}x_{m_i-1}, \Im_0),\\ &\qquad \mathcal{D}_\theta(x_{n_i-1}, \mathcal{G}x_{n_i-1}, \Im_0),\\ &\qquad \frac{\mathcal{D}_\theta(x_{m_i-1}, \mathcal{G}x_{m_i-1}, \Im_0)\mathcal{D}_\theta(x_{n_i-1}, \mathcal{G}x_{n_i-1}, \Im_0)}{\mathcal{D}_\theta(x_{m_i-1}, x_{n_i-1}, \Im_0)}\}\\ &= \max\{\mathcal{D}_\theta(x_{m_i-1}, x_{n_i-1}, \Im_0), \mathcal{D}_\theta(x_{m_i-1}, x_{m_i}, \Im_0),\\ &\qquad \mathcal{D}_\theta(x_{n_i-1}, x_{n_i}, \Im_0),\\ &\qquad \frac{\mathcal{D}_\theta(x_{m_i-1}, x_{m_i}, \Im_0)\mathcal{D}_\theta(x_{n_i-1}, x_{n_i}, \Im_0)}{\mathcal{D}_\theta(x_{m_i-1}, x_{n_i-1}, \Im_0)}\}\end{aligned} \tag{22}$$

Letting $i \to +\infty$ and applying (13) and (18), we obtain

$$\lim_{i\to+\infty} \Lambda_\theta(x_{m_i-1}, x_{n_i-1}, \Im_0) = \hbar \tag{23}$$

Now, taking limit as $i \to +\infty$ in (19), using (14)and (21), we obtain

$$\hbar \le \lim_{i\to+\infty} \mathcal{D}_\theta(x_{m_i}, x_{n_i}, \Im_0) \le \alpha\hbar < \hbar.$$

Which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence.
Since $(X, \mathcal{D}_\theta)$ is a complete θ-parametric metric space, there exists $u \in X$ such that $x_n \to u$ as $n \to +\infty$. By the continuity of $\mathcal{G}$, we have

$$u = \lim_{i\to+\infty} x_{n+1} = \lim_{i\to+\infty} \mathcal{G}x_n = \mathcal{G}u$$

So we can conclude that u is a fixed point of $\mathcal{G}$. Finally, we prove by contradiction the uniqueness of the fixed of $\mathcal{G}$. Suppose that $\mathcal{G}$ has another fixed point $z \neq u$. It follows from (9) that

$$\begin{aligned}\mathcal{D}_\theta(u, z, \Im) &= \mathcal{D}_\theta(\mathcal{G}u, \mathcal{G}z, \Im)\\ &\le \alpha\Lambda_\theta(u, z, \Im)\end{aligned}$$

Where

$$\begin{aligned}\Lambda_\theta(u, z, \Im) &= \max\{\mathcal{D}_\theta(u, z, \Im), \mathcal{D}_\theta(u, \mathcal{G}u, \Im), \mathcal{D}_\theta(z, \mathcal{G}z, \Im),\\ &\qquad \frac{\mathcal{D}_\theta(u, \mathcal{G}u, \Im)\mathcal{D}_\theta(z, \mathcal{G}z, \Im)}{\mathcal{D}_\theta(u, z, \Im)}\}\\ &= \max\{\mathcal{D}_\theta(u, z, \Im), 0, 0, 0\}\\ &= \mathcal{D}_\theta(u, z, \Im).\end{aligned}$$

Which implies that

$$\mathcal{D}_\theta(u, z, \Im) \le \alpha\mathcal{D}_\theta(u, z, \Im) < \mathcal{D}_\theta(u, z, \Im), \tag{24}$$

having in this way a contradiction, completing therefore the proof. □

Corollary 3.4. *Let $(X, \mathcal{D}_\theta)$ be a complete θ-parametric metric space and let $\mathcal{G} : X \to X$ be a continuous a mapping such that satisfies the following :*

$$\mathcal{D}_\theta(\mathcal{G}x, \mathcal{G}y, \Im) \le \lambda \max\{\mathcal{D}_\theta(x, \mathcal{G}x, \Im), P_\theta(y, \mathcal{G}y, \Im)\} \tag{25}$$

for each $x, y \in X$ and all $\Im > 0$, where $\lambda \in [0, 1[$. Then $\mathcal{G}$ has a unique fixed point.

Application

As an application, we will prove that a continuous mapping $\mathcal{G} : X \to X$ in a complete θ-parametric metric space $(X, \mathcal{D}_\theta)$ and satisfying the following inequality

$$\mathcal{D}_\theta(\mathcal{G}x, \mathcal{G}y, t) \le \eta\frac{\mathcal{D}_\theta(x, \mathcal{G}x, \Im)\mathcal{D}_\theta(y, \mathcal{G}y, \Im)}{\mathcal{D}_\theta(x, y, \Im)} + \delta\mathcal{D}_\theta(x, y, \Im) \tag{26}$$

for all $x, y \in X$ with $x \neq y$ and all $\Im > 0$, where $\eta, \delta \ge 0$, with $\eta + \delta < 1$. Then $\mathcal{G}$ has a unique fixed point.

Proof. Applying (26), for all $x, y \in X$ with $x \neq y$, $\mathfrak{I} > 0$, we get that

$$\begin{aligned}\mathcal{D}_\theta(\mathcal{G}x, \mathcal{G}y, \mathfrak{I}) &\leq \eta \frac{\mathcal{D}_\theta(x, \mathcal{G}x, \mathfrak{I})\mathcal{D}_\theta(y, \mathcal{G}y, \mathfrak{I})}{\mathcal{D}_\theta(x, y, \mathfrak{I})} + \delta \mathcal{D}_\theta(x, y, \mathfrak{I}) \\ &\leq (\eta + \delta) \max\{\frac{\mathcal{D}_\theta(x, \mathcal{G}x, \mathfrak{I})\mathcal{D}_\theta(y, \mathcal{G}y, \mathfrak{I})}{\mathcal{D}_\theta(x, y, \mathfrak{I})}, \mathcal{D}_\theta(x, y, \mathfrak{I})\} \\ &\leq (\eta + \delta) \max\{\mathcal{D}_\theta(x, y, \mathfrak{I}), \mathcal{D}_\theta(x, \mathcal{G}x, \mathfrak{I}), \mathcal{D}_\theta(y, \mathcal{G}y, \mathfrak{I}), \\ &\qquad \frac{\mathcal{D}_\theta(x, \mathcal{G}x, \mathfrak{I})\mathcal{D}_\theta(y, \mathcal{G}y, \mathfrak{I})}{\mathcal{D}_\theta(x, y, \mathfrak{I})}\}.\end{aligned}$$

Thus, all conditions of Theorem 3.3 are satisfied and $\mathcal{G}$ has unique fixed point. □

Conclusion

In this study, we introduced a new metric space, namely the concept of θ-parametric metric space as a generalization of metric and parametric metric spaces. Then, we investigated the existence and uniqueness of fixed point for various contraction principles. Since a θ-parametric metric space is a parametric space when $\theta(u, v) = u + v$, our results can be considered as generalization of several comparable results. The approach we propose may pave the way for new developments in generalized metrical structures and fixed point theory. The results obtained can be further used to investigate coincidence, common, and relation-theoretic fixed point results.

References

[1] Mathews, GS: Partial metric topology. Reseach Report 212, Dept. of Computer Science University of Warwick (1992)

[2] Romaguera, S, Valero, O: A quantitative computational model for complete partial metric spaces via formal balls. Math. Struct. Comput. Sci. 19(3), 541-563 (2009)

[3] Border, KC: Fixed Point Theorems with Applications to Economics and Game Theory. Cambridge University Press, New York (1985)

[4] Ok, EA: Real Analysis with Economic Applications. Princeton University Press, Princeton (2007)

[5] D. Dukić, Z. Kadelburg, S. Radenović, Fixed points of Geraghty-type mappings in various generalized metric spaces, Abstr. Appl. Anal., 2011 (2011), 13 pages.2.1

[6] S. Czerwik, Contraction mappings in b-metric space, Acta Math. Inf. Univ. Ostraviensis,1(1993), 5-11.

[7] A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen, 57:1-2(2000), 31-37.

[8] D.W. Boyd, J.S.Wong, On nonlinear contractions, Proc. Amer. Math. Soc., 20, 1969, 458-464.

[9] V. Gregori and A. Sapena, On fixed-point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 125 (2002), 245–252.

[10] F. Khojasteh, E. Karapinar and S. Radenović, θ-metric space: A generalization, Math.Probl. Eng. 2013, article ID 504609.

[11] N. Hussain, S. Khaleghizadeh, P. Salimi, Afrah A.N. Abdou, A new approach to fixed point results in triangular intuitionistic fuzzy metric spaces, Abstract and Applied Analysis, Article ID 690139, 2014.

[12] V. Berinde, General constructive fixed point theorems for Ćirić-type almost contraction in metric spaces, Carpathian J. Math., 24 (2008), 10-19.

[13] S.Melliani and A.Moussaoui, Fixed point theorem using a new class of fuzzy contractive mappings, Universal Mathematics, Vol.1 No.2 pp.148-154 (2018).

[14] N. Hussain, P.Salimi and V.Parvaneh, Fixed point results for various contractions in parametric and fuzzy b-metric spaces, J. Nonlinear Sci. Appl., 8(2015), 719-739.

[15] A. Moussaoui, N. Hussain and S. Melliani, *Global optimal solutions for proximal fuzzy contractions involving control functions,* J. Math. (2021), Article ID 6269304.

[16] A. Moussaoui, N. Saleem, S. Melliani and M. Zhou, *Fixed point results for new types of fuzzy contractions via admissible functions and FZ-simulation functions,* Axioms, 11 (**2022**), Paper No. 87.

[17] N.Priyobarta, R.Yumnam, S.Radenović, Fixed Point Theorems on Parametric A-metric Space, American Journal of Applied Mathematics and Statistics, 2018, Vol.6, No.1, 1-5

[18] A. Moussaoui, N. Hussain, S. Melliani, N. Hayel and M. Imdad, *Fixed point results via extended $\mathcal{FZ}$-simulation functions in fuzzy metric spaces,* J. Inequal. Appl. 2022 (2022), Paper No. 69.

[19] A. Moussaoui, V. Todorčević, M. Pantović, S. Radenović, S. Melliani, Fixed Point Results via $\mathcal{G}$-Transitive Binary Relation and Fuzzy $\mathcal{L}$-$\mathcal{R}$-Contraction. Mathematics 2023, 11, 1768.

[20] Nihal Tas and Nihal Yilmaz Ozgur, On parametric S-metric spaces and fixed point type theorems for expansive mappings, Journal of Mathematics, vol. 2016, Article ID 4746732, 6 pages.

[21] A. Roldán, J. Martínez-Moreno, C. Roldán, Y.J. Cho, Multidimensional coincidence point results for compatible mappings in partially ordered fuzzy metric spaces. Fuzzy Sets Syst. 251, 71–82 (2014)

[22] S. Etemad, M.M. Matar, M.A. Ragusa, S. Rezapour, Tripled fixed points and existence study to a tripled impulsive fractional differential system via measures of non- compactness, Mathematics, 10 (1) (2022).

[23] V. Ozturk, Common fixed point theorems for generalized orthogonal contractions, Filomat, 38 (5), 1859–1867, (2024).

[24] Abdelhamid Moussaoui, Stojan Radenović, Said Melliani, A nonlinear fuzzy contraction principle via control functions Filomat 38:6 (2024), 1963–1972.

[25] A. Kari, A. Al-Rawashdeh, New fixed point theorems for $\theta - \omega$-contraction on (λ, μ)- generalized metric spaces, Journal of Function Spaces, vol. 2023.

[26] A. Moussaoui, S. Melliani, S. Radenović, Fixed point theorems involving $\mathcal{F}\mathcal{Z}$-ϑ_f-contractions in GV-fuzzy metrics. Filomat 38:6 (2024), 1973–1985.