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On the identric mean of two accretive matrices

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Abstract. Intensive studies aiming to extend some matrix means from positive matrices to accretive matrices and to establish some of their properties have been carried out recently. The contribution of this work falls within this framework. We introduce the identric mean of two accretive matrices and we study its properties. Some inequalities involving this identric mean when the variables are sector matrices are presented as well.

1. Introduction

Throughout this manuscript, the notation $\mathbb{M}_n$ refers to the space of $n \times n$ matrices with real or complex entries.

• Every $A \in \mathbb{M}_n$ can be written in the following form

$$A = \Re A + i\Im A, \quad \text{with} \quad \Re A := \frac{A + A^*}{2} \quad \text{and} \quad \Im A := \frac{A - A^*}{2i}, \tag{1}$$

where the notation A^* refers to the adjoint of A. The decomposition (1) is known, in the literature, as the Cartesian decomposition of A and the matrices $\Re A$ and $\Im A$ are called the real part and the imaginary part of A, respectively.

• As usual, if $A \in \mathbb{M}_n$ is Hermitian, i.e. $A^* = A$, we say that A is positive semidefinite (in short $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{C}^n$ and, A is positive definite (in short $A > 0$) if A is positive semidefinite and invertible. For $A, B \in \mathbb{M}_n$ Hermitian, we write $A \leq B$ or $B \geq A$ for meaning that $B - A$ is positive semidefinite. We say that A is accretive if its real part $\Re A$ is positive definite. It is clear that if A and B are accretive then so is $A + B$ but, in general, AB may be not accretive. Also, A accretive does not ensure that A^k is accretive, for $k \geq 2$ integer. However, it is well known that every accretive matrix $A \in \mathbb{M}_n$ is invertible and A^{-1} is also accretive, [9].

• We also need to define the sector S_θ in the complex plane by

$$S_\theta := \Big\{z \in \mathbb{C} : \ \Re z > 0, \ |\Im z| \leq (\Re z) \tan\theta\Big\},$$

for some $\theta \in [0, \pi/2)$.

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The numerical range of $A \in \mathbb{M}_n$ is defined as follows

$$W(A) := \Big\{x^*Ax : \ x \in \mathbb{C}^n, \ x^*x = 1\Big\}.$$

We say that A is a sector matrix whenever $W(A) \subset S_\theta$ for some $\theta \in [0, \pi/2)$. It is obvious that if A is a sector matrix then A is accretive. For further details about the properties and applications of accretive matrices/sector matrices, we refer the reader to [1–5, 9, 10, 12–15, 17] and the related references cited therein.

• The exponential of $A \in \mathbb{M}_n$ is defined by the infinite series

$$\exp A := \sum_{k=0}^{\infty} \frac{A^k}{k!}. \tag{2}$$

Such series is convergent for any $A \in \mathbb{M}_n$ and $\exp A$ is always invertible with $(\exp A)^{-1} = \exp(-A)$. If A is invertible then the equation $A = \exp X$ has the solution, [18]

$$X = \log A := \Big(e^{i\theta}A - \mathcal{I}\Big) \int_0^1 \Big((1-t)I + te^{i\theta}A\Big)^{-1} dt - i\theta\mathcal{I}, \tag{3}$$

where θ is any angle such that $e^{i\theta}A$ has no singular value on $(-\infty, 0]$. Here, $\mathcal{I}$ denotes the $n \times n$ identity matrix. By definition, $\log A$ defined by (3) is the logarithm of A. The basic properties of the logarithm and exponential matrix functions are embodied in the following result. For further details, we refer the reader to the interesting paper of A. Wouk [18].

Proposition 1.1. *Let $A, B \in \mathbb{M}_n$. Then the following assertions hold true:*
(i) The relation $\log(\exp A) = A$ holds for any $A \in \mathbb{M}_n$ while $\exp(\log A) = A$ holds when A is invertible.
(ii) $\exp A$ is always invertible with $(\exp A)^{-1} = \exp(-A)$.
(iii) For any invertible $A \in \mathbb{M}_n$ we have $\log A^{-1} = -\log A$ and $\log(aA) = (\log a)\mathcal{I} + \log A$, where $a > 0$ is a real number.
(iv) The exponential and logarithm are analytic matrix functions. Further, the matrices $\exp A$ and $\log A$ commute with any matrix which commutes with A.
(v) For any $A \in \mathbb{M}_n$ we have $\det\Big(\exp A\Big) = \exp(TrA)$, where $\det A$ and TrA refer to the determinant and trace of A, respectively.
(vi) For any invertible $A \in \mathbb{M}_n$ we have $Tr\Big(\log A\Big) = \log\Big(\det A\Big)$.
(vii) The map $X \longmapsto \log X$ is matrix monotone increasing and concave for $X > 0$ while $X \longmapsto \exp X$ is neither matrix monotone nor convex for $X > 0$.

• For $A, B \in \mathbb{M}_n$ positive definite, we write $A \preceq B$ (or $B \succeq A$) for meaning that $\log A \leq \log B$. Then $\preceq$ defines a partial order (called chaotic order) on the set of positive definite matrices. Since the map $X \longmapsto \log X$ is matrix monotone increasing for $X > 0$ then, $A \leq B$ implies $A \preceq B$ while the converse in not always true since the map $X \longmapsto \exp X$ is not matrix monotone for $X > 0$.

The present paper will be organized as follows: In Section 2, we recall some standard means of two accretive matrices recently introduced in the literature that will be needed throughout this manuscript. In Section 3 we use (3) to write the logarithm of an accretive matrix $A \in \mathbb{M}_n$. Afterwards, we give other forms of $\log A$ in integral representations that allowed us to establish the inequality $\Re \log(A) \geq \log\Big(\Re A\Big)$, valid for every accretive matrix $A \in \mathbb{M}_n$. This latter inequality, whose the proof seems to be difficult from (3), will be a good tool for proving some inequalities in the next sections. In Section 4, we introduce the so-called chaotic geometric matrix mean of two accretive matrices. Section 5 deals with the identric mean of two accretive matrices together with a study of its properties, specially when the involved matrices are sector matrices. Section 6 displays with further properties for the logarithmic and identric means of two accretive matrices.

2. Means of accretive matrices

As already pointed out before, every accretive matrix $A \in \mathbb{M}_n$ is invertible and A^{-1} is also accretive. Further, it is easy to check that the set of all accretive matrices is a convex cone of $\mathbb{M}_n$. Let $A, B \in \mathbb{M}_n$ be accretive and $\lambda \in [0,1]$. The following expressions

$$A\nabla_\lambda B := (1-\lambda)A + \lambda B, \quad A!_\lambda B := \Big((1-\lambda)A^{-1} + \lambda B^{-1}\Big)^{-1} = \Big(A^{-1}\nabla_\lambda B^{-1}\Big)^{-1}, \tag{4}$$

are known, in the literature, as the λ-weighted arithmetic mean and the λ-weighted harmonic mean of A and B, respectively. The λ-weighted geometric mean of A and B is defined by, see [15]

$$A\sharp_\lambda B := \int_0^1 A!_t B \, d\nu_\lambda(t), \tag{5}$$

where, for fixed $\lambda \in [0,1]$, $\nu_\lambda(t)$ is the probability measure on $(0,1)$ defined by

$$d\nu_\lambda(t) := \frac{\sin(\lambda\pi)}{\pi} \frac{t^{\lambda-1}}{(1-t)^\lambda} dt. \tag{6}$$

For $\lambda = 1/2$, the previous matrix means are simply denoted by $A\nabla B$, $A!B$ and $A\sharp B$, respectively. From their definitions, it is clear that $A\nabla_\lambda B, A!_\lambda B$ and $A\sharp_\lambda B$ are accretive whenever A and B are. Further, it is clear that $\Re\Big(A\nabla_\lambda B\Big) = \Big(\Re A\Big)\nabla_\lambda\Big(\Re B\Big)$. Otherwise, we have [15]

$$\Re\Big(A!_\lambda B\Big) \geq \Big(\Re A\Big)!_\lambda\Big(\Re B\Big), \quad \Re\Big(A\sharp_\lambda B\Big) \geq \Big(\Re A\Big)\sharp_\lambda\Big(\Re B\Big). \tag{7}$$

The logarithmic mean of two accretive matrices A and B is defined by, [17]

$$L(A,B) := \int_0^1 A\sharp_t B \, dt. \tag{8}$$

From (8), $L(A,B)$ is accretive and by the second inequality in (7) we get, [17]

$$\Re L(A,B) \geq L\Big(\Re A, \Re B\Big). \tag{9}$$

If $A, B \in \mathbb{M}_n$ are positive definite, the following inequalities hold, [17]

$$A!B \leq A\sharp B \leq L(A,B) \leq A\nabla B. \tag{10}$$

An analog of (10) for sector matrices was proved in [17, Theorem 3.5].

Theorem 2.1. *Let $A, B \in \mathbb{M}_n$ be accretive. Assume that $W(A), W(B) \subset S_\theta$ for some $\theta \in [0, \pi/2)$. Then we have*

$$(\cos\theta)^2 \Re(A!B) \leq (\cos\theta)^2 \Re(A\sharp B) \leq \Re L(A,B) \leq (\sec\theta)^2 \Re(A\nabla B). \tag{11}$$

Some reverses of (7) and (9) can be found in [17, Lemma 3.3, Proposition 3.4] and we have the following result.

Proposition 2.2. *Let $A, B \in \mathbb{M}_n$ be accretive with $W(A), W(B) \subset S_\theta$ for some $\theta \in [0, \pi/2)$. If m denotes one of the three matrix means $!_\lambda, \sharp_\lambda$ and L then we have*

$$\Re(AmB) \leq (\sec\theta)^2 (\Re A) m (\Re B). \tag{12}$$

We end this section by stating the following remark which may be of interest.

Remark 2.3. *(i) All the previous matrix means, as well as the following ones, are defined when the involved arguments are accretive matrices. However, the definition of the weighted arithmetic mean* ∇_λ, *for* $\lambda \in [0,1]$, *can be extended for any* $n \times n$ *matrices. Henceforth, and for the sake of simplicity, we may use the formula* $A\nabla_\lambda B := (1-\lambda)A + \lambda B$ *when* $A, B \in \mathbb{M}_n$ *are accretive or not.*
(ii) In [11], the authors explored a general theory about the class of operator/matrix means later known as monotone operator/matrix means in the Kubo-Ando sense. In [1], the authors extended the Kubo-Ando theory from positive definite matrices to accretive matrices in an analogous and general point of view.
(iii) A weighted logarithmic mean of two accretive matrices has been introduced in [16].
(iv) As we will see later, the identric mean investigated here does not belong to the class of matrix means discussed in [1, 11].

3. Chaotic identric mean

3.1. Logarithm of accretive matrices.

If $A \in \mathbb{M}_n$ is accretive then we can take $\theta = 0$ in (3). We then state the following definition.

Definition 3.1. *Let* $A \in \mathbb{M}_n$ *be accretive. The logarithm of* A *is defined by*

$$\log A := (A - \mathcal{I}) \int_0^1 \Big((1-t)\mathcal{I} + tA\Big)^{-1} dt = (A - \mathcal{I}) \int_0^1 \Big(t\mathcal{I} + (1-t)A\Big)^{-1} dt. \tag{13}$$

Of course, $\log A$ may be not accretive. The following result will be of interest for studying some properties of the logarithm matrix function.

Proposition 3.2. *Let* $A \in \mathbb{M}_n$ *be accretive. Then there hold:*

$$\log A = \int_0^1 \frac{\mathcal{I}!_t A - \mathcal{I}}{t} dt, \tag{14}$$

$$\log A = \int_0^1 \frac{\mathcal{I} - \mathcal{I}!_t A^{-1}}{t} dt. \tag{15}$$

Proof. Simple manipulation leads to

$$\mathcal{I}!_t A - \mathcal{I} = t\Big(\mathcal{I} - A^{-1}\Big)\Big(\mathcal{I}!_t A\Big) = t(A - \mathcal{I})\Big(t\mathcal{I} + (1-t)A\Big)^{-1}.$$

This, when combined with (13), yields (14). We then deduce (15) by using (14) with the help of the identity $\log A = -\log A^{-1}$. □

Remark 3.3. *Adding* (14) *and* (15) *side by side we obtain another integral representation for* $\log A$ *which is of symmetric character in* A *and* A^{-1}.

$$\log A = \int_0^1 \frac{\mathcal{I}!_t A - \mathcal{I}!_t A^{-1}}{2t} dt.$$

The following result may be stated as well.

Proposition 3.4. *Let* $A \in \mathbb{M}_n$ *be accretive. Then*

$$\Re \log(A) \geq \log\Big(\Re A\Big). \tag{16}$$

Proof. By (14) we easily obtain

$$\Re \log(A) - \log\Big(\Re A\Big) = \int_0^1 \frac{\Re\Big(\mathcal{I}!_t A\Big) - \mathcal{I}!_t \Re A}{t} dt,$$

which, according to the first inequality in (7), immediately yields (16). □

A reverse of (16) reads as follows.

Proposition 3.5. *Let $A \in \mathbb{M}_n$ be accretive with $W(A) \subset S_\theta$ for some $\theta \in [0, \pi/2)$. Then we have*

$$0 \leq \Re\Big(\log A\Big) - \log\Big(\Re A\Big) \leq 2\log\Big(\sec\theta\Big)\mathcal{I}, \tag{17}$$

or, equivalently,

$$\exp \Re \log(A) \leq (\sec^2\theta)\Big(\Re A\Big).$$

Proof. We recall that the map $X \longmapsto \log X$ is matrix monotone increasing on the convex cone of positive definite matrices. This, with the following inequality [12, Lemma 3] $\Big(\Re A\Big)^{-1} \leq (\sec\theta)^2 \Re A^{-1}$, implies that

$$-\log\Big(\Re A\Big) \leq 2\log(\sec\theta)\mathcal{I} + \log\Big(\Re A^{-1}\Big).$$

Therefore we can write

$$\Re\Big(\log A\Big) - \log\Big(\Re A\Big) \leq -\Re\Big(\log A^{-1}\Big) + 2\log(\sec\theta)\mathcal{I} + \log\Big(\Re A^{-1}\Big),$$

which, with (16), immediately gives (17). □

3.2. Chaotic geometric mean of accretive matrices.

Let $A, B \in \mathbb{M}_n$ be accretive. We define the chaotic geometric mean $A\sharp_C B$ of A and B as follows

$$A\sharp_C B := \exp\Big(\frac{\log A + \log B}{2}\Big) := \exp\Big((\log A)\nabla(\log B)\Big). \tag{18}$$

The basic properties of $A\sharp_C B$ are embodied in the following result.

Proposition 3.6. *The following statements hold:*
(i) If $A, B \in \mathbb{M}_n$ are positive definite then so is $A\sharp_C B$.
(ii) The inequality $A\sharp_C B \leq A\nabla B$ does not in general hold when A and B are positive definite.
(iii) In general $A\sharp B \neq A\sharp_C B$, even if A and B are positive definite. If $AB = BA$ then $A\sharp_C B = A\sharp B$.
(iv) $(A\sharp_C B)^{-1} = A^{-1}\sharp_C B^{-1}$ for any accretive $A, B \in \mathbb{M}_n$.
(v) $A\sharp_C B = B\sharp_C A$ and $(aA)\sharp_C(bB) = \sqrt{ab}A\sharp_C B$ for any accretive $A, B \in \mathbb{M}_n$ and $a, b > 0$ real numbers.
(vi) $\det\Big(A\sharp_C B\Big) = (\det A)\sharp_C(\det B) = (\det A)\sharp(\det B)$.

Proof. (i) If $A, B \in \mathbb{M}_n$ are positive definite then $\log A$ and $\log B$ are Hermitian and so $A\sharp_C B$ is positive definite as exponential of Hermitian matrix.
(ii) Assume that $A\sharp_C B \leq A\nabla B$ holds for any A, B positive definite. That is,

$$\exp\Big(\frac{\log A + \log B}{2}\Big) \leq \frac{A + B}{2},$$

which means that $X \longmapsto \exp X$ is matrix midconvex and so matrix convex, since $X \longmapsto \exp X$ is continuous. This contradicts Proposition 1.1,(vii).
(iii) Since $A\sharp B \leq A\nabla B$, for any $A, B \in \mathbb{M}_n$ positive definite, we then conclude by (ii).
(iv) and (v) are immediate from (18) with the help of the basic properties of the matrix logarithm and exponential pointed out in Proposition 1.1.
(vi) Follows by using the properties (v) and (vi) of Proposition 1.1. □

Note that $A\sharp_C B$ may be not accretive when A and B are accretive. We have the following result as well.

Proposition 3.7. *Let $A, B \in \mathbb{M}_n$ be accretive. Then we have*

$$\Re \log(A\sharp_C B) \geq \log\Big((\Re A)\sharp_C(\Re B)\Big), \tag{19}$$

or, equivalently,

$$\exp \Re \log(A\sharp_C B) \geq (\Re A)\sharp_C(\Re B).$$

Proof. By (18), with the help of (16), we get

$$\begin{aligned}\Re \log(A\sharp_C B) = \Re\Big((\log A)\nabla(\log B)\Big) = \Big(\Re \log A\Big)\nabla(\Re \log B)\\ \geq \Big(\log \Re A\Big)\nabla\Big(\log \Re B\Big) = \log\Big((\Re A)\sharp_C(\Re B)\Big).\end{aligned}$$

Hence (19). □

A reverse of (19) is stated in the following result.

Proposition 3.8. *Let $A, B \in \mathbb{M}_n$ be accretive with $W(A), W(B) \subset S_\theta$. Then we have*

$$\Re \log(A\sharp_C B) - \log\Big((\Re A)\sharp_C(\Re B)\Big) \leq 2\log\Big(\sec\theta\Big)\mathcal{I}. \tag{20}$$

Or, equivalently,

$$\exp \Re \log(A\sharp_C B) \leq (\sec^2\theta)(\Re A)\sharp_C(\Re B).$$

Proof. By (18) we get

$$\Re \log(A\sharp_C B) - \log\Big((\Re A)\sharp_C(\Re B)\Big) = \Re\Big((\log A)\nabla(\log B)\Big) - \log(\Re A)\nabla\log(\Re B).$$

This, with the definition of the matrix mean ∇, yields

$$\Re \log(A\sharp_C B) - \log\Big((\Re A)\sharp_C(\Re B)\Big) = \frac{\Re(\log A) - \log(\Re A) + \Re(\log B) - \log(\Re B)}{2},$$

which, when combined with (17), gives (20). □

3.3. Identric mean of accretive matrices.

We start this subsection by stating the following main definition.

Definition 3.9. *Let $A, B \in \mathbb{M}_n$ be accretive. The identric mean of A and B is defined by*

$$I(A, B) := \exp\left(\int_0^1 \log\Big(A\nabla_t B\Big)dt\right). \tag{21}$$

The elementary properties of $I(A, B)$ are included in the following proposition.

Proposition 3.10. *The following statements hold:*
(i) If $A, B \in \mathbb{M}_n$ are positive definite then so is $I(A, B)$.
(ii) $I(A, B) = I(B, A)$ for any accretive $A, B \in \mathbb{M}_n$.
(iii) For any accretive $A, B \in \mathbb{M}_n$, $I(A, B)$ is invertible and we have

$$\Big(I(A, B)\Big)^{-1} = \exp\int_0^1 \log(A^{-1}!_t B^{-1})dt.$$

Proof. (i) Similar to that of Proposition 3.6, (i).
(ii) Follows from the relationship $A\nabla_t B = B\nabla_{1-t}A$, valid for any $A, B \in \mathbb{M}_n$ and $t \in [0,1]$, with a simple change of variables in the integral of (21).
(iii) Follows from Proposition 1.1,(ii) and the definition of $!_t$. □

As for $A\sharp_C B$, $I(A,B)$ may be not accretive when A and B are accretive. Further, it is worth mentioning that, for A and B positive definite matrices, $I(A,B)$ is not a monotone matrix mean in the Kubo-Ando sense. That is, the following relationship

$$I(A,B) = A^{1/2} I\Big(\mathcal{I}, A^{-1/2}BA^{-1/2}\Big)A^{1/2}$$

does not in general hold. To show this, it is enough [11, Theorem 4.5] to show that the inequality $I(A,B) \leq A\nabla B$ is not always true for some $A, B \in \mathbb{M}_n$ positive definite. This follows by the same arguments as for $A\sharp_C B$. We omit the details here for the reader.

An interesting result which justifies more the previous discussion and gives an analog of (10) for $I(A,B)$ is recited in the following.

Proposition 3.11. *Let $A, B \in \mathbb{M}_n$ be positive definite. Then*

$$A!B \leq A\sharp_C B \leq I(A,B) \leq A\nabla B. \tag{22}$$

Proof. Recall that the map $X \longmapsto \log X$ is matrix concave for $X > 0$. This, with the definition of $A!B$, yields

$$\log(A!B) = -\log\Big(\frac{A^{-1}+B^{-1}}{2}\Big) \leq \frac{\log A + \log B}{2},$$

which, with the definition of $A\sharp_C B$, gives $A!B \leq A\sharp_C B$. For the same reason we have

$$\int_0^1 \log(A\nabla_t B)dt \geq \int_0^1 \Big((1-t)\log A + t\log B\Big)dt = \frac{\log A + \log B}{2},$$

which, with the definition of $I(A,B)$ and that of $A\sharp_C B$, gives $A\sharp_C B \leq I(A,B)$. By the same reason and according to the integral Jensen inequality for matrix concave maps, [7], we get

$$\int_0^1 \log(A\nabla_t B)dt \leq \log \int_0^1 (A\nabla_t B)dt = \log\frac{A+B}{2},$$

which gives $I(A,B) \leq A\nabla B$. Summarizing, the proof of (22) is finished. □

The following remark may be of interest.

Remark 3.12. *We can prove* (22) *by another different way as explained in what follows. Since the map $x \longmapsto \log x$ is matrix concave on* $(0,\infty)$, *by the Hermite-Hadamard inequalities for matrix maps [8] we have*

$$-\log\frac{A^{-1}+B^{-1}}{2} \leq \frac{\log A + \log B}{2} =: \frac{\log(A\nabla_0 B) + \log(A\nabla_1 B)}{2}$$
$$\leq \int_0^1 \log(A\nabla_t B)\,dt \leq \log\Big(A\nabla_{1/2}B\Big) =: \log(A\nabla B).$$

This, with (18), (22) *and the definition of* ! *and* $\leq$, *immediately yields* (22). *Such way stems its importance in the fact that it brings us an interesting idea for refining and reversing* (21), *since refinements and reverses of the Hermite-Hadamard inequalities have been investigated in the literature, see [8] for instance. We omit the details about this latter point which is out of the purpose of this paper.*

The following result may be stated as well.

Proposition 3.13. *Let $A, B \in \mathbb{M}_n$ be accretive. Then we have*

$$\Re \log I(A,B) \geq \log I\Big(\Re A, \Re B\Big), \tag{23}$$

or, equivalently,

$$\exp\Big(\Re \log I(A,B)\Big) \geq I\Big(\Re A, \Re B\Big).$$

Proof. From (21) we deduce

$$\log I(A,B) = \int_0^1 \log\Big(A\nabla_t B\Big)dt. \tag{24}$$

This, with (16), implies that

$$\Re \log I(A,B) = \int_0^1 \Re \log\Big(A\nabla_t B\Big)dt \geq \int_0^1 \log \Re(A\nabla_t B)dt = \int_0^1 \log\Big(\Re A\nabla_t \Re B\Big)dt,$$

which, with (24), yields (23). □

A reverse of (23) reads as follows.

Proposition 3.14. *Let $A, B \in \mathbb{M}_n$ be accretive with $W(A), W(B) \subset S_\theta$. Then we have*

$$0 \leq \Re \log I(A,B) - \log I\Big(\Re A, \Re B\Big) \leq 2\log\Big(\sec\theta\Big)\mathcal{I}, \tag{25}$$

or, equivalently,

$$\exp\Big(\Re \log I(A,B)\Big) \leq (\sec^2\theta) I\Big(\Re A, \Re B\Big).$$

Proof. By (21) we get

$$\Re \log I(A,B) - \log I\Big(\Re A, \Re B\Big) = \int_0^1 \Big(\Re \log(A\nabla_t B) - \log\Big(\Re(A\nabla_t B)\Big)\Big)dt,$$

which, with (17), immediately gives (25). □

The following result concerns an analog of (22) for accretive matrices.

Theorem 3.15. *Let $A, B \in \mathbb{M}_n$ be accretive with $W(A), W(B) \subset S_\theta$. Then there holds*

$$(\Re A)!(\Re B) \leq (\Re A)\sharp_C(\Re B) \leq \exp \Re\Big(\log I(A,B)\Big) \leq (\sec\theta)^2 \Re(A\nabla B). \tag{26}$$

Proof. By (25) and then the right inequality in (22) we have

$$\Re\Big(\log I(A,B)\Big) \leq \log I\Big(\Re A, \Re B\Big) + 2\log\Big(\sec\theta\Big)\mathcal{I} \leq \log \Re(A\nabla B) + 2\log\Big(\sec\theta\Big)\mathcal{I}.$$

Hence the right inequality in (26). Now, by (21) and then (16) we have

$$\Re\Big(\log I(A,B)\Big) = \int_0^1 \Re \log(A\nabla_t B)dt \geq \int_0^1 \log\Big((1-t)\Re A + t\Re B\Big)dt,$$

which, with the fact that the map $x \longmapsto \log x$ is matrix concave for $x > 0$, yields

$$\Re\Big(\log I(A,B)\Big) \geq \int_0^1 \Big((1-t)\log \Re A + t\log \Re B\Big)dt = \frac{\log \Re A + \log \Re B}{2} = \log\Big(\Re A \sharp_C \Re B\Big),$$

whence the second inequality in (26). The first inequality in (26) follows from the left inequality in (22). The proof is finished. □

4. More properties for $L(A,B)$ and $I(A,B)$

In this section we investigate further properties for the logarithmic and identric means of two accretive matrices previously studied. We need the following lemma.

Lemma 4.1. *For any real number $x > 0$ we have*

$$\phi(x) := \int_0^1 x^t \sin(\pi t)dt = \frac{(x+1)\pi}{\pi^2 + \left(\log x\right)^2}. \tag{27}$$

Proof. We consider $\psi(x) := \int_0^1 x^t \cos(\pi t)dt$ and we compute $\psi(x) + i\phi(x)$, where $i^2 = -1$. By an elementary computation of integral we get

$$\psi(x) + i\phi(x) = \int_0^1 x^t e^{i\pi t}dt = \int_0^1 \exp t\Big(i\pi + \log x\Big)dt = \frac{-x-1}{i\pi + \log x}.$$

Separating the real and imaginary parts, we obtain the desired result. □

In (8), $L(A,B)$ is defined in terms of the weighted geometric matrix mean. The following result states another expression of $L(A,B)$ in terms of the weighted harmonic matrix mean.

Theorem 4.2. *Let $A, B \in \mathbb{M}_n$ be accretive. Then we have*

$$L(A,B) = \int_0^1 A!_t B \, d\mu(t),$$

where $\mu(t)$ denotes the probability measure on $(0,1)$ defined through

$$d\mu(t) := \frac{dt}{t(1-t)\Big(\pi^2 + \left(\log \frac{t}{1-t}\right)^2\Big)}.$$

Proof. According to (8) and then (5) with (6) we get

$$L(A,B) = \int_0^1 A\sharp_t B \, dt = \int_0^1 \frac{\sin(\pi t)}{\pi} \int_0^1 \frac{s^{t-1}}{(1-s)^t} A!_s B \, ds \, dt,$$

or, equivalently,

$$L(A,B) = \frac{1}{\pi}\int_0^1 \frac{A!_s B}{s} \int_0^1 \sin(\pi t)\Big(\frac{s}{1-s}\Big)^t \, dt \, ds.$$

Thanks to (27), with a simple reduction, we obtain

$$L(A,B) = \int_0^1 \frac{A!_s B \, ds}{s(1-s)\Big(\pi^2 + \left(\log \frac{s}{1-s}\right)^2\Big)} =: \int_0^1 A!_s B \, d\mu(s).$$

The fact that $L(A,A) = A$ and $A!_s B = A$ for any $s \in (0,1)$ imply that $\int_0^1 d\mu(s) = 1$ i.e. $\mu(s)$ is a probability measure on $(0,1)$. The proof is finished. □

In order to give a result for $I(A,B)$ we need the following lemma.

Lemma 4.3. *Let $\mathcal{T}$ be a nonempty convex subset of $\mathbb{M}_n$ and let $\Phi : \mathcal{T} \longrightarrow \mathbb{M}_n$ be a continuous matrix map. Then the following equality*

$$\int_0^1 \Phi\Big((1-t)A + t\frac{A+B}{2}\Big)dt + \int_0^1 \Phi\Big((1-t)\frac{A+B}{2} + tB\Big)dt = 2\int_0^1 \Phi\Big((1-t)A + tB\Big)dt \quad (28)$$

holds true for all $A, B \in \mathcal{T}$.

Proof. One making the change of variable $t = 2s$, we get after a simple reduction

$$\int_0^1 \Phi\Big((1-t)A + t\frac{A+B}{2}\Big)dt = 2\int_0^{1/2} \Phi\Big((1-s)A + sB\Big)ds.$$

Now, with the change of variable $t = 2s - 1$ we obtain

$$\int_0^1 \Phi\Big((1-t)\frac{A+B}{2} + tB\Big)dt = 2\int_{1/2}^1 \Phi\Big((1-s)A + sB\Big)ds.$$

Adding these latter equalities side by side, we get the desired result. □

Finally, we state the following result which gives a relationship between the three matrix means $\nabla, \sharp_C$ and I.

Proposition 4.4. *For any accretive $A, B \in \mathbb{M}_n$ we have the following identity*

$$I(A,B) = I\Big(A, A\nabla B\Big)\sharp_C I\Big(A\nabla B, B\Big). \quad (29)$$

Proof. With $\Phi(X) = \log X$, (28) can be written in the following form

$$\int_0^1 \log\Big(A\nabla_t(A\nabla B)\Big)dt + \int_0^1 \log\Big((A\nabla B)\nabla_t B\Big)dt = 2\int_0^1 \log\Big(A\nabla_t B\Big)dt,$$

which, with (21), is equivalent to

$$\log I\Big(A, A\nabla B\Big) + \log I\Big(A\nabla B, B\Big) = 2\log I(A,B),$$

and by (18) we get

$$\log\Big(I\Big(A, A\nabla B\Big)\sharp_C I\Big(A\nabla B, B\Big)\Big) = \log I(A,B).$$

Hence (29). □

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