Filomat 38:15 (2024), 5499–5506
https://doi.org/10.2298/FIL2415499Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Fractional approximation by Bernstein operators

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Abstract. We estimate the fractal approximation rate for approximation by Bernstein operators by using the Ditzian-Totik type modulus of continuity. Our results improve the previous theorem of Anastassiou. In the proof, the equivalence between the K-functional and the Ditzian-Totik type modulus, and the absolute moment estimates with fractal order play important roles.

1. Introduction

The non-integer order calculus, usually known as the fractional calculus, is a classical and important subject of Analysis. The fractional calculus has been applied in some fields of physics, chemistry, biochemistry and many other disciplines. The literatures [5] and [14] are good references on the theory of the fractional calculus. As we know, there are many different types of definitions of fractional derivatives. Recently, Sales Teodoro Tenreiro Machado and de Oliveira give an interesting review of definitions of fractional derivatives ([17]). Here, we recall the definition of the left Caputo fractional derivative and the right Caputo fractional derivative.

Definition 1.1. *Let $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We define the left Caputo fractional derivative of f as the function*

$$D^{\nu}_{*a} f(x) := \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \quad x \in [a,b],$$

where Γ is the gamma function $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt, \nu > 0$.

Note that $D^{\nu}_{*a} f \in L_1([a,b])$ and $D^{\nu}_{*a} f$ exists a.e on $[a,b]$.

We set $D^0_{*a} f(x) = f(x)$ for any $x \in [a,b]$.

Definition 1.2. *Let $f \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order α is given by*

$$D^{\alpha}_{b-} f(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (t-x)^{m-\alpha-1} f^{(m)}(t)\, dt, \ x \in [a,b].$$

2020 *Mathematics Subject Classification*. Primary 41A36, 41A25; Secondary 41A35 26A33.
Keywords. Fractional approximation, Bernstein operators, Voronovaskaja type estimates, Ditzian-Totik type theorem.
Received: 30 May 2023; Revised: 19 December 2023; Accepted: 01 January 2024
Communicated by Miodrag Spalević
The work is partially supported by National Natural Science Foundation of China (NSFC12271133).
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We set $D_{b-}^{0}f(x)=f(x)$. *Note that* $D_{b-}^{\alpha}f\in L_1([a,b])$ *and* $D_{b-}^{\alpha}f$ *exists a.e on* $[a,b]$.

In approximation theory, the results involving fractional derivatives are very rare. Some early results mainly focused on the best approximation of functions by algebraic and trigonometric polynomials (see [8], [15]). Anastassiou ([1]-[3]) and Păltănea ([16]) investigated the fractal approximation by linear positive operators including the well known Bernstein operators and some neural network operators.

For any $f(x)\in C_{[0,1]}$, the Bernstein operator is defined by

$$B_n(f,x):=\sum_{i=0}^{n}f\left(\frac{i}{n}\right)p_{ni}(x),$$

where $p_{ni}(x):=\binom{n}{i}x^i(1-x)^{n-i},\ i=0,1,\cdots,n$. It is well known that $B_n(f,x)$ converges to f uniformly and preserves the monotonicity and convexity of the approximated function. Bernstein operators have been used in many branches of mathematics and computer science. The approximation properties of Bernstein operators have been studied very extensively (see [6],[7],[10]-[13], [18], [19] for reference).

Anastassiou obtained the following approximation rate estimates for the fractal approximation by Bernstein operators ([Corollary 36, 2]):

Theorem 1.3. *Let* $0<\alpha<1,\ r>0,\ f\in AC_{[0,1]}, f'\in L_\infty([0,1])$. *Then*

$$\|B_n(f)-f\|_\infty\le\frac{1}{\Gamma(\alpha+1)}\left(1+\frac{1}{(\alpha+1)r}\right)\Big[\sup_{x\in[0,1]}\omega_1\left(D_{x-}^{\alpha}f,r\|B_n(|\cdot-x|^{\alpha+1}\chi_{[0,x]}(\cdot),x)\|_\infty^{\frac{1}{\alpha+1}}\right)_{[0,x]}$$

$$\times\|B_n(|\cdot-x|^{\alpha+1}\chi_{[0,x]}(\cdot),x)\|_\infty^{\frac{\alpha}{\alpha+1}}+\sup_{x\in[0,1]}\omega_1\left(D_{*x}^{\alpha}f,r\|B_n(|\cdot-x|^{\alpha+1}\chi_{[x,1]}(\cdot),x)\|_\infty^{\frac{1}{\alpha+1}}\right)_{[x,1]}$$

$$\times\|B_n(|\cdot-x|^{\alpha+1}\chi_{[x,1]}(\cdot),x)\|_\infty^{\frac{\alpha}{\alpha+1}}\Big],$$

where $\|f\|_\infty$ *is the usual uniform norm on* $[0,1]$ *and* $\omega(f,t)_{[a,b]}$ *is the modulus of continuity of* f *on* $[a,b]$.

For convenience, we write $\|f\|$ to replace $\|f\|_\infty$. By using the moments estimate

$$B_n(|t-x|^{3/2},x)\le\frac{1}{(4n)^{3/4}},$$

Anastassiou ([Discussion 39,2]) showed that, for $f\in C^1[0,1]$, there are some $x_1,x_2\in[0,1]$ such that

$$\|B_n(f)-f\|\le C\left(\omega\left(D_{x_1-}^{\frac{1}{2}}f,\frac{1}{3\sqrt{n}}\right)_{[0,1]}+\omega\left(D_{*x_2}^{\frac{1}{2}}f,\frac{1}{3\sqrt{n}}\right)_{[0,1]}\right).$$

Anastassiou ([2]) also pointed out that the above estimate is essentially better than the usual approximation rate estimate.

Write $\varphi(x)=\sqrt{x(1-x)},\ \delta_n(x):=\varphi(x)+\frac{1}{\sqrt{n}}$. For any continuous function f, the so called Ditzian-Totik type modulus of continuity of f is defined by

$$\omega_{\varphi^\lambda}(f,t):=\sup_{0<h\le t}\ \sup_{x\pm\frac{h\varphi^\lambda(x)}{2}\in[0,1]}\left|f\left(x+\frac{h\varphi^\lambda(x)}{2}\right)-f\left(x-\frac{h\varphi^\lambda(x)}{2}\right)\right|,$$

where $\lambda\in[0,1]$ is a given number.

Our purpose in this paper is to improve Theorem 1 by using the Ditzian-Totik type modulus of continuity. In fact, we have the following pointwise approximation rate estimate described by using the Ditzian-Totik type modulus.

Theorem 1.4. *Let $f \in AC^m([0,1])$, $f^{(m)} \in L_\infty([0,1])$, $m = \lceil \alpha \rceil, \alpha > 0, \alpha \notin \mathbb{N}$.*
(i). If $0 < \alpha < 1$, $0 \le \lambda \le 1$, then

$$\left|B_n(f,x) - f(x)\right| \le C_\alpha \left(\frac{\varphi(x)}{\sqrt{n}}\right)^\alpha \left(\omega_{\varphi^\lambda}\left(D^\alpha_{x-}f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right) + \omega_{\varphi^\lambda}\left(D^\alpha_{*x}f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)\right). \tag{1}$$

(ii) If $1 < \alpha < 2$, $0 \le \lambda \le 1$, then

$$\left|B_n(f,x) - f(x)\right| \le C_\alpha \frac{\varphi(x)}{\sqrt{n}} \left(\frac{\delta_n(x)}{\sqrt{n}}\right)^{\alpha-1} \left(\omega_{\varphi^\lambda}\left(D^\alpha_{x-}f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right) + \omega_{\varphi^\lambda}\left(D^\alpha_{*x}f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)\right). \tag{2}$$

(iii) If $\alpha > 2$, $0 \le \lambda \le 1$, then

$$\left|B_n(f,x) - f(x) - \sum_{j=2}^{m-1} \frac{f^{(k)}(x)}{k!} B_n(|t-x|^j, x)\right|$$

$$\le C_\alpha \frac{\varphi(x)}{\sqrt{n}} \left(\frac{\delta_n(x)}{\sqrt{n}}\right)^{\alpha-1} \left(\omega_{\varphi^\lambda}\left(D^\alpha_{x-}f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right) + \omega_{\varphi^\lambda}\left(D^\alpha_{*x}f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)\right). \tag{3}$$

Corollary 1.5. *Let $f \in AC^m([0,1])$, $f^{(m)} \in L_\infty([0,1])$, $m = \lceil \alpha \rceil, \alpha > 2, \alpha \notin \mathbb{N}$. Then*

$$\begin{aligned}\left|B_n(f,x) - f(x)\right| \quad &\le \quad C \sum_{j=2}^{m-1} \frac{|f^{(k)}(x)|}{k!} \frac{\varphi(x)}{\sqrt{n}} \left(\frac{\delta_n(x)}{\sqrt{n}}\right)^{j-1} \\ &\quad + C_\alpha \frac{\varphi(x)}{\sqrt{n}} \left(\frac{\delta_n(x)}{\sqrt{n}}\right)^{\alpha-1} \left(\omega_{\varphi^\lambda}\left(D^\alpha_{x-}f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right) + \omega_{\varphi^\lambda}\left(D^\alpha_{*x}f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)\right).\end{aligned}$$

2. Some Auxiliary Lemmas

The following Lemma 1-4 are established in [2].

Lemma 2.1. *Let $\nu > 0, \nu \notin \mathbb{N}, n = \lceil \nu \rceil, f \in C^{n-1}([a,b])$ and $f^{(n)} \in L_\infty([a,b])$. Then $D^\nu_{*a} f(a) = 0$.*

Lemma 2.2. *Let $f \in C^{m-1}([a,b]), f^{(m)} \in L_\infty([a,b])$. Then $D^\alpha_{b-} f(b) = 0$.*

Lemma 2.3. *Let $f \in C^{m-1}([a,b]), f^{(m)} \in L_\infty([a,b]), m = \lceil \alpha \rceil, \alpha > 0$, and*

$$D^\alpha_{*x_0} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt$$

for all $x, x_0 \in [a,b]; x \ge x_0$.Then $D^\alpha_{*x_0} f(x)$ is continuous in x_0.

Lemma 2.4. *Let $f \in C^{m-1}([a,b]), f^{(m)} \in L_\infty([a,b]), m = \lceil \alpha \rceil, \alpha > 0$, and*

$$D^\alpha_{x_0-} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^{x_0} (t-x)^{m-\alpha-1} f^{(m)}(t)\, dt$$

for all $x, x_0 \in [a,b]; x \le x_0$.Then $D^\alpha_{x_0-} f(x)$ is continuous in x_0.

The following absolute moment estimates of Bernstein operators with fractal order play important roles in our proof of Theorem 2.

Lemma 2.5. *Let $\beta > 0$. Then,*

$$\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{\beta} p_{nk}(x) \le \left(\frac{\varphi(x)}{\sqrt{n}}\right)^{\beta} \tag{4}$$

$$\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{\beta+1} p_{nk}(x) \le C_{\beta} \left(\frac{\varphi(x)}{\sqrt{n}}\right)^{\beta} \frac{\delta_n(x)}{\sqrt{n}} \tag{5}$$

for $0 < \beta \le 1$, and

$$\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{\beta} p_{nk}(x) \le C_{\beta} \frac{\varphi(x)}{\sqrt{n}} \left(\frac{\delta_n(x)}{\sqrt{n}}\right)^{\beta-1} \tag{6}$$

for $\beta > 1$.

Proof. When $0 < \beta \le 1$, by Hölder's inequality, we have

$$\begin{aligned} \sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{\beta} p_{nk}(x) &\le \left(\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{2} p_{nk}(x)\right)^{\frac{\beta}{2}} \left(\sum_{k=0}^{n} p_{nk}(x)\right)^{\frac{2-\beta}{2}} \\ &= \left(\frac{\varphi(x)}{\sqrt{n}}\right)^{\beta}, \end{aligned}$$

which proves (4).

By Hölder's inequality again,

$$\begin{aligned} \sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{\beta+1} p_{nk}(x) &\le \left(\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{2} p_{nk}(x)\right)^{\frac{\beta}{2}} \left(\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{\frac{2}{2-\beta}} p_{nk}(x)\right)^{\frac{2-\beta}{2}} \\ &= \left(\frac{\varphi(x)}{\sqrt{n}}\right)^{\beta} \left(\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{\frac{2}{2-\beta}} p_{nk}(x)\right)^{\frac{2-\beta}{2}}. \end{aligned}$$

By using the inequality (3.8) in [7], we have

$$\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{\gamma} p_{nk}(x) \le C_{\gamma} \left(\frac{\delta_n(x)}{\sqrt{n}}\right)^{\gamma} \tag{7}$$

for all $\gamma > 0$, we have (4).

Similarly, when $\beta > 1$, we have by Hölder's inequality and (7) that

$$\begin{aligned} \sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{\beta} p_{nk}(x) &\le \left(\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{2} p_{nk}(x)\right)^{\frac{1}{2}} \left(\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{2(\beta-1)} p_{nk}(x)\right)^{\frac{1}{2}} \\ &= \frac{\varphi(x)}{\sqrt{n}} \left(\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{2(\beta-1)} p_{nk}(x)\right)^{\frac{1}{2}} \\ &\le C_{\beta} \frac{\varphi(x)}{\sqrt{n}} \left(\frac{\delta_n(x)}{\sqrt{n}}\right)^{(\beta-1)}, \end{aligned}$$

which proves (6). □

Obviously, our estimates on the absolute moments of Bernstein operators in Lemma 5 is better than the well known estimates in (7). We believe that Lemma 5 has its own great value, which may play important roles in approximation by Bernstein operators.

3. Proofs of the Results

3.1. *Proof of Theorem 2*

Proof of (i). Under the assumptions of theorem, by Lemma 1-4, both $D^{\alpha}_{x-}f$ and $D^{\alpha}_{*x}f$ are continuous on $[0, 1]$, and $D^{\alpha}_{x-}f(x) = D^{\alpha}_{*x}f(x) = 0$. From [5], we have by the left Caputo fractional Taylor formula that

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x - t)^{\alpha-1} D^{\alpha}_{*x_0} f(t)dt \tag{8}$$

for all $x_0 < x \le 1$.

Also from [1], using the right Caputo fractional Taylor formula, we have

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x}^{x_0} (t - x)^{\alpha-1} D^{\alpha}_{x_0-} f(t)dt \tag{9}$$

for all $0 \le x < x_0$. Therefore, with $m = 1$ in the case when $0 < \alpha < 1$, by (8) and (9), we deduce that

$$\begin{aligned} |B_n(f, x) - f(x)| &= \sum_{k=0}^{n} \left(f\left(\frac{k}{n}\right) - f(x) \right) p_{nk}(x) \\ &= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{[nx]} \left(\int_{\frac{k}{n}}^{x} \left(t - \frac{k}{n} \right)^{\alpha-1} (D^{\alpha}_{x-}f(t) - D^{\alpha}_{x-}f(x))dt \right) p_{nk}(x) \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{k=[nx]+1}^{n} \left(\int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - t \right)^{\alpha-1} (D^{\alpha}_{*x}f(t) - D^{\alpha}_{*x}f(x))dt \right) p_{nk}(x) \\ &=: I_1 + I_2. \end{aligned} \tag{10}$$

Define the K-functional:

$$K_{\varphi^{\lambda}}(f, t) := \inf_{g \in W_{\lambda}} \left(\|f - g\| + t\|\varphi^{\lambda} g'\| + t^{\frac{1}{1-\frac{\lambda}{2}}} \|g'\| \right),$$

where $W_{\lambda} := \{f : f \in AC_{loc},\ \|\varphi^{\lambda} f'\| < \infty,\ \|f'\| < \infty\}$. It is well knwon that (page 25, [6])

$$K_{\varphi^{\lambda}}(f, t) \sim \omega_{\varphi^{\lambda}}(f, t),\ 0 \le \lambda \le 1. \tag{11}$$

By (11), for any fixed n, x and λ, there is a $g(x) \in W_{\lambda}$ such that

$$\|D^{\alpha}_{x-}f - g\| \le C\omega_{\varphi^{\lambda}} \left(D^{\alpha}_{x-}f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}} \right), \tag{12}$$

$$\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}} \|\varphi^{\lambda} g'\| \le C\omega_{\varphi^{\lambda}} \left(D^{\alpha}_{x-}f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}} \right), \tag{13}$$

$$\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}} \right)^{\frac{1}{1-\frac{\lambda}{2}}} \|g'\| \le C\omega_{\varphi^{\lambda}} \left(D^{\alpha}_{x-}f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}} \right), \tag{14}$$

By (12) and (4), we have

$$\begin{aligned}
|I_1| &\leq \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{[nx]}\left(\int_{\frac{k}{n}}^{x}\left(t-\frac{k}{n}\right)^{\alpha-1}|D_{x-}^{\alpha}f(t)-g(t)+g(t)-g(x)+g(x)-D_{x-}^{\alpha}f(x)|dt\right)p_{nk}(x)\\
&\leq C_{\alpha}\sum_{k=0}^{[nx]}\left(\int_{\frac{k}{n}}^{x}\left(t-\frac{k}{n}\right)^{\alpha-1}\left(\omega_{\varphi^{\lambda}}\left(D_{x-}^{\alpha}f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)+|g(t)-g(x)|\right)dt\right)p_{nk}(x)\\
&\leq C_{\alpha}\omega_{\varphi^{\lambda}}\left(D_{x-}^{\alpha}f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)\sum_{k=0}^{[nx]}\left|x-\frac{k}{n}\right|^{\alpha}p_{nk}(x)\\
&\quad +C_{\alpha}\sum_{k=0}^{[nx]}\left(\int_{\frac{k}{n}}^{x}\left(t-\frac{k}{n}\right)^{\alpha-1}\int_{t}^{x}|g'(u)|dudt\right)p_{nk}(x)\\
&\leq C_{\alpha}\left(\frac{\varphi(x)}{\sqrt{n}}\right)^{\alpha}\omega_{\varphi^{\lambda}}\left(D_{x-}^{\alpha}f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)+C_{\alpha}\sum_{k=0}^{[nx]}\left(x-\frac{k}{n}\right)^{\alpha}\left(\int_{\frac{k}{n}}^{x}|g'(u)|du\right)p_{nk}(x). \qquad (15)
\end{aligned}$$

We further estimate I_1 by considering the following two cases.

Case 1. $x\in\left[0,\frac{1}{n}\right]\cup\left[1-\frac{1}{n},1\right]$. In this case, we have $\delta_n(x)\sim\frac{1}{\sqrt{n}}$. Then (note that $\delta_n(x)\leq C\delta_n\left(\frac{k}{n}\right)$)

$$\begin{aligned}
\sum_{k=0}^{[nx]}\left(x-\frac{k}{n}\right)^{\alpha}\left(\int_{\frac{k}{n}}^{x}|g'(u)|du\right)p_{nk}(x) &\leq \sum_{k=0}^{[nx]}\left(x-\frac{k}{n}\right)^{\alpha}\left(\int_{\frac{k}{n}}^{x}\frac{|\delta_n^{\lambda}(u)g'(u)|}{\delta_n^{\lambda}(u)}du\right)p_{nk}(x)\\
&\leq \|\delta_n^{\lambda}g'\|\sum_{k=0}^{[nx]}\left(x-\frac{k}{n}\right)^{\alpha}\left(\int_{\frac{k}{n}}^{x}\frac{1}{\delta_n^{\lambda}(u)}du\right)p_{nk}(x)\\
&\leq \|\delta_n^{\lambda}g'\|\sum_{k=0}^{[nx]}\left(x-\frac{k}{n}\right)^{\alpha+1}\left(\frac{1}{\delta_n^{\lambda}(x)}+\frac{1}{\delta_n\left(\frac{k}{n}\right)}\right)p_{nk}(x)\\
&\leq \frac{\|\delta_n^{\lambda}g'\|}{\delta_n^{\lambda}(x)}\sum_{k=0}^{[nx]}\left(x-\frac{k}{n}\right)^{\alpha+1}\left(1+\frac{\delta_n^{\lambda}(x)}{\delta_n\left(\frac{k}{n}\right)}\right)p_{nk}(x)\\
&\leq C\frac{\|\delta_n^{\lambda}g'\|}{\delta_n^{\lambda}(x)}\sum_{k=0}^{[nx]}\left(x-\frac{k}{n}\right)^{\alpha+1}p_{nk}(x)\\
&\leq C_{\alpha}\left(\frac{\varphi(x)}{\sqrt{n}}\right)^{\alpha}\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\delta_n^{\lambda}g'\|, \qquad (16)
\end{aligned}$$

where in the last inequality, (5) is applied. Now, by (13), (14), (16) and the following fact:

$$\begin{aligned}
\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\left\|\delta_n^{\lambda}g'\right\| &\leq C\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\left\|\varphi^{\lambda}g'\right\|+\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}\right)^{\lambda}\left\|g'\right\|\right)\\
&\leq C\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\left\|\varphi^{\lambda}g'\right\|+\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{1}{1-\frac{\lambda}{2}}}\left\|g'\right\|\right), \qquad (17)
\end{aligned}$$

we get

$$\sum_{k=0}^{[nx]}\left(x-\frac{k}{n}\right)^{\alpha}\left(\int_{\frac{k}{n}}^{x}|g'(u)|du\right)p_{nk}(x)\leq C_{\alpha}\left(\frac{\varphi(x)}{\sqrt{n}}\right)^{\alpha}\omega_{\varphi^{\lambda}}\left(D_{x-}^{\alpha}f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \qquad (18)$$

Case 2. $x\in\left(\frac{1}{n},1-\frac{1}{n}\right)$. In this case, we have $\delta_n(x)\sim\varphi(x)$. For any $t\in(0,1)$, we have

$$\begin{aligned}\left|\int_x^t \frac{1}{\varphi^\lambda(u)}du\right| &\le \left|\int_x^t \frac{1}{\sqrt{u(1-u)}}du\right|^\lambda |t-x|^{1-\lambda}\\ &\le |t-x|^{1-\lambda}\left|\int_x^t\left(\frac{1}{\sqrt{u}}+\frac{1}{\sqrt{1-u}}\right)du\right|^\lambda\\ &\le |t-x|^{1-\lambda}\left(2\left(\left|\sqrt{t}-\sqrt{x}\right|+\left|\sqrt{1-t}-\sqrt{1-x}\right|\right)\right)^\lambda\\ &\le 2^\lambda|t-x|\left(\frac{1}{\sqrt{t}+\sqrt{x}}+\frac{1}{\sqrt{1-t}+\sqrt{1-x}}\right)^\lambda\\ &\le 2^\lambda|t-x|\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{1-x}}\right)^\lambda\\ &\le C\frac{|t-x|}{\varphi^\lambda(x)}.\end{aligned}$$

Therefore, by (5) and (13), we have

$$\begin{aligned}\sum_{k=0}^{[nx]}\left(x-\frac{k}{n}\right)^\alpha\left(\int_{\frac{k}{n}}^x|g'(u)|du\right)p_{nk}(x) &\le \sum_{k=0}^{[nx]}\left(x-\frac{k}{n}\right)^\alpha\left(\int_{\frac{k}{n}}^x\frac{|\varphi^\lambda(u)g'(u)|}{\varphi^\lambda(u)}du\right)p_{nk}(x)\\ &\le C\frac{\|\varphi^\lambda g'\|}{\varphi^\lambda(x)}\sum_{k=0}^{[nx]}\left(x-\frac{k}{n}\right)^{\alpha+1}p_{nk}(x)\\ &\le C_\alpha\left(\frac{\varphi(x)}{\sqrt{n}}\right)^\alpha\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\varphi^\lambda g'\|\\ &\le C_\alpha\left(\frac{\varphi(x)}{\sqrt{n}}\right)^\alpha\omega_{\varphi^\lambda}\left(D_{x-}^\alpha f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \qquad (19)\end{aligned}$$

Combining (16), (18) and (19), we have

$$|I_1|\le C_\alpha\left(\frac{\varphi(x)}{\sqrt{n}}\right)^\alpha\omega_{\varphi^\lambda}\left(D_{x-}^\alpha f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \qquad (20)$$

Similarly, we can prove that

$$|I_2|\le C_\alpha\left(\frac{\varphi(x)}{\sqrt{n}}\right)^\alpha\omega_{\varphi^\lambda}\left(D_{*x}^\alpha f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \qquad (21)$$

Thus, (1) follows from (10), (20) and (21).

Proof of (ii) and (iii). When $\alpha>1$, by using the Caputo fractional Taylor formulas (8) and (9), and the fact that $B_n(t-x,x)\equiv 0$, we can prove (2) and (3) in a way similar to that of (1). We omit the details here.

3.2. *Proof of Corollary 1*

It follows from (6) that

$$B_n(|t-x|^j,x)\le C\frac{\varphi(x)}{\sqrt{n}}\left(\frac{\delta_n(x)}{\sqrt{n}}\right)^{j-1},\ j=2,3,\cdots;\ x\in[0,1].$$

Consequently, we see that Corollary 1 follows from (iii) in Theorem 2.

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