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Univariate Shepard operators combined with least squares fitting polynomials

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Abstract. We obtain new univariate Shepard operators using polynomials that are constructed such that they fit the interpolation data in a weighted least squares approximation way. We study the degree of exactness, the linearity and the remainder for the corresponding interpolation formula.

1. Introduction

D. Shepard introduced in 1968 in [13] a very powerful method for approximating a given function f on a set of scattered data, method that nowadays is named after him. The procedure has an easy implementation and it is expressed as a combination between some basis functions and the values of the function f on a given set of interpolation nodes. However, two of its major drawbacks are the high computational cost and the low degree of exactness. Several authors have studied them and proposed different solutions to overcome them, such as modifying the basis functions or combining the Shepard operator with other interpolation operators for an increased degree of exactness (see, e.g., [1–10]).

In the univariate case, when f is a real-valued function defined on a subset X of $\mathbb{R}$, for a given set of K interpolation nodes, $x_i \in X,\ i = 1, ..., K$, the Shepard operator is defined as

$$S_\mu f(x) = \sum_{i=1}^{K} A_{i,\mu}(x) \cdot f(x_i), \tag{1}$$

with the basis functions $A_{i,\mu}$ given by

$$A_{i,\mu}(x) = \frac{|x - x_i|^{-\mu}}{\sum\limits_{j=1}^{K} \left|x - x_j\right|^{-\mu}}, \quad i = 1, ..., K,\ x_i \neq x_j,\ \text{for } i \neq j,\ j = 1, ..., K, \tag{2}$$

$x \in X$ and $\mu > 0$ an arbitrary parameter.

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This paper focuses on introducing a new univariate Shepard operator, combined with polynomials constructed based on the least squares approach. In Section 2, after we construct these polynomials, we study several properties of them and of the combined Shepard operators (interpolation property, degree of exactness, linearity). Finally, we study the errors, based on Peano's Theorem. Section 3 is dedicated to numerical examples that show the benefits of these Shepard operators.

2. Shepard operators combined with polynomials constructed by the least squares method

R. J. Renka introduced in 1988 in [11] an algorithm for improving the bivariate Shepard operator, considering a quadratic polynomial that interpolates the function f on a set of given nodes and also approximates the data in a weighted least squares way. Later on, in 1999 in [12] he improved this method by replacing the quadratic polynomial with a cubic one. In 2010 in [14], W. I. Thacker et al. emphasized the main disadvantages of these two methods and suggested the combination of the Shepard operator with a linear polynomial that still fits the data in a weighted least squares sense.

Using some ideas for the bivariate case presented in the above mentioned papers, we are going to consider an improvement for the classical Shepard operator in the univariate case, by combining it with polynomials of degree $n,\ n \in \mathbb{N}$, constructed following the weighted least squares approximation technique.

Consider $X \subset \mathbb{R}$, $f : X \to \mathbb{R}$ and K given real nodes, denoted by x_j, $j = 1, ..., K$. The values of the function f on the given nodes are known and denoted by $f_j = f(x_j),\ j = 1, ..., K$.

Under these assumptions, for a point $x \in X$, let us define the nth degree polynomial function $C_j^n[f]$, $j = 1, ..., K$, $n \in \mathbb{N}$, as

$$C_j^n[f](x) = f_j + \sum_{k=1}^{n} a_{j,k}(x - x_j)^k, \tag{3}$$

where the coefficients $a_{j,k}$ are found such that they minimize the sum of the weighted squared residuals

$$E_j = \sum_{\substack{i=1\\ i\neq j}}^{K} \lambda_{i,j} \left[C_j^n[f](x_i) - f_i\right]^2, \tag{4}$$

where

$$\lambda_{i,j} = \frac{\left|x_i - x_j\right|^{-\mu}}{\sum\limits_{\substack{k=1\\ k\neq i}}^{K} |x_i - x_k|^{-\mu}}, \tag{5}$$

for $i, j = 1, ..., K$ and $\mu > 0$.

In order to find the coefficients $a_{j,k}$ (i.e, obtain the minimum of expression (4)), we follow the weighted least squares reasoning, take the partial derivatives of E_j with respect to each unknown, set them to zero and solve the resulting system:

$$\frac{\partial E_j}{\partial a_{j,k}} = 0,\ \text{for each } k = 1, ..., n \text{ and } j = 1, ..., K.$$

Further, for every $j = 1, ..., K$ one obtains

$$\frac{\partial E_j}{\partial a_{j,k}} = 2\sum_{\substack{i=1\\ i\neq j}}^{K} \lambda_{i,j} \left[\sum_{p=1}^{n} a_{j,p}(x_i - x_j)^p + (f_j - f_i)\right] \cdot (x_i - x_j)^k = 0,\ \text{for each } k = 1, ..., n.$$

Let us make the notation

$$x_{i,j}^{p}=\sum_{\substack{i=1\\ i\neq j}}^{K}\lambda_{i,j}\cdot(x_i-x_j)^p.$$

Then, the system of normal equations that has to be solved in order to find the coefficients $a_{j,k},\ k=1,...,n$, has the form

$$\begin{cases} a_{j,1}x_{i,j}^{2}+a_{j,2}x_{i,j}^{3}+...+a_{j,n}x_{i,j}^{n+1}=\sum\limits_{\substack{i=1\\ i\neq j}}^{K}\lambda_{i,j}\cdot(x_i-x_j)\cdot(f_i-f_j)\\ ...\\ a_{j,1}x_{i,j}^{k+1}+a_{j,2}x_{i,j}^{k+2}+...+a_{j,n}x_{i,j}^{k+n}=\sum\limits_{\substack{i=1\\ i\neq j}}^{K}\lambda_{i,j}\cdot(x_i-x_j)^k\cdot(f_i-f_j)\\ ...\\ a_{j,1}x_{i,j}^{n+1}+a_{j,2}x_{i,j}^{n+2}+...+a_{j,n}x_{i,j}^{2n}=\sum\limits_{\substack{i=1\\ i\neq j}}^{K}\lambda_{i,j}\cdot(x_i-x_j)^n\cdot(f_i-f_j), \end{cases} \tag{6}$$

for each $j=1,...,K$.

For every $j=1,...,K$, we can write the normal equations that appear above in a matricial form as

$$M_j\cdot a_j=b_j, \tag{7}$$

where M_j is a $n\times n$ matrix having on the entry (r,s) the element $\sum\limits_{\substack{i=1\\ i\neq j}}^{K}\lambda_{i,j}\cdot(x_i-x_j)^{r+s}$, b_j is a vector of n elements with $\sum\limits_{\substack{i=1\\ i\neq j}}^{K}\lambda_{i,j}\cdot(x_i-x_j)^k\cdot(f_i-f_j)$ on the kth entry and $a_j=\left(a_{j,1},\ a_{j,2},\ ...,a_{j,n}\right)^T$ is the vector of unknowns.

Theorem 2.1. *The operator $C_j^n[f]$ defined in (3) satisfies the following interpolation property*

$$C_j^n[f](x_j)=f_j,\ j=1,...,K.$$

Proof. For any $j=1,...,K$, one has

$$C_j^n[f](x_j)=f_j+\sum_{k=1}^{n}a_{j,k}(x_j-x_j)^k=f_j.$$

□

Theorem 2.2. *The operator $C_j^n[f],\ j=1,...,K$, has the degree of exactness n, i.e.,*

$$\mathrm{dex}(C_j^n[f])=n,\ j=1,...,K,$$

*where "*dex*" denotes the degree of exactness.*

Proof. For $x\in X$ we have the following cases for $C_j^n[f],\ j=1,...,K$:

Case 1. $f(x)=e_0(x)=x^0$. We get $a_{j,k}=0$, $k=1,...,n$, and obviously

$$C_j^n[e_0](x)=e_0(x_j)+0\cdot\sum_{k=1}^{n}(x-x_j)^k=1=e_0(x).$$

Case 2. $f(x) = e_n(x) = x^n$. We obtain the following solution for the coefficients $a_{j,k}$:

$$a_{j,k} = \binom{n}{n-k} x_j^{n-k}, \; k = 1, ..., n \tag{8}$$

and

$$\begin{aligned} C_j^n[e_n](x) &= e_n(x_j) + \sum_{k=1}^{n} \binom{n}{n-k} x_j^{n-k}(x - x_j)^k \\ &= \binom{n}{n} x_j^{n-0}(x - x_j)^0 + \sum_{k=1}^{n} \binom{n}{n-k} x_j^{n-k}(x - x_j)^k \\ &= \sum_{k=0}^{n} \binom{n}{k} x_j^{n-k}(x - x_j)^k = (x_j + x - x_j)^n = x^n = e_n(x). \end{aligned}$$

Case 3. $f(x) = e_p(x), \; p = 1, ..., n-1$. In this situation we have

$$a_{j,r} = \binom{p}{p-r} x_j^{p-r} \text{ for } r = 1, ..., p \text{ and } a_{j,s} = 0 \text{ for } s = p+1, ..., n$$

and

$$C_j^n[e_p](x) = e_p(x_j) + \sum_{r=1}^{p} \binom{p}{p-r} x_j^{p-r}(x - x_j)^r = x^p = e_p(x).$$

Case 4. $f(x) = e_{n+1}(x)$. It is obvious that $C_j^n[e_{n+1}](x) \neq x^{n+1}$ since

$$C_j^n[e_{n+1}](x) = e_{n+1}(x_j) + \sum_{k=1}^{n} a_{j,k}(x - x_j)^k$$

and no term x^{n+1} appears.

In conclusion, $\text{dex}(C_j^n[f]) = n, \; j = 1, ..., K$. □

Theorem 2.3. *The operator* $C_j^n[f]$ *is linear.*

Proof. We have to show that for $g_1, \; g_2 : X \to \mathbb{R}$, arbitrarily chosen, and $\alpha, \; \beta \in \mathbb{R}$, one has for $x \in X$:

$$C_j^n[\alpha g_1 + \beta g_2](x) = \alpha C_j^n[g_1](x) + \beta C_j^n[g_2](x), \; j = 1, ..., K. \tag{9}$$

Let us define the terms that appear in (9):

$$C_j^n[g_1](x) = g_1(x_j) + \sum_{k=1}^{n} a'_{j,k}(x - x_j)^k, \; j = 1, ..., K,$$

$$C_j^n[g_2](x) = g_2(x_j) + \sum_{k=1}^{n} a''_{j,k}(x - x_j)^k, \; j = 1, ..., K,$$

$$\begin{aligned} C_j^n[\alpha g_1 + \beta g_2](x) &= (\alpha g_1 + \beta g_2)(x_j) + \sum_{k=1}^{n} a_{j,k}(x - x_j)^k \\ &= \alpha g_1(x_j) + \beta g_2(x_j) + \sum_{k=1}^{n} a_{j,k}(x - x_j)^k, \; j = 1, ..., K. \end{aligned}$$

By solving similar systems as in (6), we obtain the following relation between the coefficients that appear above

$$a_{j,k} = \alpha a'_{j,k} + \beta a''_{j,k}, \text{ for every } k = 1, ..., n \text{ and } j = 1, ..., K.$$

Now, one has

$$\begin{aligned} C_j^n[\alpha g_1 + \beta g_2](x) &= \alpha g_1(x_j) + \beta g_2(x_j) + \sum_{k=1}^{n} (\alpha a'_{j,k} + \beta a''_{j,k})(x - x_j)^k \\ &= \alpha \left[g_1(x_j) + \sum_{k=1}^{n} a'_{j,k}(x - x_j)^k \right] + \beta \left[g_2(x_j) + \sum_{k=1}^{n} a''_{j,k}(x - x_j)^k \right] \\ &= \alpha C_j^n[g_1](x) + \beta C_j^n[g_2](x), \ j = 1, ..., K, \end{aligned}$$

and so (9) is proved. □

Definition 2.4. For $f : X \to \mathbb{R}$ and the set of K interpolation nodes, using the nth degree polynomial given in (3), we can define *the univariate Shepard operator combined with a nth degree polynomial* as

$$SP_n[f](x) = \sum_{j=1}^{K} A_{j,\mu}(x) \cdot C_j^n[f](x), \tag{10}$$

with $A_{j,\mu}$ defined in (2) using the given parameter $\mu > 0$.

Theorem 2.5. *The following interpolation property holds*

$$SP_n[f](x_j) = f(x_j), \quad j = 1, ..., K.$$

Proof. It follows from Theorem 2.1 and the fact that for $A_{j,\mu}$ given in (2), we have $A_{j,\mu}(x_i) = \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}. \tag{11}$$

□

Theorem 2.6. *The operator SP_n is linear.*

Proof. For $g_1, \ g_2 : X \to \mathbb{R}$ arbitrarily chosen and $\alpha, \ \beta \in \mathbb{R}$, using the linearity of C_j^n showed in Theorem 2.3, one has

$$\begin{aligned} SP_n[\alpha g_1 + \beta g_2](x) &= \sum_{j=1}^{K} A_{j,\mu}(x) \cdot C_j^n[\alpha g_1 + \beta g_2](x) \\ &= \sum_{j=1}^{K} A_{j,\mu}(x) \cdot \left[\alpha C_j^n[g_1](x) + \beta C_j^n[g_2](x) \right] \\ &= \alpha \sum_{j=1}^{K} A_{j,\mu}(x) \cdot C_j^n[g_1](x) + \beta \sum_{j=1}^{K} A_{j,\mu}(x) \cdot C_j^n[g_2](x) \\ &= \alpha SP_n[g_1](x) + \beta SP_n[g_2](x), \end{aligned}$$

the linearity of SP_n being proved. □

Theorem 2.7. *The Shepard operator SP_n has degree of exactness n.*

Proof. We know that for some arbitrary operators $R_i,\ i = 1, ..., K$, with $\text{dex}(R_i) = r_i,\ i = 1, ..., K$, we have $\text{dex}(S_R) = \min\{r_1, ..., r_K\}$, where

$$S_R f(x) = \sum_{i=1}^{K} A_{i,\mu}(x) \cdot R_i(x).$$

Taking into account this property and the fact that $\text{dex}(C_j^n) = n,\ \forall j = 1, ..., K$, we obtain the desired conclusion. □

We now introduce the interpolation formula for the univariate Shepard combined with a polynomial, that is given by

$$f = SP_n[f] + R_n[f],$$

with $R_n[f]$ denoting the remainder.

Considering the space $H^m[a, b]$, $m \in \mathbb{N} \setminus \{0\}$ of functions $f \in C^{m-1}[a, b]$ (continuously differentiable up to order $m - 1$, inclusively) with $f^{(m-1)}$ absolutely continuous on $[a, b]$, we obtain the following result for the remainder of the formula:

Theorem 2.8. *If $f \in H^{n+1}[a, b]$, then*

$$R_n[f](x) = \int_a^b \phi_n(x, t) \cdot f^{(n+1)}(t)\, dt,$$

where

$$\phi_n(x, t) = \frac{(x - t)_+^n}{n!} - \sum_{j=1}^{K} A_{j,\mu}(x) \cdot \left[\frac{(x_j - t)_+^n}{n!} + \sum_{k=1}^{n} a_{j,k}(x - x_j)^k \right], \tag{12}$$

with $a_{j,k}$ given as solutions of $\frac{\partial E_j}{\partial a_{j,k}} = 0$, for each $k = 1, ..., n$, for

$$E_j = \sum_{\substack{i=1 \\ i \neq j}}^{K} \lambda_{i,j} \left[\frac{(x_j - t)_+^n}{n!} + \sum_{k=1}^{n} a_{j,k}(x_i - x_j)^k - \frac{(x_i - t)_+^n}{n!} \right]^2$$

and $\lambda_{i,j}$ given in (5), $j = 1, ..., K$.

Proof. The degree of exactness for the Shepard operator SP_n is n. Using now the Peano's theorem, one gets

$$R_n[f](x) = \int_a^b \phi_n(x, t) \cdot f^{(n+1)}(t)\, \mathrm{d}t,$$

with

$$\phi_n(\cdot, t) = R_n \left[\frac{(\cdot - t)_+^n}{n!} \right] = \frac{(\cdot - t)_+^n}{n!} - \sum_{j=1}^{K} A_{j,\mu}(\cdot) \cdot C_j^n \left[\frac{(\cdot - t)_+^n}{n!} \right].$$

Finally, for all $x \in [a, b]$, we obtain

$$\phi_n(x, t) = \frac{(x - t)_+^n}{n!} - \sum_{j=1}^{K} A_{j,\mu}(x) \cdot \left[\frac{(x_j - t)_+^n}{n!} + \sum_{k=1}^{n} a_{j,k}(x - x_j)^k \right],$$

where $a_{j,k}$ are solutions of $\frac{\partial E_j}{\partial a_{j,k}} = 0$, for each $k = 1, ..., n$, for

$$E_j = \sum_{\substack{i=1 \\ i \neq j}}^{K} \lambda_{i,j} \left[\frac{(x_j - t)_+^n}{n!} + \sum_{k=1}^{n} a_{j,k}(x_i - x_j)^k - \frac{(x_i - t)_+^n}{n!} \right]^2$$

concluding in this way the proof. □

3. Test results

For the numerical experiments we consider four well-known real-valued test functions (see, e.g., [9]):

$$\begin{aligned} &\text{Cliff:} && f_1(x) = \tfrac{1}{2}\tanh(-9x+1) + 0.5, \\ &\text{Gentle:} && f_2(x) = \tfrac{1}{3}\exp[-\tfrac{81}{16}(x-0.5)^2], \\ &\text{Saddle:} && f_3(x) = \tfrac{1.25}{6+6(3x-1)^2}, \\ &\text{Steep:} && f_4(x) = \tfrac{1}{3}\exp[-\tfrac{81}{4}(x-0.5)^2]. \end{aligned} \tag{13}$$

For each function $f_i,\ i = 1, ..., 4$, we compare the test results obtained by considering the linear, quadratic and cubic interpolants $SP_j[f_i],\ j = 1, 2, 3$, with the ones obtained for some other combined Shepard operators, well-known in the literature. We study the maximum approximation errors for the Shepard operators of Lagrange, Taylor and Bernoulli type, for all of them considering the first, second and third order. They are denoted by SL_k, ST_k and SB_k, $k = 1, 2, 3$, respectively. We recall their definitions in the sequel.

3.1. *Univariate Shepard-Lagrange operator (see, e.g., [5])*

For K distinct points x_i that belong to $X \subset \mathbb{R}$ and the real-valued function f defined on X such that the data $f(x_i),\ i \in \{1, \ldots, K\}$ are known, the univariate Shepard-Lagrange operator is defined as

$$SL_k[f](x) = \sum_{j=1}^{K} A_{j,\mu}(x) \cdot L_k^j[f](x), \tag{14}$$

with

$$L_k^j[f](x) = \sum_{i=0}^{k} \frac{\prod\limits_{\alpha=0,\ \alpha\neq i}^{k} (x - x_{j+\alpha})}{\prod\limits_{\alpha=0,\ \alpha\neq i}^{k} (x_{j+i} - x_{j+\alpha})} \cdot f(x_{j+i}), \tag{15}$$

$x_{K+i} = x_{K-i},\ i = 1, ..., k$, and $A_{j,\mu}$ defined in (2), $\mu > 0$.

3.2. *Univariate Shepard-Taylor operator (see, e.g., [5])*

For $f : X \to \mathbb{R}$ and K distinct interpolation nodes $x_j,\ j \in \{1, \ldots, K\}$, consider the sets

$$\Delta = \{\eta_{j,i} \mid \eta_{j,i}(f) = f^{(i)}(x_j) \text{ with } j = 1, \ldots, K;\ i = 0, \ldots, k;\ k \in \mathbb{N}^*\}$$

and

$$\Delta_j(f) = \{\eta_{j,p} \mid p = 0, \ldots, k\}$$

such that $\Delta_j \subset \Delta$ is a subset of Δ associated to η_j, having $\eta_j \in \Delta_j$, for all $j = 1, ..., K$.

Then, the univariate Shepard-Taylor operator is defined as

$$ST_k[f](x) = \sum_{j=1}^{K} A_{j,\mu}(x) \cdot T_k^j[f](x), \tag{16}$$

with

$$T_k^j[f](x) = \sum_{i=0}^{k} \frac{(x - x_j)^i}{i!} \cdot f^{(i)}(x_j) \tag{17}$$

and $A_{j,\mu}$ defined in (2), $\mu > 0$.

3.3. Univariate Shepard-Bernoulli operator (see, e.g., [6])

Suppose there are K given points x_j in $X \subset \mathbb{R}$ and $x_{K+1} = x_{K-1}$. Then, we can define the univariate Shepard-Bernoulli operator as follows

$$SB_k[f](x) = \sum_{j=1}^{K} A_{j,\mu}(x) \cdot B_k[f; x_j, x_{j+1}](x), \tag{18}$$

with $A_{j,\mu}$ defined in (2) and the Bernoulli operators B_k given by

$$B_k[f; a, b] = f(a) + \sum_{j=1}^{k} \frac{h^{j-1}}{j!} \cdot \left(B_j\left(\frac{x-a}{h}\right) - B_j \right) \cdot \left(f^{(j-1)}(a) - f^{(j-1)}(b) \right)$$

for $f \in C^k[a, b],\ k \geq 1, h = b - a$.

B_n are the Bernoulli numbers, i.e. the values of the Bernoulli polynomials $B_n(x)$ at $x = 0$. The Bernoulli polynomials are defined recursively as

$$\begin{cases} B_0(x) = 1, \\ B_n'(x) = nB_{n-1}(x),\ n \geq 1, \\ \int_0^1 B_n(x)\, dx = 0,\ n \geq 1. \end{cases}$$

We consider a set of $K = 50$ equally spaced interpolation nodes from the interval $X = [0, 1]$ and set the μ parameter's value to 2. Table 1 presents the maximum interpolation errors for the classical Shepard operator $S_\mu f$ introduced in (1) and the linear SP_1, quadratic SP_2 and cubic SP_3 Shepard operators, introduced in (10) for $n = 1, 2, 3$, respectively. In addition, we present the maximum approximation errors for the Shepard operators of Lagrange, Taylor and Bernoulli type, of order 1, 2 and 3, respectively. We can observe that the three new operators produce better approximation results than the classical Shepard operator. Moreover, as it was expected, higher degrees polynomials produce smaller approximation errors. We can observe that in the linear and quadratic cases, the approximation results for the Shepard operators combined with least squares fitting polynomials are close to the best approximation results for most of the functions. In the cubic cases the new Shepard operators obtained produce the smallest interpolation errors.

	f_1	f_2	f_3	f_4
$S_\mu f$	0.0247	0.0043	0.0024	0.0084
SP_1	0.0157	0.0025	0.0024	0.0041
SL_1	0.0081	0.0020	0.0013	0.0030
ST_1	0.0067	0.0020	0.0012	0.0027
SB_1	0.0081	0.0020	0.0013	0.0030
SP_2	0.0066	0.0012	8.7983e-04	0.0041
SL_2	0.0048	9.4582e-04	0.0014	0.0027
ST_2	0.0050	0.0010	7.7720e-04	0.0026
SB_2	0.0048	0.0010	7.7771e-04	0.0027
SP_3	0.0053	5.4032e-04	6.7304e-04	0.0023
SL_3	0.0671	0.0050	0.0049	0.0024
ST_3	0.0094	8.7148e-04	8.7393e-04	0.0024
SB_3	0.0104	8.8533e-04	8.7689e-04	0.0024

Table 1: Maximum approximation errors, 50 equidistant nodes.

We also test these operators on a second set of $K = 50$ Chebyshev nodes, defined as

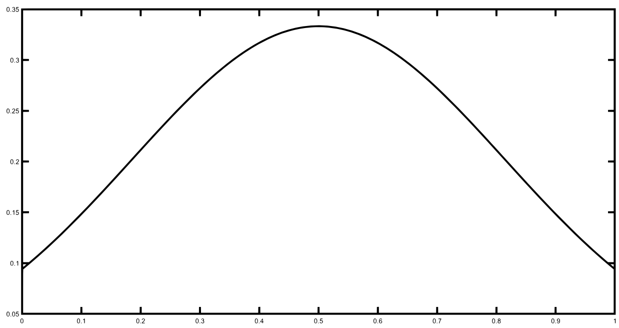
$$x_j = \frac{1}{2} + \frac{1}{2} \cos\left(\frac{2j-1}{2K}\pi\right),\ j = 1, ..., K.$$

We present the maximum approximation errors in Table 2. In this case we can see that the new operators produce the best results in the quadratic case for all the functions, except for f_4. In the cubic case they produce the smallest interpolation errors for all test functions. Very good results are obtained in the linear case as well.

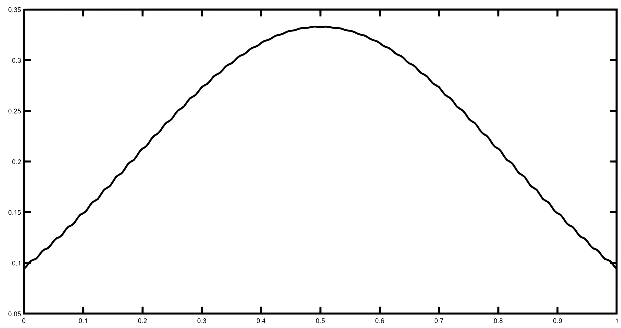
	f_1	f_2	f_3	f_4
$S_\mu f$	0.0246	0.0064	0.0046	0.0160
SP_1	0.0119	0.0018	0.0019	0.0066
SL_1	0.0078	0.0034	0.0016	0.0031
ST_1	0.0094	0.0035	0.0017	0.0021
SB_1	0.0083	0.0035	0.0016	0.0031
SP_2	0.0054	9.3156e-04	0.0011	0.0065
SL_2	0.0119	0.0027	0.0013	0.0040
ST_2	0.0139	0.0029	0.0015	0.0037
SB_2	0.0133	0.0029	0.0015	0.0038
SP_3	0.0046	2.6850e-04	7.0098e-04	0.0027
SL_3	0.0070	7.3503e-04	7.9055e-04	0.0051
ST_3	0.0089	9.3807e-04	9.8040e-04	0.0054
SB_3	0.0082	9.3521e-04	9.8137e-04	0.0054

Table 2: Maximum approximation errors, 50 Chebyshev nodes.

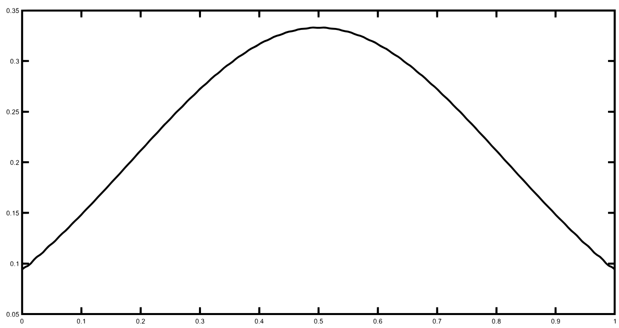
Finally, we present the graphical results for the Gentle and the Saddle functions using the set of 50 equally spaced nodes. Figures 1–2 illustrates the functions f_2 and f_3 and their corresponding polynomial Shepard interpolants SP_1, SP_2 and SP_3.



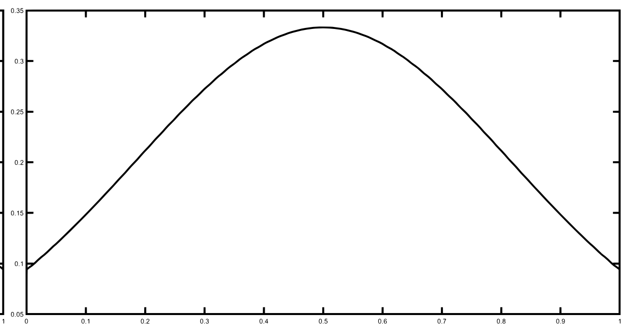
(a) Function f_2.



(b) Interpolant $SP_1[f_2]$.

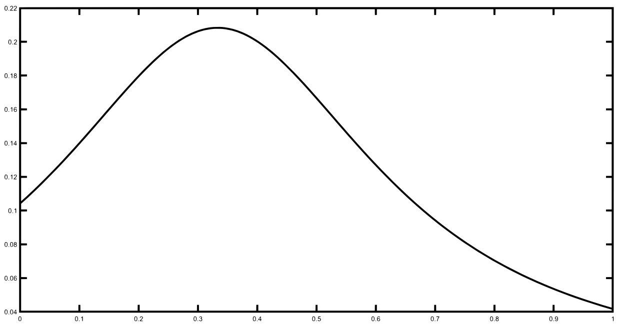


(c) Interpolant $SP_2[f_2]$.

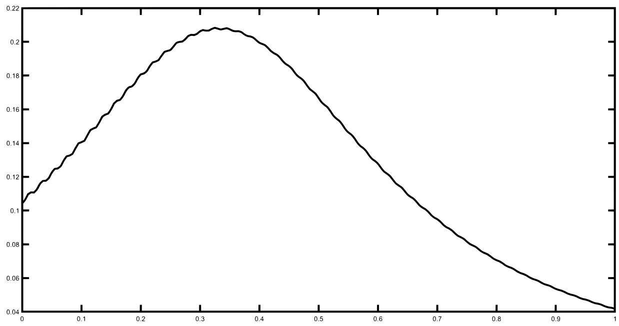


(d) Interpolant $SP_3[f_2]$.

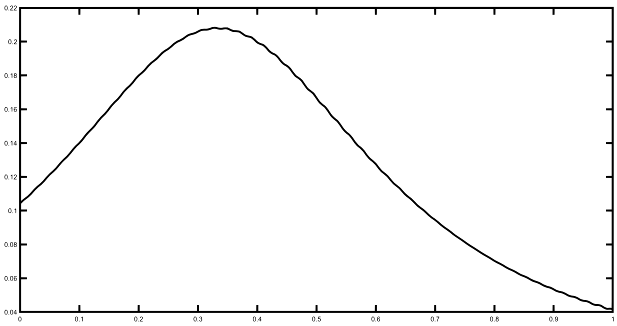
Figure 1: Graphs for the Gentle function f_2.



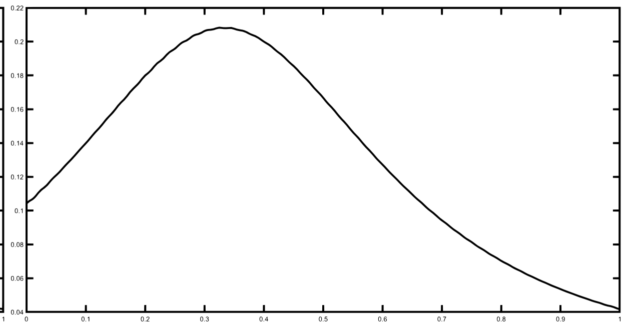
(a) Function f_3.



(b) Interpolant $SP_1[f_3]$.



(c) Interpolant $SP_2[f_3]$.



(d) Interpolant $SP_3[f_3]$.

Figure 2: Graphs for the Saddle function f_3.

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