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On mixed multiset ideal topological spaces

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Abstract. The concept of mixed multiset topology was introduced and investigated by different researchers from different aspects. In this paper, we introduce the notion of mixed multiset ideal topological space. Further, we define the concepts of $\tau_1(\tau_2)$-pre-$\mathcal{I}$-open mset, $\tau_1(\tau_2)$-semi-$\mathcal{I}$-open mset, $\tau_1(\tau_2)$-α-$\mathcal{I}$-open mset and $\tau_1(\tau_2)$-δ-$\mathcal{I}$-open mset in mixed multiset ideal topological space. We investigate on these generalized open multisets.

1. Introduction

Over the last five decades, different concepts of topological space have been developed and expanded in several ways. Among these, two major developments are the notions of mixed topology and bitopology. Mixed topology is a technique of mixing two topologies on a set to get a third topology, which lies in the theory of strict topology. Mixed topology in the context of multiset has recently been studied by Shravan and Tripathy [26], Ray and Dey [23]. For a comprehensive study on mixed topology, one may refer to [3–5, 22, 29, 30, 32]. A collection of objects that may appear more than once is referred to as a multiset (briefly, mset). These objects are called the elements of such collection. A multiset is characterized by a count function, which maps every element of a set from which a multiset is drawn to a non-negative integer and that describes how frequently it appears in a multiset. The notion of multiset is considered when repeated elements are significant, such as data analysis, probability theory and algorithms involving item frequencies. Blizard [1, 2] provided an excellent overview of the literature on multiset theories. The basic properties of multiset can be found in [11, 12, 17, 33]. Girish and John [12] established the notion of multiset topology. Thereafter, multiset topological spaces have been carried out by several researchers [10, 13, 15, 21, 24, 28]. The authors [7–9] applied the notions of multiset and multiset topology in deoxyribonucleic acid (DNA) and ribonucleic acid (RNA) mutations. By extending the notion of multiset in fuzzy environment, Hoque et al. [14] studied the concept of fuzzy multiset topological space. Unlike in the case of general topology, in multiset topology we can define two subspace M-topologies on a submset in terms of open and closed msets [18]. Also, these two subspaces do not behave like similar concepts in general topology and thus, many results in multiset topology vary from general topology via subspace topology. By applying this concept, Kumar and John [18] defined two types of connectedness and Kumar et al. [19] defined two types of compactness in multiset topology.

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The concept of an ideal topological spaces has been studied by several researchers. We may refer to [6, 16, 20, 31]. Zakaria et al. [34] presented some of the ideal concepts in the multiset trend. Thereafter, it was carried out by Shravan and Tripathy [25, 27]. On the other hand, in a mixed multiset topological space, every $\tau_1(\tau_2)$-open mset is τ_2-open mset (Theorem 3.10, Ray and Dey [23]). This result certainly plays a connection between parent topologies and mixed topology. In this paper, we present the notion of mixed multiset ideal topological space and study some of its basic properties. Thereafter, we define the notions of $\tau_1(\tau_2)$-pre-$\mathcal{I}$-open mset, $\tau_1(\tau_2)$-semi-$\mathcal{I}$-open mset, $\tau_1(\tau_2)$-α-$\mathcal{I}$-open mset, $\tau_1(\tau_2)$-δ-$\mathcal{I}$-open mset and investigate some of their properties in mixed multiset ideal topological space. We prove that the notions $\tau_1(\tau_2)$-pre-$\mathcal{I}$-open mset and τ_2-pre-$\mathcal{I}$-open mset; $\tau_1(\tau_2)$-semi-$\mathcal{I}$-open mset and τ_2-semi-$\mathcal{I}$-open mset; $\tau_1(\tau_2)$-α-$\mathcal{I}$-open mset and τ_2-α-$\mathcal{I}$-open mset are completely independent. We also prove that a submset is $\tau_1(\tau_2)$-α-$\mathcal{I}$-open if and only if it is both $\tau_1(\tau_2)$-δ-$\mathcal{I}$-open and $\tau_1(\tau_2)$-pre-$\mathcal{I}$-open.

2. Preliminaries

We now recall some important definitions and results for the developing of this article.

Definition 2.1. *[33] Let X be a base set. An mset M drawn from X is characterized by a function Count M or C_M, is defined by $C_M : X \longrightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of all non-negative integers.*

Here, $C_M(x)$ denote the multiplicity of the element x in the mset M. If $X = \{a_1, a_2, ..., a_k\}$ and multiplicity of a_i is r_i, then an mset M is represented by $M = \{r_1/a_1, r_2/a_2, ..., r_k/a_k\}$.

Remark 2.2. *(i) If $C_M(x) = 0$ for some $x \in X$, those elements will not be considered in the mset M.*
(ii) If $C_M(x) = 1$ for every $x \in X$, then M becomes a crisp set. Therefore, an mset is the generalization of the crisp set.

Definition 2.3. *[33] Let X be a base set. The family of all msets drawn from X is denoted by $[X]^\omega$, where ω is the highest multiplicity of an element in an mset. On the other hand, $[X]^\infty$ means the family of all msets with there is no restriction on the multiplicity.*
If $X = \{a_1, a_2, ..., a_k\}$, then
$[X]^\omega = \{\{r_1/a_1, r_2/a_2, ..., r_k/a_k\} : \text{for } i = 1, 2, ..., k; r_i \in \{0, 1, 2, ..., \omega\}\}$.
Let $K, L \in [X]^\omega$. Then,
(i) $K = L$ if $C_K(x) = C_L(x)$, $\forall x \in X$.
(ii) $K \subseteq L$ if $C_K(x) \leq C_L(x)$, $\forall x \in X$.
(iii) $W = K \cup L$ if $C_W(x) = max\{C_K(x), C_L(x)\}$, $\forall x \in X$.
(iv) $W = K \cap L$ if $C_W(x) = min\{C_K(x), C_L(x)\}$, $\forall x \in X$.
(v) $W = K \ominus L$ if $C_W(x) = max\{C_K(x) - C_L(x), 0\}$, $\forall x \in X$.

Definition 2.4. *[17] Let $M \in [X]^\omega$. Then the complement M^c of M is defined by $C_{M^c}(x) = \omega - C_M(x)$, for all $x \in X$.*

Definition 2.5. *[12] (i) A submset K of M is called a whole submset of M if $C_K(x) = C_M(x)$, $\forall x \in K^*$.*
(ii) A submset K of M is called a partial whole submset of M if $C_K(x) = C_M(x)$, for some $x \in K^*$.
(iii) A submset K of M is called a full submset of M if $K^* = M^*$ with $C_K(x) \leq C_M(x)$, $\forall x \in K^*$.

Definition 2.6. *[12] Let $M \in [X]^\omega$ and $\tau \subseteq P^*(M)$. Then τ is called a multiset topology (M-topology) on M if τ satisfies the following properties:*
(i) $\emptyset, M$ in τ;
(ii) $N_1, N_2 \in \tau \Rightarrow N_1 \cap N_2 \in \tau$;
(iii) $\cup_{\gamma\in\Lambda} N_\gamma \in \tau$ for every $\{N_\gamma : \gamma \in \Lambda\} \subseteq \tau$.
The pair (M, τ) is called multiset topological space.

Definition 2.7. *[12] Let (M, τ) be an M-topological space. Let A be a submset of M. The intersection of all closed msets containing A is defined as the closure of an mset A, and is denoted by cl(A), i.e., $cl(A) = \cap\{K \subseteq M : K$ is a closed mset and $A \subseteq K\}$ and $C_{cl(A)}(x) = min\{C_K(x) : A \subseteq K\}$.*

Definition 2.8. *[12] Let (M, τ) be an M-topological space. Let A be a submset of M. The union of all open msets contained in A is defined as the interior of an mset A, and is denoted by $int(A)$, i.e., $int(A) = \cup\{K \subseteq M : K$ is an open mset and $K \subseteq A\}$ and $C_{int(A)}(x) = max\{C_K(x) : K \subseteq A\}$.*

Definition 2.9. *[26] An mset K is said to be quasi-coincident with L, i.e., KqL at x iff $C_K(x) > C_{L^c}(x)$.*

Definition 2.10. *[26] An mset N in an M-topological space (M, τ) is said to be q-neighbourhood (q-nbd) of r/a if and only if there exists an open mset W such that $r/aqW \subset N$. The set of all q-nbd of r/a will be denoted by $\mathcal{N}(r/a)$*

Definition 2.11. *[34] A non-empty family $\mathcal{I}$ of submsets of an mset M is said to be an mset ideal on M, if*
(i) $K \in \mathcal{I}$ and $C_L(x) \leq C_K(x)$, $\forall x \in X \Rightarrow L \in \mathcal{I}$;
(ii) $K \in \mathcal{I}$ and $L \in \mathcal{I} \Rightarrow K \cup L \in \mathcal{I}$.

Theorem 2.12. *[23] Let (M, τ_1) and (M, τ_2) be two M-topological spaces on M. Let $\tau_1(\tau_2) = \{A \subseteq M :$ for any submset B of M with BqA, there exists a τ_2-open mset C such that BqC and τ_1-closure $cl(C) \subseteq A\}$.*
The collection $\tau_1(\tau_2)$ forms an M-topology on M and this topology $\tau_1(\tau_2)$ is called mixed multiset topology (mixed M-topology) on M and the pair $(M, \tau_1(\tau_2))$ as mixed multiset topological space (briefly, MMTS).

Theorem 2.13. *[23] Let (M, τ_1) and (M, τ_2) be two M-topological spaces on M. Then the mixed M-topology $\tau_1(\tau_2)$ is coarser than τ_2, i.e., $\tau_1(\tau_2) \subseteq \tau_2$.*

The $\tau_1(\tau_2)$-closure and $\tau_1(\tau_2)$-interior of an mset in a mixed multiset topology are defined in the similar way to the closure and interior in multiset context. If N be a submset of M, then $\tau_1(\tau_2)cl(N)$ and $\tau_1(\tau_2)int(N)$, respectively, denote the mset closure and mset interior of N in a mixed multiset topological space $(M, \tau_1(\tau_2))$. Every member of $\tau_1(\tau_2)$ will be called a $\tau_1(\tau_2)$-open mset. An mset is $\tau_1(\tau_2)$-closed if and only if its complement is $\tau_1(\tau_2)$-open.

3. Main results

In this section, we introduce the notion of mixed multiset ideal topological space.

Definition 3.1. *Let $(M, \tau_1(\tau_2))$ be an MMTS, and $\mathcal{I}$ be an mset ideal on M. Then the triplet $(M, \tau_1(\tau_2), \mathcal{I})$ is called a mixed multiset ideal topological space (briefly, MMITS).*

Example 3.2. *Let $X = \{p, q, r\}$ and $M = \{2/p, 3/q, 1/r\} \in [X]^3$. Consider two M-topologies τ_1 and τ_2 on M defined by:*
$\tau_1 = \{M, \emptyset, \{2/p\}, \{3/q\}, \{2/p, 3/q\}\}$, and
$\tau_2 = \{M, \emptyset, \{2/p\}, \{3/q\}, \{1/r\}, \{2/p, 3/q\}, \{3/q, 1/r\}, \{2/p, 1/r\}\}$.
By Theorem 2.12, we obtain $\tau_1(\tau_2) = \{M, \emptyset, \{1/r\}, \{3/q, 1/r\}, \{2/p, 1/r\}\}$.
Let $\mathcal{I} = \{\emptyset, M\}$, then $\mathcal{I}$ is an mset ideal on M. Thus, $(M, \tau_1(\tau_2), \mathcal{I})$ is an MMITS.

Example 3.3. *Let $X = \{p, q, r\}$ and $M = \{3/p, 4/q, 5/r\} \in [X]^5$. Consider two M-topologies τ_1 and τ_2 on M defined by:*
$\tau_1 = \{M, \emptyset, \{3/p\}, \{4/q\}, \{3/p, 4/q\}\}$, and
$\tau_2 = \{M, \emptyset, \{3/p\}, \{5/r\}, \{3/p, 5/r\}\}$.
By Theorem 2.12, we obtain $\tau_1(\tau_2) = \{M, \emptyset, \{5/r\}, \{3/p, 5/r\}\}$.
Let $\mathcal{I} = \{\emptyset, \{2/p, 2/q, 3/r\}\}$, then $\mathcal{I}$ is an mset ideal on M. Thus, $(M, \tau_1(\tau_2), \mathcal{I})$ is an MMITS.

Definition 3.4. *Let $(M, \tau_1(\tau_2))$ be an MMTS with mset ideal $\mathcal{I}$ on M. Then the mset local function $N^{**}(\tau_1(\tau_2), \mathcal{I})$ of N is the union of all multipoints r_i/a_i such that if $W \in \mathcal{N}(r_i/a_i)$ and $I \in \mathcal{I}$ then there is at least one $a_j \in X$ such that $C_W(a_j) - C_{N^c}(a_j) > C_I(a_j)$. Sometimes we shall write N^{**} or $N^{**}(\mathcal{I})$ for $N^{**}(\tau_1(\tau_2), \mathcal{I})$.*

Remark 3.5. *The class of mset local functions with respect to $\tau_1(\tau_2)$ contains the class of mset local functions with respect to τ_2 in the sense of Shravan and Tripathy [25], that is $N^*(\tau_2, \mathcal{I}) \subseteq N^{**}(\tau_1(\tau_2), \mathcal{I})$ for every submset N of M and the converse need not be true in general as shown in the example below.*

Example 3.6. *Consider the MMTS $(M, \tau_1(\tau_2))$ as shown in Example 3.2. Let $\mathcal{I} = \{\emptyset\}$. Then it is an mset ideal in M. Let $N = \{2/p, 1/r\}$. Then one can deduce that $N^{**}(\tau_1(\tau_2), \mathcal{I}) = M$ and $N^{*}(\tau_2, \mathcal{I}) = N$. Clearly $N^{**}(\tau_1(\tau_2), \mathcal{I}) \nsubseteq N^{*}(\tau_2, \mathcal{I})$.*

The following result gives the relation between mset interior and mset closure of an mset via mixed M-topology and M-topology, which can be obtained from Theorem 2.13.

Remark 3.7. *$\tau_1(\tau_2)int(N) \subseteq \tau_2 int(N) \subseteq N \subseteq \tau_2 cl(N) \subseteq \tau_1(\tau_2)cl(N)$ for every submset N of M.*

Lemma 3.8. *In an MMITS $(M, \tau_1(\tau_2), \mathcal{I})$, if $\mathcal{I} = \{\emptyset\}$ then $N^{**}(\mathcal{I}) = \tau_1(\tau_2)cl(N)$ for any submset N of M.*

Proof. Suppose that $\mathcal{I} = \{\emptyset\}$. Then by Definition 3.4, we have
$N^{**}(\tau_1(\tau_2), \mathcal{I}) = \cup\{r_i/a_i \in M : C_W(a_j) - C_{N^c}(a_j) > 0, I \in \mathcal{I}, \forall W \in \mathcal{N}(r_i/a_i)$ with at least $a_j \in X\}$
$= \cup\{r_i/a_i \in M : C_W(a_j) > C_{N^c}(a_j), I \in \mathcal{I}, \forall W \in \mathcal{N}(r_i/a_i)$ with at least $a_j \in X\} = \tau_1(\tau_2)cl(N)$. □

Lemma 3.9. *In an MMITS $(M, \tau_1(\tau_2), \mathcal{I})$, if $\mathcal{I} = P^*(M)$ then $N^{**}(\mathcal{I}) = \emptyset$ for any submset N of M.*

Proof. Since for every multipoint r_i/a_i of M, there is at least one $W \in \mathcal{N}(r_i/a_i)$ such that for every $a_j \in X$, $C_W(a_j) - C_{N^c}(a_j) \leq C_I(a_j)$ for $I = P^*(M)$, we have $N^{**}(\mathcal{I}) = \emptyset$. □

The following theorem gives some basic properties of mset local function.

Theorem 3.10. *Let $(M, \tau_1(\tau_2))$ be an MMTS, and $\mathcal{I}, \mathcal{J}$ be any two mset ideals on M. Then for any two submsets N, P of $(M, \tau_1(\tau_2))$, the following results hold:*
(i) $N \subseteq P \Rightarrow N^{**}(\mathcal{I}) \subseteq P^{**}(\mathcal{I})$.
(ii) $\mathcal{I} \subseteq \mathcal{J} \Rightarrow N^{**}(\mathcal{J}) \subseteq N^{**}(\mathcal{I})$.
(iii) $(N \cup P)^{**}(\mathcal{I}) = N^{**}(\mathcal{I}) \cup P^{**}(\mathcal{I})$.
(iv) $(N \cap P)^{**}(\mathcal{I}) \subseteq N^{**}(\mathcal{I}) \cap P^{**}(\mathcal{I})$.

Proof. We prove only (i) and (ii). The proofs for the rest will follow similarly.
(i) Let r_i/a_i be a multipoint in M such that $r_i/a_i \in N^{**}(\mathcal{I})$. Then for every $W \in \mathcal{N}(r_i/a_i)$, $I \in \mathcal{I}$ there exists $a_j \in X$ such that $C_I(a_j) < C_W(a_j) - C_{N^c}(a_j)$.
Now, for all $a_j \in X$, $W \in \mathcal{N}(r_i/a_i)$, $I \in \mathcal{I}$ we have

$$\begin{aligned}
& N \subseteq P \\
& \Rightarrow C_N(a_j) \leq C_P(a_j) \\
& \Rightarrow C^c_P(a_j) \leq C^c_N(a_j) \\
& \Rightarrow C_W(a_j) - C_{N^c}(a_j) \leq C_W(a_j) - C_{P^c}(a_j).
\end{aligned}$$

Therefore, for every $W \in \mathcal{N}(r_i/a_i)$, $I \in \mathcal{I}$ there exists $a_j \in X$ such that $C_I(a_j) < C_W(a_j) - C_{P^c}(a_j)$ and eventually $r_i/a_i \in P^{**}(\mathcal{I})$, which establishes the proof.

(ii) Let N be any submset of M and for any two mset ideals $\mathcal{I}, \mathcal{J}$ on M with $\mathcal{I} \subseteq \mathcal{J}$. Let r_i/a_i be a multipoint in M such that $r_i/a_i \in N^{**}(\mathcal{J})$. Then for every $W \in \mathcal{N}(r_i/a_i)$, $J \in \mathcal{J}$ there exists $a_j \in X$ such that $C_J(a_j) < C_W(a_j) - C_{N^c}(a_j)$. Since $\mathcal{I} \subseteq \mathcal{J}$, $C_I(a_j) < C_W(a_j) - C_{N^c}(a_j)$ for every $W \in \mathcal{N}(r_i/a_i)$, $I \in \mathcal{I}$. Therefore, $r_i/a_i \in N^{**}(\mathcal{I})$. □

Remark 3.11. *The equality in Theorem 3.10(iv) does not hold in general, which is illustrated by the following example.*

Example 3.12. *Let $X = \{p, q, r\}$ and $M = \{2/p, 3/q, 5/r\} \in [X]^5$. Consider two M-topologies τ_1 and τ_2 on M defined by:*
$\tau_1 = \{M, \emptyset, \{1/p, 1/q, 2/r\}, \{1/p, 1/q, 5/r\}, \{1/p, 3/q, 5/r\}, \{2/p, 1/q, 5/r\}\}$, and
$\tau_2 = \{M, \emptyset, \{1/p\}, \{2/q\}, \{1/p, 2/q\}, \{1/p, 2/q, 3/r\}\}$.

By Theorem 2.12, we obtain $\tau_1(\tau_2) = \{M, \emptyset, \{1/p\}, \{2/q\}, \{1/p, 2/q\}, \{1/p, 2/q, 3/r\}\}$.
Let $\mathcal{I} = \{\emptyset\}$*, then* $\mathcal{I}$ *is an mset ideal in M. Thus,* $(M, \tau_1(\tau_2), \mathcal{I})$ *is an MMITS.*
Consider two submsets $N = \{1/p, 2/q, 1/r\}$ *and* $P = \{2/p, 1/q, 3/r\}$ *of M. By using Definition 3.4, one can deduce that* $N^{**}(\mathcal{I}) = \{1/p, 3/q, 5/r\}$, $P^{**}(\mathcal{I}) = \{2/p, 1/q, 5/r\}$ *and* $(N \cap P)^{**}(\mathcal{I}) = \{1/p, 1/q, 2/r\}$*. Clearly* $(N \cap P)^{**}(\mathcal{I}) \neq N^{**}(\mathcal{I}) \cap P^{**}(\mathcal{I})$.

Definition 3.13. *Let* $(M, \tau_1(\tau_2))$ *be an MMTS, and N be a submset of M. A function* $cl^{**}(.) : P^*(M) \rightarrow P^*(M)$ *is said to be an mset closure operator on N, defined by* $cl^{**}(N) = N \cup N^{**}(\mathcal{I})$*, where* $N^{**}(\mathcal{I})$ *is the mset local function of N.*

Remark 3.14. *It follows from Remark 3.5 that the mset closure operator on N with respect to* τ_2 *in the sense of Shravan and Tripathy [25] contained in that of with respect to* $\tau_1(\tau_2)$ *that is* $cl_2^*(N) \subseteq cl^{**}(N)$ *for any submset N of M.*

The following theorem gives some basic properties of mset closure operator.

Theorem 3.15. *Let* $(M, \tau_1(\tau_2), \mathcal{I})$ *be an MMITS. Then for any two submsets N, P of M, the following results hold:*
(i) If $N \subseteq P$*, then* $cl^{**}(N) \subseteq cl^{**}(P)$.
(ii) $cl^{**}(N \cup P) = cl^{**}(N) \cup cl^{**}(P)$.
(iii) $\tau_1(\tau_2)int(N) \subseteq cl^{**}(N)$.

Proof. (i) Let $N \subseteq P$, then for all $x \in X$, $C_N(x) \subseteq C_P(x)$.
Now, for all $x \in X$, we have

$$\begin{aligned} C_{cl^{**}(N)}(x) &= C_{N \cup N^{**}(\mathcal{I})}(x) \\ &= max\{C_N(x), C_{N^{**}(\mathcal{I})}(x)\} \\ &\leq max\{C_P(x), C_{P^{**}(\mathcal{I})}(x)\} \\ &= C_{cl^{**}(P)}(x). \end{aligned}$$

Therefore, $N \subseteq P \Rightarrow cl^{**}(N) \subseteq cl^{**}(P)$.
(ii) By using Definition 3.13, for any two submsets N, P of M, we obtain

$$\begin{aligned} cl^{**}(N \cup P) &= (N \cup P) \cup (N \cup P)^{**}(\mathcal{I}) \\ &= (N \cup P) \cup (N^{**}(\mathcal{I}) \cup P^{**}(\mathcal{I})) \\ &= (N \cup N^{**}(\mathcal{I})) \cup (P \cup P^{**}(\mathcal{I})) \\ &= cl^{**}(N) \cup cl^{**}(P). \end{aligned}$$

Therefore, $cl^{**}(N \cup P) = cl^{**}(N) \cup cl^{**}(P)$.
(iii) Let N be any submset of M. Since $\tau_1(\tau_2)int(N) \subseteq N$ and $N \subseteq N \cup N^{**}(\mathcal{I})$, we have $\tau_1(\tau_2)int(N) \subseteq N \cup N^{**}(\mathcal{I}) = cl^{**}(N)$. □

Definition 3.16. *Let* $(M, \tau_1(\tau_2), \mathcal{I})$ *be an MMITS. A submset N of M is said to be*
(i) $\tau_1(\tau_2)$*-pre-*$\mathcal{I}$*-open mset (briefly,* $\tau_1(\tau_2)$*-**PIO**) if* $N \subseteq \tau_1(\tau_2)int(cl^{**}(N))$.
(ii) $\tau_1(\tau_2)$*-semi-*$\mathcal{I}$*-open mset (briefly,* $\tau_1(\tau_2)$*-**SIO**) if* $N \subseteq cl^{**}(\tau_1(\tau_2)int(N))$.
(iii) $\tau_1(\tau_2)$*-*α*-*$\mathcal{I}$*-open mset (briefly,* $\tau_1(\tau_2)$*-*α***IO**) if* $N \subseteq \tau_1(\tau_2)int(cl^{**}(\tau_1(\tau_2)int(N)))$.

Definition 3.17. *Let* $(M, \tau_1, \mathcal{I})$ *and* $(M, \tau_2, \mathcal{I})$ *be two mset ideal topological spaces. A submset N of M is said to be*
(i) τ_k*-pre-*$\mathcal{I}$*-open mset (briefly,* τ_k*-**PIO**), where* $k \in \{1, 2\}$ *if* $N \subseteq \tau_k int(cl_k^*(N))$;
(ii) τ_k*-semi-*$\mathcal{I}$*-open mset (briefly,* τ_k*-**SIO**), where* $k \in \{1, 2\}$ *if* $N \subseteq cl_k^*(\tau_k int(N))$;
(iii) τ_k*-*α*-*$\mathcal{I}$*-open mset (briefly,* τ_k*-*α***IO**), where* $k \in \{1, 2\}$ *if* $N \subseteq \tau_k int(cl_k^*(\tau_k int(N)))$.
Here, $cl_k^*(N) = N \cup N^*(\tau_k, \mathcal{I})$ *and* $N^*(\tau_k, \mathcal{I})$ *is the mset local function of N with respect to* τ_k*, where* $k \in \{1, 2\}$ *and mset ideal* $\mathcal{I}$ *[25].*

Theorem 3.18. *Let* $(M, \tau_1(\tau_2), \mathcal{I})$ *be an MMITS. Then every* $\tau_1(\tau_2)$*-open mset is* $\tau_1(\tau_2)$*-**PIO** (resp.* $\tau_1(\tau_2)$*-**SIO**,* $\tau_1(\tau_2)$*-*α***IO**).*

Proof. We prove the result for the case $\tau_1(\tau_2)$-**PIO**. The others can be established in a similar technique.
Let N be a $\tau_1(\tau_2)$-open mset in $(M, \tau_1(\tau_2), \mathcal{I})$. Then $N = \tau_1(\tau_2)int(N)$. Since $N \subseteq cl^{**}(N)$ and $N = \tau_1(\tau_2)int(N)$, we have $N \subseteq \tau_1(\tau_2)int(cl^{**}(N))$. Therefore, N is $\tau_1(\tau_2)$-**PIO**. □

Theorem 3.19. *Let $(M, \tau_1(\tau_2), \mathcal{I})$ be an MMITS. Then every $\tau_1(\tau_2)$-open mset is τ_2-**PIO** (resp. τ_2-**SIO**, τ_2-α**IO**).*

Proof. We prove the result for the case τ_2-**PIO**. The proofs for the rest will follow similarly.
Let N be a $\tau_1(\tau_2)$-open mset in $(M, \tau_1(\tau_2), \mathcal{I})$. Then $N = \tau_1(\tau_2)int(N)$. By using Remark 3.7, we have $N = \tau_1(\tau_2)int(N) \subseteq \tau_2 int(N)$. Since $N \subseteq cl_2^*(N)$, we get $N \subseteq \tau_2 int(cl_2^*(N))$. Therefore, N is τ_2-**PIO**. □

Result 3.20. *Let $(M, \tau_1, \mathcal{I})$ and $(M, \tau_2, \mathcal{I})$ be two mset ideal topological spaces. If a submset N of M is $\tau_1(\tau_2)$-**PIO**, then it need not be a τ_2-**PIO**.*

Example 3.21. *Let $X = \{p, q\}$ and $M = \{10/p, 10/q\} \in [X]^{10}$. Consider two M-topologies τ_1 and τ_2 on M defined by:*
$\tau_1 = \{M, \emptyset, \{4/p, 6/q\}, \{6/p, 8/q\}\}$, and
$\tau_2 = \{M, \emptyset, \{2/q\}, \{6/p, 4/q\}, \{4/p, 2/q\}\}$.
By Theorem 2.12, we obtain $\tau_1(\tau_2) = \{M, \emptyset, \{6/p, 4/q\}, \{4/p, 2/q\}\}$.
*Let $\mathcal{I} = \{\emptyset\}$. Then it is an mset ideal on M. If $N = \{8/p, 7/q\}$, then one can obtain that N is a $\tau_1(\tau_2)$-**PIO** but not a τ_2-**PIO**.*

Result 3.22. *Let $(M, \tau_1, \mathcal{I})$ and $(M, \tau_2, \mathcal{I})$ be two mset ideal topological spaces. If a submset N of M is $\tau_1(\tau_2)$-**SIO** then it need not be a τ_2-**SIO**.*

Example 3.23. *Let $X = \{p, q\}$ and $M = \{2/p, 3/q\} \in [X]^3$. Consider two M-topologies τ_1 and τ_2 on M defined by:*
$\tau_1 = \{M, \emptyset, \{1/p, 3/q\}, \{2/p, 1/q\}, \{1/p, 1/q\}\}$, and
$\tau_2 = \{M, \emptyset, \{1/p\}, \{1/q\}, \{2/q\}, \{1/p, 1/q\}, \{1/p, 2/q\}\}$.
By Theorem 2.12, we obtain $\tau_1(\tau_2) = \{M, \emptyset, \{1/p\}, \{2/q\}, \{1/p, 2/q\}\}$.
*Let $\mathcal{I} = \{\emptyset\}$. Then it is an mset ideal on M. If $N = \{3/q\}$, then one can verify that N is a $\tau_1(\tau_2)$-**SIO** but not a τ_2-**SIO**.*

Result 3.24. *Let $(M, \tau_1, \mathcal{I})$ and $(M, \tau_2, \mathcal{I})$ be two mset ideal topological spaces. If a submset N of M is $\tau_1(\tau_2)$-α**IO** then it need not be a τ_2-α**IO**.*

Example 3.25. *Let $X = \{p, q, r\}$ and $M = \{2/p, 3/q, 1/r\} \in [X]^3$. Consider two M-topologies τ_1 and τ_2 on M defined by:*
$\tau_1 = \{M, \emptyset, \{2/p\}, \{3/q\}, \{2/p, 3/q\}\}$, and
$\tau_2 = \{M, \emptyset, \{2/p\}, \{3/q\}, \{1/r\}, \{2/p, 3/q\}, \{3/q, 1/r\}, \{2/p, 1/r\}\}$.
By Theorem 2.12, we obtain $\tau_1(\tau_2) = \{M, \emptyset, \{1/r\}, \{3/q, 1/r\}, \{2/p, 1/r\}\}$.
*Let $\mathcal{I} = \{\emptyset\}$. Then it is an mset ideal on M. If $N = \{2/r\}$, then one can show that N is a $\tau_1(\tau_2)$-α**IO** but not a τ_2-α**IO**.*

Proposition 3.26. *Let $(M, \tau_1(\tau_2), \mathcal{I})$ be an MMITS. If $\mathcal{I} = P^*(M)$, then N is $\tau_1(\tau_2)$-**PIO** (resp. $\tau_1(\tau_2)$-**SIO**, $\tau_1(\tau_2)$-α**IO**) if and only if N is $\tau_1(\tau_2)$-open mset.*

Proof. We prove the result for the case $\tau_1(\tau_2)$-**PIO**. The proofs for the rest will follow similarly.
Let $\mathcal{I} = P^*(M)$. Then by Lemma 3.9, we have $N^{**}(\mathcal{I}) = \emptyset$. Clearly, $cl^{**}(N) = N \cup N^{**}(\mathcal{I})$ implies $cl^{**}(N) = N$. Thus, $\tau_1(\tau_2)int(cl^{**}(N)) = \tau_1(\tau_2)int(N)$. Therefore, N is $\tau_1(\tau_2)$-**PIO** if and only if it is $\tau_1(\tau_2)$-open. □

Proposition 3.27. *Let $(M, \tau_1(\tau_2), \mathcal{I})$ be an MMITS. If $\mathcal{I} = P^*(M)$, then every $\tau_1(\tau_2)$-**PIO** (resp. $\tau_1(\tau_2)$-**SIO**, $\tau_1(\tau_2)$-α**IO**) is τ_2-**PIO** (resp. τ_2-**SIO**, τ_2-α**IO**).*

Proof. We only prove that every $\tau_1(\tau_2)$-**PIO** is τ_2-**PIO**. The others can be established in a similar technique.
Let N be a submset of M and $\mathcal{I} = P^*(M)$. Then by Lemma 3.9, we have $N^{**}(\mathcal{I}) = \emptyset$. Clearly, $cl^{**}(N) = N \cup N^{**}(\mathcal{I})$ implies $cl^{**}(N) = N$.
By using Remark 3.7, we obtain

$$\begin{aligned}\tau_1(\tau_2)int(cl^{**}(N)) &= \tau_1(\tau_2)int(N)\\ &\subseteq \tau_2 int(N)\\ &\subseteq \tau_2 int(cl_2^*(N)).\end{aligned}$$

Therefore, $N \subseteq \tau_1(\tau_2)int(cl^{**}(N)) \Rightarrow N \subseteq \tau_2 int(cl_2^*(N))$. This proves that N is τ_2-**PIO**, whenever N is $\tau_1(\tau_2)$-**PIO**. □

Remark 3.28. *The converses of the above proposition need not be true in general, as illustrated by the following example.*

Example 3.29. *(i) Consider the MMTS $(M, \tau_1(\tau_2))$ as in Example 3.21. Let $\mathcal{I} = P^*(M)$ be an mset ideal on M. Then, $(M, \tau_1(\tau_2), \mathcal{I})$ is an MMITS. One can easily justify that the submset $N = \{2/q\}$ of M is a τ_2-**PIO** but not a $\tau_1(\tau_2)$-**PIO**.*
*(ii) Consider the MMTS $(M, \tau_1(\tau_2))$ as in Example 3.23. Let $\mathcal{I} = P^*(M)$ be an mset ideal on M. Then, $(M, \tau_1(\tau_2), \mathcal{I})$ is an MMITS. One can prove that the submset $N = \{1/p, 1/q\}$ of M is a τ_2-**SIO** but not a $\tau_1(\tau_2)$-**SIO**.*
*(iii) Consider the MMTS $(M, \tau_1(\tau_2))$ as in Example 3.25. Let $\mathcal{I} = P^*(M)$ be an mset ideal on M. Then, $(M, \tau_1(\tau_2), \mathcal{I})$ is an MMITS. It can be verified that the submset $N = \{2/p, 3/q\}$ of M is a τ_2-α**IO** but not a $\tau_1(\tau_2)$-α**IO**.*

Note 3.30. *By using Results 3.20, 3.22, 3.24 and Remark 3.28, we conclude that the notions $\tau_1(\tau_2)$-**PIO** and τ_2-**PIO**; $\tau_1(\tau_2)$-**SIO** and τ_2-**SIO**; $\tau_1(\tau_2)$-α**IO** and τ_2-α**IO** are completely independent.*

Theorem 3.31. *Let $(M, \tau_1(\tau_2), \mathcal{I})$ be an MMITS. Then every $\tau_1(\tau_2)$-α**IO** is $\tau_1(\tau_2)$-**PIO**.*

Proof. Let N be a $\tau_1(\tau_2)$-α**IO** in an MMITS $(M, \tau_1(\tau_2), \mathcal{I})$, then $N \subseteq \tau_1(\tau_2)int(cl^{**}(\tau_1(\tau_2)int(N)))$.
Now,

$$\begin{aligned}
&\tau_1(\tau_2)int(N) \subseteq N\\
&\Rightarrow cl^{**}(\tau_1(\tau_2)int(N)) \subseteq cl^{**}(N)\\
&\Rightarrow \tau_1(\tau_2)int(cl^{**}(\tau_1(\tau_2)int(N))) \subseteq \tau_1(\tau_2)int(cl^{**}(N)).
\end{aligned}$$

Therefore, we get $N \subseteq \tau_1(\tau_2)int(cl^{**}(N))$. This shows that N is $\tau_1(\tau_2)$-**PIO**. □

Remark 3.32. *The converse of the above theorem is not true in general, which follows from the following example.*

Example 3.33. *Consider the MMITS $(M, \tau_1(\tau_2), \mathcal{I})$ as shown in Example 3.21. One can deduce that the submset $N = \{8/p, 7/q\}$ of M is $\tau_1(\tau_2)$-**PIO** but not a $\tau_1(\tau_2)$-α**IO**.*

Theorem 3.34. *Let $(M, \tau_1(\tau_2), \mathcal{I})$ be an MMITS. Then every $\tau_1(\tau_2)$-α**IO** is $\tau_1(\tau_2)$-**SIO**.*

Proof. Let N be a $\tau_1(\tau_2)$-α**IO** in $(M, \tau_1(\tau_2), \mathcal{I})$, then $N \subseteq \tau_1(\tau_2)int(cl^{**}(\tau_1(\tau_2)int(N)))$.
Clearly, $\tau_1(\tau_2)int(cl^{**}(\tau_1(\tau_2)int(N))) \subseteq cl^{**}(\tau_1(\tau_2)int(N))$. Therefore, $N \subseteq cl^{**}(\tau_1(\tau_2)int(N))$. This shows that N is $\tau_1(\tau_2)$-**SIO**.
□

Remark 3.35. *The converse of the above theorem is not true in general, which follows from the following example.*

Example 3.36. *Consider the MMITS $(M, \tau_1(\tau_2), \mathcal{I})$ as in Example 3.23. One can easily justify that the submset $N = \{3/q\}$ of M is a $\tau_1(\tau_2)$-**SIO** but not a $\tau_1(\tau_2)$-α**IO**.*

Definition 3.37. *A submset N in an MMITS $(M, \tau_1(\tau_2), \mathcal{I})$ is called $\tau_1(\tau_2)$-δ-$\mathcal{I}$-open mset (briefly, $\tau_1(\tau_2)$-δ**IO**) if $\tau_1(\tau_2)int(cl^{**}(N)) \subseteq cl^{**}(\tau_1(\tau_2)int(N))$.*

Proposition 3.38. *In an MMITS $(M, \tau_1(\tau_2), \mathcal{I})$, every $\tau_1(\tau_2)$-**SIO** is $\tau_1(\tau_2)$-δ**IO**.*

Proof. Let N be a $\tau_1(\tau_2)$-**SIO** in $(M, \tau_1(\tau_2), \mathcal{I})$, then $N \subseteq cl^{**}(\tau_1(\tau_2)int(N))$.
Now,

$$\begin{aligned}
\tau_1(\tau_2)int(cl^{**}(N)) &\subseteq cl^{**}(N)\\
&\subseteq cl^{**}(cl^{**}(\tau_1(\tau_2)int(N)))\\
&= cl^{**}(\tau_1(\tau_2)int(N)).
\end{aligned}$$

Hence, the proof is completed. □

Proposition 3.39. *Let $(M, \tau_1(\tau_2), \mathcal{I})$ be an MMITS. Then a submset of M is $\tau_1(\tau_2)$-α**IO** if and only if it is both $\tau_1(\tau_2)$-δ**IO** and $\tau_1(\tau_2)$-**PIO**.*

Proof. Let N be a $\tau_1(\tau_2)$-α**IO**. Then it is $\tau_1(\tau_2)$-**SIO**. Hence, by above theorem it is $\tau_1(\tau_2)$-δ**IO**. On the other hand, by Theorem 3.31, every $\tau_1(\tau_2)$-α**IO** is $\tau_1(\tau_2)$-**PIO**.

Conversely, let N be a $\tau_1(\tau_2)$-δ**IO** and $\tau_1(\tau_2)$-**PIO**. Then, we have $\tau_1(\tau_2)int(cl^{**}(N)) \subseteq cl^{**}(\tau_1(\tau_2)int(N))$ and hence $\tau_1(\tau_2)int(cl^{**}(N)) \subseteq \tau_1(\tau_2)int(cl^{**}(\tau_1(\tau_2)int(N)))$. Since N is $\tau_1(\tau_2)$-pre-$\mathcal{I}$-open, we have $N \subseteq \tau_1(\tau_2)int(cl^{**}(N))$. Thus, we obtain $N \subseteq \tau_1(\tau_2)int(cl^{**}(\tau_1(\tau_2)int(N)))$. This shows that N is a $\tau_1(\tau_2)$-α**IO**. □

Proposition 3.40. *Let K, L be submsets of an MMITS $(M, \tau_1(\tau_2), \mathcal{I})$. If $K \subseteq L \subseteq cl^{**}(K)$ and K is $\tau_1(\tau_2)$-δ**IO**, then L is $\tau_1(\tau_2)$-δ**IO**.*

Proof. Suppose that $K \subseteq L \subseteq cl^{**}(K)$ and K is $\tau_1(\tau_2)$-δ**IO**.
Then,

$$\tau_1(\tau_2)int(cl^{**}(K)) \subseteq cl^{**}(\tau_1(\tau_2)int(K)) \tag{1}$$

By using Eq. (1), we have

$$\begin{aligned} & K \subseteq L \\ & \Rightarrow \tau_1(\tau_2)int(K) \subseteq \tau_1(\tau_2)int(L) \\ & \Rightarrow cl^{**}(\tau_1(\tau_2)int(K)) \subseteq cl^{**}(\tau_1(\tau_2)int(L)) \\ & \Rightarrow \tau_1(\tau_2)int(cl^{**}(K)) \subseteq cl^{**}(\tau_1(\tau_2)int(L)). \end{aligned} \tag{2}$$

Also,

$$\begin{aligned} & L \subseteq cl^{**}(K) \\ & \Rightarrow cl^{**}(L) \subseteq cl^{**}(cl^{**}(K)) = cl^{**}(K) \\ & \Rightarrow \tau_1(\tau_2)int(cl^{**}(L)) \subseteq \tau_1(\tau_2)int(cl^{**}(K)). \end{aligned} \tag{3}$$

By using Eqs. (2) and (3), we obtain $\tau_1(\tau_2)int(cl^{**}(L)) \subseteq cl^{**}(\tau_1(\tau_2)int(L))$. This proves that L is a $\tau_1(\tau_2)$-δ**IO**. □

Remark 3.41. *The concepts of $\tau_1(\tau_2)$-δ**IO** and $\tau_1(\tau_2)$-**PIO** are independent notions as illustrated by the following example.*

Example 3.42. *Consider the MMITS $(M, \tau_1(\tau_2), \mathcal{I})$ as shown in Example 3.21. If $N = \{8/p, 7/q\}$, then it can be verified that it is a $\tau_1(\tau_2)$-**PIO** but not a $\tau_1(\tau_2)$-δ**IO**. On the other hand, by considering Example 3.23, one can deduce that the submset $N = \{3/q\}$ of M is a $\tau_1(\tau_2)$-δ**IO** but not a $\tau_1(\tau_2)$-**PIO**.*

Theorem 3.43. *Let K, L and W be three submsets of an MMITS $(M, \tau_1(\tau_2), \mathcal{I})$ with $\tau_1(\tau_2)int(cl^{**}(W))$ is a whole submset of M. If W is a $\tau_1(\tau_2)$-δ**IO**, then $W = K \cup L$, where K is a $\tau_1(\tau_2)$-α**IO**, $\tau_1(\tau_2)int(cl^{**}(L)) = \emptyset$ and $L \cap K = \emptyset$.*

Proof. Suppose that W is a $\tau_1(\tau_2)$-δ**IO** in $(M, \tau_1(\tau_2), \mathcal{I})$.
Then,

$$\begin{aligned} & \tau_1(\tau_2)int(cl^{**}(W)) \subseteq cl^{**}(\tau_1(\tau_2)int(W)) \\ & \Rightarrow \tau_1(\tau_2)int(cl^{**}(W)) \subseteq \tau_1(\tau_2)int(cl^{**}(\tau_1(\tau_2)int(W))) \end{aligned} \tag{4}$$

Now, we have $W = [W \cap \tau_1(\tau_2)int(cl^{**}(W))] \cup [W \ominus \tau_1(\tau_2)int(cl^{**}(W))]$.
We now set $K = W \cap \tau_1(\tau_2)int(cl^{**}(W))$ and $L = W \ominus \tau_1(\tau_2)int(cl^{**}(W))$.
We first show that K is a $\tau_1(\tau_2)$-α**IO**.
Now,

$$\begin{aligned} & \tau_1(\tau_2)int(cl^{**}(\tau_1(\tau_2)int(K))) \\ & = \tau_1(\tau_2)int(cl^{**}(\tau_1(\tau_2)int[W \cap \tau_1(\tau_2)int(cl^{**}(W))))] \\ & = \tau_1(\tau_2)int(cl^{**}[(\tau_1(\tau_2)int(W) \cap \tau_1(\tau_2)int(cl^{**}(W))))] \\ & = \tau_1(\tau_2)int(cl^{**}(\tau_1(\tau_2)int(W). \end{aligned} \tag{5}$$

Since $K = W \cap \tau_1(\tau_2)int(cl^{**}(W))$, we have $K \subseteq \tau_1(\tau_2)int(cl^{**}(W))$.
Therefore, from Eqs. (4) and (5), we have $K \subseteq \tau_1(\tau_2)int(cl^{**}(\tau_1(\tau_2)int(K)))$. This shows that K is $\tau_1(\tau_2)$-α**IO**.
Next we prove that $\tau_1(\tau_2)int(cl^{**}(L)) = \emptyset$.
We have,

$$\begin{aligned}
&\tau_1(\tau_2)int(cl^{**}(L))\\
&= \tau_1(\tau_2)int(cl^{**}[W \ominus \tau_1(\tau_2)int(cl^{**}(W)])\\
&= \tau_1(\tau_2)int(cl^{**}[W \cap (M \ominus \tau_1(\tau_2)int(cl^{**}(W))])\\
&\subseteq \tau_1(\tau_2)int(cl^{**}(W) \cap \tau_1(\tau_2)int(cl^{**}(M \ominus \tau_1(\tau_2)int(cl^{**}(W)))\\
&\subseteq \tau_1(\tau_2)int(cl^{**}(W) \cap \tau_1(\tau_2)int(\tau_1(\tau_2)cl(M \ominus \tau_1(\tau_2)int(cl^{**}(W)))\\
&\subseteq \tau_1(\tau_2)int(cl^{**}(W) \cap [M \ominus \tau_1(\tau_2)int(cl^{**}(W)]\\
&= \emptyset.
\end{aligned}$$

Also, it is clear that $[W \cap \tau_1(\tau_2)int(cl^{**}(W))] \cap [W \ominus \tau_1(\tau_2)int(cl^{**}(W))] = \emptyset$.
Hence, the proof is completed. □

Example 3.44. *Consider $X = \{p, q, r\}$ and $M = \{2/p, 3/q, 4/r\} \in [X]^4$. Consider two M-topologies τ_1 and τ_2 on M defined by:*
$\tau_1 = \{M, \emptyset, \{3/q\}, \{4/r\}, \{3/q, 4/r\}\}$, *and*
$\tau_2 = \{M, \emptyset, \{1/p\}, \{2/p\}, \{2/p, 3/q\}, \{2/p, 4/r\}\}$.
By Theorem 2.12, we obtain $\tau_1(\tau_2) = \{M, \emptyset, \{2/p\}, \{2/p, 3/q\}, \{2/p, 4/r\}\}$.
Consider an mset ideal $\mathcal{I} = \{\emptyset\}$ on M. Then $(M, \tau_1(\tau_2), \mathcal{I})$ is an MMITS.
*Let $W = \{2/p, 2/q\}$ be a submset of M. We first show that W is a $\tau_1(\tau_2)$-δ**IO**.*
By using Definition 3.13, one can deduce that $cl^{**}(W) = M$ and thus, $\tau_1(\tau_2)int(cl^{**}(W)) = M$. Also, $cl^{**}(\tau_1(\tau_2)int(W)) = M$.
*Clearly, $\tau_1(\tau_2)int(cl^{**}(W)) \subseteq cl^{**}(\tau_1(\tau_2)int(W))$. Therefore, W is a $\tau_1(\tau_2)$-δ**IO**.*
Let us write $W = K \cup L$, where $K = W$ and $L = \emptyset$.
*We now verify that K is a $\tau_1(\tau_2)$-α**IO**.*
*As $cl^{**}(\tau_1(\tau_2)int(K)) = M$, we have $\tau_1(\tau_2)int(cl^{**}(\tau_1(\tau_2)int(K))) = M$. Therefore, K is $\tau_1(\tau_2)$-α**IO**. On the other hand, $\tau_1(\tau_2)int(cl^{**}(L)) = \emptyset$ and $K \cap L = \emptyset$.*

Remark 3.45. *We now show that the condition, $\tau_1(\tau_2)int(cl^{**}(W))$ is a whole submset of M in the above theorem is mandatory.*

Example 3.46. *Consider the MMITS $(M, \tau_1(\tau_2), \mathcal{I})$ as shown in Example 3.23. Consider $W = \{3/q\}$. By using Definition 3.13, we obtain $cl^{**}(W) = \{1/p, 3/q\}$ and then $\tau_1(\tau_2)int(cl^{**}(W)) = \{1/p, 2/q\}$. Also, $cl^{**}(\tau_1(\tau_2)int(W)) = \{1/p, 3/q\}$. Therefore, W is a $\tau_1(\tau_2)$-δ**IO**. But, there do not exist two submsets K, L of M such that $W = K \cup L$, where K is a $\tau_1(\tau_2)$-α**IO**, $\tau_1(\tau_2)int(cl^{**}(L)) = \emptyset$ and $K \cap L = \emptyset$.*

4. Conclusion

In this article, we have introduced the notion of mixed multiset ideal topological spaces. We have defined the notions of $\tau_1(\tau_2)$-pre-$\mathcal{I}$-open mset, $\tau_1(\tau_2)$-semi-$\mathcal{I}$-open mset, $\tau_1(\tau_2)$-α-$\mathcal{I}$-open mset and $\tau_1(\tau_2)$-δ-$\mathcal{I}$-open mset via mixed multiset ideal topological space and investigated on these generalized open msets. This article is a starting of a new structure. There are lots of investigation left in this direction and a few of the properties have been discussed here. We hope that this article will open up a new window for research fraternity in coming days.

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