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On geometric properties of topologically charged Ellis-Bronnikov-type wormhole spacetime

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Abstract. In this paper, we have studied the geometric properties of topologically charged Ellis-Bronnikov-type wormhole (briefly, TCEBW) spacetime. The TCEBW spacetime is a static and spherically symmetric solution of the Einstein field equations with a non-zero cosmological constant. We obtained several important geometric properties viz. pseudosymmetry due to conformal curvature as well as conharmonic curvature, Ricci generalized pseudosymmetry and Ricci generalized projectively pseudosymmetry. Also, it is shown that the TCEBW spacetime is generalized Roter type, 2-quasi-Einstein, Einstein manifold of level 3 and its conformal 2-forms are recurrent. As a special case, the geometric properties of Morris-Thorne wormhole spacetime are analyzed. Also, we have shown that the TCEBW spacetime admits an almost η-Ricci-Yamabe soliton and an almost η-Ricci soliton. Finally, a comparison between Morris-Thorne wormhole and TCEBW spacetime regarding their geometric structures is exhibited.

1. Introduction

Black holes are the most fascinating objects in the universe, both in observational and theoretical cases, and as the researchers throughout the globe progress in this field every moment, it gives us the better understanding of the nature. The study of black hole physics has been of growing research interest for a long time. After the theory of general relativity proposed by A. Einstein in 1915, many authors attempted to construct exact solutions of black holes in 3, 4 and higher dimensions. Some well-known black hole solutions are the non-rotating Schwarzschild vacuum solution with or without cosmological constant, rotating Kerr solution, Kerr-Newman solution, Bardeen regular solution, Hayward solution, and some other solutions in the literature.

Another interesting type of spacetime geometry that is also an exact solutions of the Einstein field equations, is the (Lorentzian) traversable wormholes. Conceptually, a wormhole can be defined as a tunnel connecting two different regions of a spacetime by a bridge or narrow throat, and the two regions may be

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from same universe (intra-universe wormhole) or two different universes (inter-universe wormhole). From a point of view, the possibility of the existence of wormholes draws attention to the questions related to the causality (time travel), the geometric structure and topology of spacetime, quantum gravity and energy constraints. The study of wormholes was commenced in the work of Flamm [44] together with the paper [39] by Einstein and Rosen. After a seminal work by Morris and Thorne [68], many authors have focused on the investigation of wormholes, which is now an active field of research in mathematical physics. The principal work [68] by Morris and Thorne has clarified that some form of exotic matter violating the weak energy conditions (WEC) as well as the null energy conditions (NEC) is necessary in order to keep the throat of the wormhole open. It is noteworthy to mention that before the Morris-Thorne wormhole, the simplest example of a traversable wormhole was constructed independently by Ellis [40] and Bronnikov [13], which is called Ellis-Bronnikov wormhole spacetime later. It is an exact solution of the Einstein field equations minimally coupled with a massless phantom scalar field and also an example of a traversable wormhole, which appears in the literature with a plenty of papers on the investigation of its geometric properties.

In the context of this study, we consider a topologically charged Ellis-Bronnikov-type wormhole spacetime whose curvature properties are the main concern here because of its simple form. The line element of a static and spherically symmetric metric in the coordinates (t, x, θ, ϕ), which describes a topologically charged Ellis-Bronnikov-type wormhole (briefly, TCEBW) spacetime (Refs. [13, 40, 70]), is given by

$$ds^2 = -dt^2 + \frac{dx^2}{\alpha^2} + (x^2 + b^2)\,(d\theta^2 + \sin^2\theta\, d\phi^2) = g_{\mu\nu}\, dx^\mu\, dx^\nu, \tag{1}$$

where $\mu, \nu = 1, 2, 3, 4$, the constant b is the radius of the wormhole throat, and $\alpha < 1$ is the topological defect parameter depending on the energy scale η. Here, the coordinates (t, x, θ, ϕ) are in the ranges $-\infty < t, x < +\infty$, $0 \leq \theta \leq \frac{\pi}{2}$, and $0 \leq \phi < 2\pi$. This wormhole solution has been studied in the context of the relativistic wave equations in Refs. [2, 7, 69, 118]. For $\alpha \to 1$, the above spacetime becomes Ellis-Bronnikov-type wormhole, also known as Morris-Thorne wormhole (briefly, MTW) spacetime [11], whose curvature properties have been studied in Ref. [42]. Whereas for $b \to 0$, the spacetime becomes a point-like global monopole whose geometric properties have been studied in Ref. [83].

Let M be an n-dimensional $(n \geq 3)$ smooth and connected manifold with a semi-Riemannian metric g of signature $(\zeta, n-\zeta)$, $0 \leq \zeta \leq n$. Now if (i) $\zeta = 1$ or $\zeta = n-1$; (ii) $\zeta = 0$ or $\zeta = n$, then (M, g) is said to be a (i) Lorentzian; (ii) Riemannian manifold respectively, and we note that M with Lorentzian signature $(1, 3)$ or $(3, 1)$ is physically treated as a spacetime. A semi-Riemannian manifold (M, g) primarily occupies three notions of curvature, specifically, Riemann curvature tensor (R) of type $(0, 4)$, Ricci tensor (S) of type $(0, 2)$ and scalar curvature (κ).

The geometric structure of a spacetime describes the geometry and the physical nature of the space. So the determination of geometric structure of spacetime is crucial in differential geometry, and such geometric structures can be obtained by investigating its curvature properties, even though extracting the curvature properties of a given metric is very cumbersome due to rigorous calculations. There are various ways to define and generalize the notion of symmetry of a manifold, such as, the notion of locally symmetric manifold has been introduced by Cartan [14] and it is defined as $\nabla_r R^m_{nop} = 0$, the notion of semisymmetric manifold has been also introduced by Cartan [15] and it is defined as $(R \cdot R)_{mnopqs} = 0$ (which were classified later by Szabó [120–122] for the Riemann case), the notion of pseudosymmetric manifold has been introduced by Adamów and Deszcz [1] and it is defined as $R \cdot R = \mathcal{L}Q(g, R)$, recurrent manifold has been introduced by Walker [127] (see also [75–77]), several generalized notions of recurrent manifold have been introduced by Shaikh and his co-authors (see [84, 106, 109–113]), curvature 2-forms of recurrent manifold have been introduced by Besse [54, 58–60], weakly symmetric manifold has been introduced by Tamássy and Binh [124, 125], etc. It may be noted that in the above geometric structures, various curvature tensors are involved along with their first or second order covariant derivatives. By extending the notion of pseudosymmetry to other curvature tensors, different kinds of pseudosymmetric type curvature conditions are obtained in TCEBW spacetime, which are significant in the literature as several spacetimes (see, [8, 36, 79–81, 86, 91]) are basically different types of pseudosymmetric manifolds. We mention that the notions of pseudosymmetries introduced by both Chaki [16] and Deszcz [20] are absolutely different.

Ricci flow, a process of evolving a Riemanninan metric over time, was introduced by Hamilton [51] in 1982 during the study of 3-dimensional compact manifolds with positive Ricci curvature. Ricci solitons are the self-similar solutions of Ricci flow, which are natural generalizations of Einstein metrics [9, 12, 78, 115]. Again, Hamilton [52] simultaneously established the concept of another geometric flow, known as Yamabe flow. Recently, Güler and Cráșmareănu [50] introduced the notion of a new geometric flow in terms of the scalar combination of Ricci flow and Yamabe flow, called Ricci-Yamabe flow, whose self-similar solutions are known as Ricci-Yamabe solitons.

If a semi-Riemannian manifold M realizes the relation

$$\frac{1}{2}\pounds_{\xi}g + S - \mu g = 0,$$

with a constant μ, then M is said to be a Ricci soliton, where $\pounds_{\xi}$ denotes the Lie derivative with respect to the soliton vector field ξ. If $\mu < 0$, $\mu = 0$ or $\mu > 0$, then it is expanding, steady or shrinking respectively. If μ is allowed to be a non-constant smooth function, M is called an almost Ricci soliton [73]. We note that the Ricci soliton turns into an Einstein manifold if ξ is a Killing vector field. Again, if M possesses the relation

$$\frac{1}{2}\pounds_{\xi}g + S - \ell_2 g + \ell_3(\eta \otimes \eta) = 0,$$

for some constants ℓ_2, ℓ_3 and non-zero 1-form η, then M is known as an η-Ricci soliton [18]. Further, M is an almost η-Ricci soliton [10] if ℓ_2, ℓ_3 are non-constant smooth functions. If the Ricci curvature S, the metric tensor g and scalar curvature κ satisfy the relation

$$\frac{1}{2}\pounds_{\xi}g + \ell_1 S + \left(\lambda - \frac{1}{2}\ell\kappa\right)g = 0,$$

with the constants ℓ_1, ℓ, λ, and the soliton vector field ξ, then M is called a Ricci-Yamabe soliton [116]. We mention that if $(\ell_1, \ell) = (0, 2)$ and $(1, 0)$, then the Ricci-Yamabe soliton becomes Yamabe soliton and Ricci soliton respectively. Moreover, M is called an almost Ricci-Yamabe soliton [116] if ℓ_1, ℓ, λ are non-constant smooth functions. In addition, if there is a non-zero 1-form η on M realizing the relation

$$\frac{1}{2}\pounds_{\xi}g + \ell_1 S + \left(\lambda - \frac{1}{2}\ell\kappa\right)g + \ell_3\eta \otimes \eta = 0,$$

with the constants ℓ_1, ℓ_3, ℓ, λ, then M is called an η-Ricci-Yamabe soliton [116], and M is said to be an almost η-Ricci-Yamabe soliton [116] if the constants ℓ_1, ℓ_3, ℓ, λ are non-constant smooth functions. During last three decades, a lot of research articles (see, [3–5, 88] and the references therein) on Ricci soliton, Yamabe soliton and their generalizations have been appeared, which turns it into an active area of research for geometers.

The motto of this paper is to investigate the geometric properties of the above mentioned TCEBW spacetime. It is shown that although the spacetime defies both the semisymmetric and pseudosymmtric properties, it agrees several pseudosymmetric type structures, such as, pseudosymmetry due to conformal curvature as well as conharmonic curvature, Ricci generalized pseudosymmetry and Ricci generalized projectively pseudosymmetry. Further, it is proved that the TCEBW spacetime is an Einstein manifold of level 3, 2-quasi-Einstein and generalized Roter type. The Ricci tensor of TCEBW spacetime is neither Codazzi type nor cyclic parallel. Also, it is shown that the TCEBW spacetime possesses almost η-Ricci soliton as well as almost η-Ricci-Yamabe soliton. We mention that as a special case of TCEBW spacetime, the curvature properties of MTW spacetime can also be obtained for $\alpha \to 1$.

The paper is composed as follows: *Section* 2 deals with the definitions and backgrounds of geometric structures as preliminaries. *Section* 3, which is the crucial part of this manuscript, presents the calculation of several tensor components and exhibits different geometric structures of TCEBW spacetime. *Section* 4 is devoted to the study of almost η-Ricci soliton and almost η-Ricci-Yamabe soliton admitted by the TCEBW spacetime. In *Section* 5, a comparison between MTW and TCEBW spacetime has been articulated with respect to their curvature restricted geometric structures. Finally, the conclusion of the paper has been drawn in the last section.

2. Preliminaries

Let E and D be two $(0,2)$-type symmetric tensors. Then their Kulkarni-Nomizu product $E \wedge D$ is defined as (see, [26, 35, 48, 53, 87, 105]):

$$(E \wedge D)_{mnop} = 2E_{m[p}D_{o]n} + 2E_{n[o}D_{p]m},$$

where [.] denotes the antisymmetrization of index pairs. The $(1,3)$-type Riemann (resp., concircular, conharmonic, conformal and projective) curvature tensor $\mathfrak{R}$ (resp., $\mathfrak{W}$, $\mathfrak{K}$, $\mathfrak{C}$ and $\mathfrak{P}$) is given as below:

$$\begin{aligned}
\mathfrak{R}^m_{nop} &= 2\left(\Gamma^s_{n[o}\Gamma^m_{p]s} + \partial_{[p}\Gamma^m_{o]n}\right),\\
\mathfrak{W}^m_{nop} &= \mathfrak{R}^m_{nop} - \kappa\frac{2}{n(n-1)}\delta^m_{[n}g_{o]p},\\
\mathfrak{K}^m_{nop} &= \mathfrak{R}^m_{nop} - \frac{2}{n-2}\left(\mathfrak{S}^u_{[n}g_{o]p} + \delta^m_{[n}\mathfrak{S}_{o]p}\right),\\
\mathfrak{C}^m_{nop} &= \mathfrak{R}^m_{nop} + \frac{2}{n-2}\left(\delta^m_{[n}\mathfrak{S}_{o]p} + \mathfrak{S}^m_{[n}g_{o]p}\right) - \kappa\frac{2}{(n-1)(n-2)}\delta^m_{[n}g_{o]p},\\
\mathfrak{P}^m_{nop} &= \mathfrak{R}^m_{nop} - \frac{2}{n-1}\delta^m_{[n}\mathfrak{S}_{o]p},
\end{aligned}$$

where $\mathfrak{S}^n_o$ is the Ricci curvature of type $(1,1)$, $\partial_m = \frac{\partial}{\partial x^m}$ and Γ^n_{op} are the connection coefficients. By lowering the indices, the $(0,4)$-type Riemann (resp., concircular, conharmonic, conformal and projective) curvature tensor R (resp., W, K, C and P) can be obtained as follows:

$$\begin{aligned}
R_{mnop} &= g_{m\alpha}(\partial_p\Gamma^\alpha_{no} - \partial_o\Gamma^\alpha_{no} + \Gamma^{\bar\alpha}_{no}\Gamma^\alpha_{\bar\alpha p} - \Gamma^{\bar\alpha}_{np}\Gamma^\alpha_{\bar\alpha o}),\\
W_{mnop} &= R_{mnop} - \frac{\bar\kappa}{2n(n-1)}(g\wedge g)_{mnop},\\
K_{mnop} &= R_{mnop} - \frac{1}{n-2}(g\wedge S)_{mnop},\\
C_{mnop} &= R_{mnop} - \frac{1}{n-2}(g\wedge S)_{mnop} + \frac{\bar\kappa}{2(n-1)(n-2)}(g\wedge g)_{mnop},\\
P_{mnop} &= R_{mnop} - \frac{1}{n-1}(g_{mp}S_{no} - g_{np}S_{mo}).
\end{aligned}$$

Let β be a tensor of rank $(0,v)$ on M with $v \geq 1$. Then the tensor $U\cdot\beta$ of $(0,v{+}2)$ type (see, [24, 25, 34, 93, 100]) is defined as

$$(U\cdot\beta)_{p_1p_2\cdots p_vab} = -\left[U^\alpha_{abp_1}\beta_{\alpha p_1\cdots p_v} + \cdots + U^\alpha_{abp_v}\beta_{p_1\cdots\alpha}\right],$$

where U^m_{nop} is a tensor of rank $(1,3)$. Again, for a tensor X of type $(0,2)$, the Tachibana tensor (see, [33, 93, 100, 123]) $Q(X,\beta)$ of type $(0,v+2)$, is defined as

$$\begin{aligned}
Q(X,\beta)_{p_1p_2\cdots p_vab} &= X_{bp_1}\beta_{ap_2\cdots p_v} + \cdots + X_{bp_v}\beta_{p_1p_2\cdots a}\\
&- X_{ap_1}\beta_{bp_2\cdots p_v} - \cdots - X_{bp_v}\beta_{p_1p_2\cdots b}.
\end{aligned}$$

If the curvature relation

$$(U\cdot\beta)_{p_1p_2\cdots p_vab} = \mathcal{F}_\beta Q(X,\beta)_{p_1p_2\cdots p_vab} \tag{2}$$

holds on (M^n, g), where $\mathcal{F}_\beta$ is some scalar function, then (M^n, g) is said to be a β-pseudosymmetric manifold due to U (see, [1, 19–21, 27, 28, 100, 103, 104]) and a β-semisymmetric type manifold due to U (see, [15, 120–122]) is defined by the relation $U \cdot \beta = 0$. In particular, if $U_{mnop} = \beta_{mnop} = R_{mnop}$ and $X_{ab} = g_{ab}$, then (M^n, g) is simply said to be pseudosymmetric manifold. Again, for $U_{mnop} = \beta_{mnop} = W_{mnop}$ (resp., K_{mnop}, C_{mnop} and P_{mnop}) and $X_{ab} = g_{ab}$, M is called concircular (resp., conharmonic, conformal and projective) pseudosymmetric manifold. Similarly, Ricci generalized pseudosymmetric manifold is defined for $X_{ab} = S_{ab}$ and several kinds of pseudosymmetric manifolds can be defined accordingly. For instance, Robertson-Walker spacetime, Friedmann-Lemaître-Robertson Walker (FLRW) spacetime, Reissner-Nordström spacetime and Kottler spacetime as well as Schwarzschild spacetime are the "old" examples of pseudosymmetric manifolds. We mention that Morris-Thorne spacetime [42] and Gödel spacetime [35] agree pseudosymmetric Weyl curvature tensor. Further, recently various spacetimes of pseudosymmetric type structures are studied in [41, 43, 47, 82, 83, 89, 92, 94–97]. It is noteworthy to mention that the notion of geodesic mappings on various generalizations of symmetric Riemannian manifolds has been investigated by Mikeš et al. [61–67].

Definition 2.1. *([78, 99, 105, 115]) A semi-Riemannian manifold M is called 2-quasi-Einstein (resp., quasi-Einstein and Einstein) manifold if rank of $(S - \alpha_1 g)$=2 (resp., 1 and 0), for some scalar α_1. In particular, one can obtain Ricci simple manifold for $\alpha_1 = 0$ in quasi-Einstein manifold. For instance, Vaidya spacetime [108] as well as Gödel spacetime [35] are Ricci simple manifolds.*

In addition, Kantowski-Sachs spacetime [87] and Som-Raychaudhuri spacetime [102] are 2-quasi-Einstein, Robertson-Walker spacetime [6, 71, 119] is quasi-Einstein and Kaigorodov spacetime [91] is Einstein.

Again, the notion of Einstein manifolds has been generalized to Ein(4) [9, 100, 105], and it is defined by the relation of linear dependency of g and S^i $(i = 1, 2, 3, 4)$ given by

$$\aleph_1 g + \aleph_2 S + \aleph_3 S^2 + \aleph_4 S^3 + S^4 = 0,$$

and Ein(3) and Ein(2) are defined respectively by the linear dependency relations

$$\aleph_5 g + \aleph_6 S + \aleph_7 S^2 + S^3 = 0 \textit{ and } \aleph_8 g + \aleph_9 S + S^2 = 0,$$

where $\aleph_i (1 \le i \le 9)$ are smooth functions on M and $S^{i+1}(V_1, V_2) = S^i(V_1, \mathfrak{S} V_2)$ for $i = 1, 2, 3$.

For example, Vaidya-Bonner spacetime [90] is Ein(3) while Robinson-Trautman spacetime [79] is Ein(2).

Definition 2.2. *If the Riemann tensor R of M entails the form*

$$\begin{aligned} R_{mnop} &= \mathcal{R}_{11}(g \wedge g)_{mnop} + \mathcal{R}_{12}(g \wedge S)_{mnop} + \mathcal{R}_{22}(S \wedge S)_{mnop} \\ &+ \mathcal{R}_{13}(g \wedge S^2)_{mnop} + \mathcal{R}_{23}(S \wedge S^2)_{mnop} + \mathcal{R}_{33}(S^2 \wedge S^2)_{mnop}, \end{aligned}$$

then M is called generalized Roter type manifold [22, 23, 29, 30, 32, 101, 105], where $\mathcal{R}_{ij}$ are some scalars. Also, it reduces to Roter type manifold ([22, 24, 31, 37, 38, 49]) for $\mathcal{R}_{13} = \mathcal{R}_{23} = \mathcal{R}_{33} = 0$.

We mention that Lifshitz spacetime [114] is a model of generalized Roter type manifold but Melvin magnetic spacetime [80] is Roter type.

Definition 2.3. *The Ricci tensor of a manifold M is called Codazzi type [45, 117] if the relation*

$$\nabla_o S_{mn} - \nabla_m S_{no} = 0$$

holds, and if M agrees the condition

$$\nabla_o S_{mn} + \nabla_n S_{om} + \nabla_m S_{no} = 0,$$

then the Ricci tensor of M is called cyclic parallel [46, 85, 98].

Definition 2.4. *[124, 125] A weakly symmetric manifold is defined the relation*

$$\nabla_q R_{mnop} = \Pi_q R_{mnop} + \gamma_m R_{qnop} + \gamma_n R_{mqop} + \lambda_o R_{mnqp} + \lambda_p R_{mnoq},$$

where Π, γ, λ are 1-forms on M. If $\frac{1}{2}\Pi_m = \gamma_m = \lambda_m$ (resp., $\gamma_m = \lambda_m = 0$), then M is called a Chaki pseudosymmetric manifold [16, 17] (resp., recurrent manifold [72, 75–77, 127]).

Definition 2.5. *([19, 25, 29, 33, 55–57]) Let Σ be a tensor of $(0,4)$-type on a manifold M. If M admits the condition*

$$\underset{m,n,o}{\mathcal{A}} \mathfrak{S}^q_m \Sigma_{nopq} = 0,$$

$\mathcal{A}$ being the cyclic sum over m, n, o, then the Ricci tensor is known as Σ-compatible. Further, an 1-form ϱ is termed as Σ-compatible if Σ-compatibility occurs for the tensor $\varrho \otimes \varrho$.

If Σ is chosen as R, C, K, P and W, one can obtain the compatibility of Ricci tensor for Riemann, conformal, conharmonic, projective and concircular curvature respectively. Again, if the relation (see, [54, 58–60])

$$\underset{m,n,o}{\mathcal{A}} \nabla_m \Sigma_{nopq} = \underset{m,n,o}{\mathcal{A}} \pi_m \Sigma_{nop}$$

holds, then the corresponding curvature 2-forms for Σ are recurrent. For a $(0,2)$-type symmetric tensor E, the 1-forms are recurrent if $\nabla_m E_{no} - \nabla_n E_{mo} = \pi_m E_{no} - \pi_n E_{mo}$, for some covector π.

Definition 2.6. *([74, 101, 126]) Let Σ be a $(0,4)$-type tensor on M and the linear space $\mathcal{V}(M)$ be defined by*

$$\mathcal{V}(M) = \left\{ \pi : \underset{m,n,o}{\mathcal{A}} \pi_a \Sigma_{nopq} = 0, \text{ where } \pi \text{ is an 1-form} \right\}.$$

If $\dim \mathcal{V}(M) \geq 1$, then M is termed as Σ-space by Venzi.

We refer to [107] for an in-depth definition of semi-Riemannian spaces using algebraic computations in Wolfram Mathematica relevant to various types of tensor, including the projective, conformal, concircular, and conharmonic curvature tensors.

3. Curvature restricted geometric structures on topologically charged Ellis-Bronnikov-type wormhole spacetime

In the coordinates (t, x, θ, ϕ), the metric tensor of TCEBW spacetime is given as follows:

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{\alpha^2} & 0 & 0 \\ 0 & 0 & (x^2+b^2) & 0 \\ 0 & 0 & 0 & (x^2+b^2)\sin^2\theta \end{pmatrix}$$

That is,

$$g_{11} = -1,\ g_{22} = \tfrac{1}{\alpha^2},\ g_{33} = (x^2+b^2),\ g_{44} = (x^2+b^2)\sin^2\theta,\ g_{ij} = 0, \text{ otherwise.}$$

The non-zero components of the second kind christoffel symbols Γ^h_{ij} are given below:

$$\Gamma^3_{23} = \Gamma^4_{24} = \tfrac{x}{b^2+x^2},\ \Gamma^2_{33} = -x\,\alpha^2,\ \Gamma^4_{34} = \cot\theta,\ \Gamma^2_{44} = \sin^2\theta\,\Gamma^2_{33},\ \Gamma^3_{44} = -\cos\theta\ \sin\theta.$$

Also, for the spacetime geometry under consideration in this work, the non-zero components (upto symmetry) of the Riemann curvature tensor $R_{\mu\nu\alpha\beta}$ are

$$R_{2323} = -\tfrac{b^2}{b^2+x^2},\ R_{2424} = \sin^2\theta R_{2323},\ R_{3434} = (b^2 - x^2(\alpha^2-1))\sin^2\theta.$$

The non-zero components of the Ricci tensor $S_{\mu\nu}$ are

$$S_{22} = \frac{2b^2}{(b^2+x^2)^2}, \ \ S_{33} = \alpha^2 - 1, \ \ S_{44} = \sin^2\theta S_{33}.$$

And the scalar curvature κ, which does not vanish, is given by

$$\kappa = \frac{2b^2(2\alpha^2-1)+2x^2(\alpha^2-1)}{(b^2+x^2)^2}.$$

Again, the components (upto symmetry) of the covariant derivatives ∇R and ∇S, which are found non-zero, are shown as follows:

$$\nabla_2 R_{2323} = \frac{4b^2x}{(b^2+x^2)^2}, \quad \nabla_4 R_{2334} = \frac{x(b^2+x^2+(b^2-x^2)\alpha^2)\sin^2\theta}{(b^2+x^2)}, \ \ \nabla_2 R_{2424} = \sin^2\theta\, \nabla_2 R_{2323},$$
$$\nabla_3 R_{2434} = \frac{x(x^2(\alpha^2-1)-b^2(\alpha^2+1))\sin^2\theta}{b^2+x^2}, \ \ \nabla_2 R_{3434} = -2\nabla_4 R_{2334};$$
$$\nabla_2 S_{22} = -\frac{8b^2x}{(b^2+x^2)^3}, \ \ \nabla_3 S_{23} = \frac{x(b^2+x^2+(b^2-x^2)\alpha^2)}{(b^2+x^2)^2}, \ \ \nabla_4 S_{24} = \sin^2\theta \nabla_3 S_{23},$$
$$\nabla_2 S_{33} = -\frac{2x(\alpha^2-1)}{(b^2+x^2)}, \nabla_2 S_{44} = \sin^2\theta \nabla_2 S_{33}.$$

The components of S^2, S^3, which do not vanish, are listed as follows:

$$S^2_{22} = \frac{4b^4\alpha^2}{(b^2+x^2)^4}, \ \ S^2_{33} = \frac{(\alpha^2-1)^2}{b^2+x^2}, \ \ S^2_{44} = \frac{(\alpha^2-1)^2}{b^2+x^2}\sin^2\theta,$$
$$S^3_{22} = \frac{8b^6\alpha^4}{(b^2+x^2)^6}, \ \ S^3_{33} = \frac{(\alpha^2-1)^3}{(b^2+x^2)^2}, \ \ S^3_{44} = \frac{(\alpha^2-1)^3}{(b^2+x^2)^2}\sin^2\theta.$$

Let $g \wedge S = L^1$, $S \wedge S = L^2$, $g \wedge S^2 = L^3$, $S^2 \wedge S^2 = L^4$, $g \wedge g = L^5$ and $S \wedge S^2 = L^6$. Then the non-vanishing components of L^i ($1 \le i \le 6$) are computed as follows:

$$L^1_{1212} = \frac{2b^2}{(b^2+x^2)^2}, \ \ L^1_{1313} = -1+\alpha^2, \ \ L^1_{3434} = -2(b^2+x^2)(-1+\alpha^2)\ \sin^2\theta,$$
$$L^1_{1414} = \sin^2\theta L^1_{1313}, \ L^1_{2323} = -1 - \frac{2b^2}{b^2+x^2} + \frac{1}{\alpha^2}, \ L^1_{2424} = \sin^2\theta\, L^1_{2323};$$
$$L^2_{2323} = -\frac{4b^2(-1+\alpha^2)}{(b^2+x^2)^2}, \ L^2_{2424} = \sin^2\theta L^2_{2323}, \ \ L^2_{3434} = -2(-1+\alpha^2)^2\sin^2\theta;$$
$$L^3_{1212} = \frac{4b^4\alpha^2}{(b^2+x^2)^4}, \ L^3_{1313} = \frac{(-1+\alpha^2)}{b^2+x^2}, \ L^3_{1414} = \sin^2\theta L^3_{1313},$$
$$L^3_{2323} = \frac{-4b^4\alpha^4-(b^2+x^2)^2(-1+\alpha^2)^2}{\alpha^2(b^2+x^2)^3}, \ L^3_{2424} = \sin^2\theta L^3_{2323}, \ L^3_{3434} = -2(-1+\alpha^2)^2\sin^2\theta;$$
$$L^4_{2323} = -\frac{8b^4\alpha^2(-1+\alpha^2)^2}{(b^2+x^2)^5}, \ L^4_{2424} = \sin^2\theta L^4_{2323}, \ \ L^4_{3434} = -\frac{2(-1+\alpha^2)^4\sin^2\theta}{(b^2+x^2)^2};$$
$$L^5_{1212} = \frac{2}{\alpha^2}, \ \ L^5_{1414} = \sin^2\theta L^5_{1313} = 2(b^2+x^2)\sin^2\theta,$$
$$L^5_{2424} = \sin^2\theta L^5_{2323} = -\frac{2(b^2+x^2)\sin^2\theta}{\alpha^2}, \ \ L^5_{3434} = -2(b^2+x^2)^2\sin^2\theta;$$
$$L^6_{2424} = -\frac{2b^2(\alpha^2-1)[x^2(\alpha^2-1)+b^2(3\alpha^2-1)]}{(b^2+x^2)^4}\sin^2\theta = \sin^2\theta L^6_{2323}, \ \ L^6_{3434} = -\frac{2(\alpha^2-1)^3\sin^2\theta}{b^2+x^2}.$$

From the above tensor components, we obtain the following relation of dependency:

$$\begin{aligned} R_{mnop} &= \mathcal{R}_{11}(g\wedge g)_{mnop} + \mathcal{R}_{12}(g\wedge S)_{mnop} + \mathcal{R}_{22}(S\wedge S)_{mnop} \\ &+ \mathcal{R}_{13}(g\wedge S^2)_{mnop} + \mathcal{R}_{23}(S\wedge S^2)_{mnop} + \mathcal{R}_{33}(S^2\wedge S^2)_{mnop}, \end{aligned} \tag{3}$$

where $\mathcal{R}_{ij}$'s ($i \le j$ with $i, j = 1, 2, 3$) are computed as follows:

$$\left.\begin{aligned} &\mathcal{R}_{11} = 1, \ \ \mathcal{R}_{23} = 0, \ \ \mathcal{R}_{13} = \frac{(b^2+x^2)^3}{b^2\alpha^2(\alpha^2-1)}, \\ &\mathcal{R}_{12} = \frac{(b^2+x^2)[b^2+x^2-\alpha^2(3b^2+x^2)]}{b^2\alpha^2(\alpha^2-1)}, \\ &\mathcal{R}_{22} = -\frac{b^2\alpha^2[b^2x^2+\alpha^2(b^2-x^2)]+2(b^2+x^2)^2[b^2+x^2-\alpha^2(3b^2+x^2)]}{4b^2\alpha^2(\alpha^2-1)^2}, \\ &\mathcal{R}_{33} = \frac{(b^2+x^2)^3[b^2\alpha^2-2(b^2+x^2)^2]}{4b^2\alpha^2(\alpha^2-1)^3}. \end{aligned}\right\} \tag{4}$$

From the above calculations of tensor components, we can state the following:

Proposition 3.1. *The TCEBW spacetime satisfies the following curvature properties:*

(i) it is 2*-quasi-Einstein as rank* $(S - \lambda g) = 2$ *for* $\lambda = \frac{1}{2}(b^2 + x^2 - \sqrt{4 + (b^2 + x^2)^2}$,

(ii) it is generalized Roter type as it possesses the relation (3), where $\mathcal{R}_{ij}$*'s* $(i, j = 1, 2, 3)$ *are given in (4),*

(iii) it is Einstein manifold of level 3 *as it realizes* $S^3 + \frac{b^2+x^2-\alpha^2(3b^2+x^2)}{(b^2+x^2)^2}S^2 + \frac{2b^2\alpha^2(\alpha^2-1)}{(b^2+x^2)^3}S = 0$,

(iv) its Ricci tensor is Riemann, projective, conharmonic, conformal and concircular compatible,

(v) its Ricci 1*-forms are recurrent for the associated 1-form* $\pi = \left\{0, -\frac{x(-b^2-x^2+3b^2\alpha^2+x^2\alpha^2)}{(b^2+x^2)^2(-1+\alpha^2)}, 0, 0\right\}$.

Again, the components (upto symmetry) of the conformal curvature C_{mnop}, which do not vanish, are given below

$$C_{1212} = \frac{x^2(\alpha^2-1)-b^2(\alpha^2+1)}{3\alpha^2(b^2+x^2)^2},\ \ C_{1313} = \frac{b^2+x^2+(b^2-x^2)\alpha^2}{6(b^2+x^2)},\ \ C_{1414} = \sin^2\theta C_{1313},$$
$$C_{2323} = -\frac{b^2+x^2+(b^2-x^2)\alpha^2}{6(b^2+x^2)\alpha^2},\ \ C_{2424} = \sin^2\theta C_{2323},\ \ C_{3434} = \frac{1}{3}(b^2 + x^2 + (b^2 - x^2)\alpha^2)\sin^2\theta.$$

The components (upto symmetry) of the projective curvature P_{mnop}, which are found non-zero, are listed as follows:

$$P_{1221} = \frac{2b^2}{3(b^2+x^2)^2},\ \ P_{1331} = \frac{1}{3}(\alpha^2 - 1),\ \ P_{1441} = \sin^2\theta P_{1331},\ \ P_{2323} = -\frac{b^2}{3(b^2+x^2)},$$
$$P_{2332} = \frac{1}{3}(-1 + \frac{3b^2}{b^2+x^2} + \frac{1}{\alpha^2}),\ \ P_{2424} = \sin^2\theta P_{2323},\ \ P_{2442} = \sin^2\theta P_{2332},$$
$$P_{3434} = \frac{1}{3}(-2x^2(\alpha^2 - 1) + b^2(\alpha^2 + 2))\sin^2\theta = -P_{3443}.$$

The components (upto symmetry) of the tensor $(R \cdot R)_{mnopqs}$, which do not vanish, are given below

$$(R \cdot R)_{243423} = \frac{b^2(b^2+x^2+(b^2-x^2)\alpha^2)\sin^2\theta}{(b^2+x^2)^2},\ \ (R \cdot R)_{233424} = -\frac{b^2(b^2+x^2+(b^2-x^2)\alpha^2)\sin^2\theta}{(b^2+x^2)^2}.$$

The non-zero components (upto symmetry) of the tensor $(P \cdot R)_{mnopqs}$ are listed as follows:

$$(P \cdot R)_{243423} = \frac{2b^2(b^2+x^2+(b^2-x^2)\alpha^2)\sin^2\theta}{3(b^2+x^2)^2} = (P \cdot R)_{233442},$$
$$(P \cdot R)_{233424} = -\frac{2b^2(b^2+x^2+(b^2-x^2)\alpha^2)\sin^2\theta}{3(b^2+x^2)^2} = (P \cdot R)_{243432}.$$

The components (upto symmetry) of the Tachibana tensor $Q(S, R)_{mnopqs}$, which do not vanish, are given below

$$Q(S, R)_{243423} = \frac{b^2(b^2+x^2+(b^2-x^2)\alpha^2)\sin^2\theta}{(b^2+x^2)^2},\ \ Q(S, R)_{233424} = -\frac{b^2(b^2+x^2+(b^2-x^2)\alpha^2)\sin^2\theta}{(b^2+x^2)^2}.$$

Let $C^1 = C \cdot C$. Then the non-vanishing components (upto symmetry) of the tensor C^1 are computed as below

$$C^1_{122313} = -\frac{(b^2+x^2+(b^2-x^2)\alpha^2)^2}{12\alpha^2(b^2+x^2)^3} = -C^1_{121323},\ \ C^1_{122414} = -\frac{(b^2+x^2+(b^2-x^2)\alpha^2)\sin^2\theta}{12\alpha^2(b^2+x^2)^3} = -C^1_{121424},$$
$$C^1_{143413} = -\frac{(b^2+x^2+(b^2-x^2)\alpha^2)^2\sin^2\theta}{12(b^2+x^2)^2} = -C^1_{133414},\ \ C^1_{243423} = \frac{(b^2+x^2+(b^2-x^2)\alpha^2)^2\sin^2\theta}{12\alpha^2(b^2+x^2)^3} = -C^1_{233424}.$$

Let $C^2 = Q(g, C)$. Then the non-zero components (upto symmetry) of the tensor C^2 are shown as follows:

$$C^2_{122313} = \frac{1}{2} - \frac{b^2}{b^2+x^2} - \frac{1}{2\alpha^2} = -C^2_{121323},\ \ C^2_{143413} = -\frac{1}{2}(b^2 + x^2 + (b^2 - x^2)\alpha^2)\sin^2\theta = -C^2_{133414},$$
$$C^2_{122414} = -\frac{(b^2+x^2+(b^2-x^2)\alpha^2)\sin^2\theta}{2\alpha^2(b^2+x^2)} = -C^2_{121424},\ \ C^2_{243423} = \frac{(b^2+x^2+(b^2-x^2)\alpha^2)\sin^2\theta}{2\alpha^2} = -C^2_{233424}.$$

From the above calculation, it can be concluded that the TCEBW spacetime agrees several geometric structures, which are stated as follows:

Theorem 3.2. *The TCEBW spacetime possesses the following curvature properties:*

(i) it admits Ricci generalized pseudosymmetry as $R \cdot R = Q(S, R)$,

(ii) *it admits* $C \cdot C = -\frac{(-b^2-x^2-b^2\alpha^2+x^2\alpha^2)}{6(b^2+x^2)^2}Q(g,C)$ *and hence* $C \cdot K = -\frac{(-b^2-x^2-b^2\alpha^2+x^2\alpha^2)}{6(b^2+x^2)^2}Q(g,K)$,

(iii) *also it admits* $K \cdot C = -\frac{(-1+\alpha^2)}{2(b^2+x^2)}Q(g,C)$ *and hence* $K \cdot K = -\frac{(-1+\alpha^2)}{2(b^2+x^2)}Q(g,K)$,

(iv) *its conformal* 2*-forms are recurrent for the* 1*-form* $\left\{0, \frac{x(b^2+x^2-(3b^2+x^2)\alpha^2)}{(b^2+x^2)(b^2+x^2+(b^2-x^2)\alpha^2)}, 0, 0\right\}$,

(v) *Ricci* 1*-forms are recurrent for the associated* 1*-form* $\pi = \left\{0, -\frac{x(-b^2-x^2+3b^2\alpha^2+x^2\alpha^2)}{(b^2+x^2)^2(-1+\alpha^2)}, 0, 0\right\}$,

(vi) 2*-quasi Einstein manifold as rank*$(S - \lambda g) = 2$ *for* $\lambda = \frac{1}{2}(b^2 + x^2 - \sqrt{4 + b^4 + 2b^2x^2 + x^4})$,

(vii) *it is generalized Roter type as it possesses the relation (3), where* $\mathcal{R}_{ij}$*'s* $(i, j = 1, 2, 3)$ *are given in (4),*

(viii) *it is* $Ein(3)$ *manifold as it satisfies* $S^3 + \frac{b^2+x^2-\alpha^2(3b^2+x^2)}{(b^2+x^2)^2}S^2 + \frac{2b^2\alpha^2(\alpha^2-1)}{(b^2+x^2)^3}S = 0$,

(ix) *the general form of R-compatible tensor is given below*

$$\begin{pmatrix} \mathcal{T}_{11} & 0 & 0 & 0 \\ \mathcal{T}_{21} & \mathcal{T}_{22} & \mathcal{T}_{23} & \mathcal{T}_{24} \\ \mathcal{T}_{31} & -\frac{(b^2+x^2-x^2\alpha^2)\mathcal{T}_{23}}{b^2\alpha^2} & \mathcal{T}_{33} & \mathcal{T}_{34} \\ \mathcal{T}_{41} & -\frac{(b^2+x^2-x^2\alpha^2)\mathcal{T}_{24}}{b^2\alpha^2} & \mathcal{T}_{34} & \mathcal{T}_{44} \end{pmatrix}$$

where $\mathcal{T}_{ij}$ *are arbitrary scalars,*

(x) *the general form of C-compatible and K-compatible tensors are given below*

$$\begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{21} & 0 & 0 \\ \mathcal{T}_{21} & \mathcal{T}_{22} & 0 & 0 \\ 0 & 0 & \mathcal{T}_{33} & \mathcal{T}_{34} \\ 0 & 0 & \mathcal{T}_{34} & \mathcal{T}_{44} \end{pmatrix}$$

where $\mathcal{T}_{ij}$ *are arbitrary scalars,*

(xi) *the general form of P-compatible tensor is given below*

$$\begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} & 0 & 0 \\ -\frac{(b^2+x^2+2b^2\alpha^2-x^2\alpha^2)\mathcal{T}_{12}}{(b^2+x^2)(-1+\alpha^2)} & \mathcal{T}_{22} & -\frac{(b^2+x^2+2b^2\alpha^2-x^2\alpha^2)\mathcal{T}_{32}}{2b^2+2x^2+b^2\alpha^2-2x^2\alpha^2} & -\frac{(b^2+x^2+2b^2\alpha^2-x^2\alpha^2)\mathcal{T}_{42}}{2b^2+2x^2+b^2\alpha^2-2x^2\alpha^2} \\ 0 & \mathcal{T}_{32} & \mathcal{T}_{33} & \mathcal{T}_{34} \\ 0 & \mathcal{T}_{42} & \mathcal{T}_{34} & \mathcal{T}_{44} \end{pmatrix}$$

where $\mathcal{T}_{ij}$ *are arbitrary scalars,*

(xii) *the general form of W-compatible tensor is given below*

$$\begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} & 0 & 0 \\ -\frac{(b^2+x^2+4b^2\alpha^2-x^2\alpha^2)\mathcal{T}_{12}}{-b^2-x^2+2b^2\alpha^2+x^2\alpha^2} & \mathcal{T}_{22} & 0 & 0 \\ 0 & 0 & \mathcal{T}_{33} & \mathcal{T}_{43} \\ 0 & 0 & \mathcal{T}_{43} & \mathcal{T}_{44} \end{pmatrix}$$

where $\mathcal{T}_{ij}$ *are arbitrary scalars,*

(xiii) *its Ricci tensor is compatible for R, P, W, K and C.*

Corollary 3.3. *Since the TCEBW spacetime is Ricci generalized pseudosymmetric (i.e.,* $R \cdot R = Q(S, R)$*), it follows that the spacetime also realizes the relation* $P \cdot R = \frac{2}{3}Q(S, R)$*, i.e., it is Ricci generalized projectively pseudosymmetric spacetime.*

Remark 3.4. *The TCEBW spacetime has the following curvature restricted geometric properties of disagreement: The TCEBW spacetime does not satisfy*

(i) any semisymmetric type condition,

(ii) cyclic parallel Ricci tensor and Codazzi type Ricci tensor,

(iii) Chaki pseudosymmetry,

(iv) $K \cdot R = f_R Q(g, R)$ *for any smooth function* f_R*, and consequently it is not pseudosymmetric as well as projectively, conformally and concircularly pseudosymmetric.*

Also, the TCEBW spacetime is not

(v) Roter type,

(vi) Σ*-Venzi space for* $\Sigma = R, P, W, C$ *and* K*,*

(vii) recurrent for P*,* W *and* K*.*

4. Ricci soliton admitted by topologically charged Ellis-Bronnikov-type wormhole spacetime

The set $\mathcal{K}(M)$ of all Killing vector fields on M constitutes a Lie subalgebra of the Lie algebra $\chi(M)$ of all smooth vector fields on M. It is well-known that $\mathcal{K}(M)$ consists of maximum $n(n+1)/2$ linearly independent Killing vector fields and M is called maximally symmetric if $\mathcal{K}(M)$ contains exactly $n(n+1)/2$ linearly independent Killing vector fields. A sufficient condition for M to be maximally symmetric is that the scalar curvature of M is constant. We must mention that the TCEBW spacetime has non-constant scalar curvature given by $\kappa = \frac{2b^2(2\alpha^2-1)+2x^2(\alpha^2-1)}{(b^2+x^2)^2}$. By proceeding a little straightforward calculation, it can be obtained that the vector fields $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ are Killing (i.e., $\pounds_\xi g = 0$ for $\xi = \frac{\partial}{\partial t}, \frac{\partial}{\partial \phi}$) on TCEBW spacetime. Hence, the vector field $\lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial}{\partial \phi}$ is also Killing for each real number λ_1 and λ_2.

If $\mathcal{D} = \pounds_\xi g$ for the non-Killing vector field $\xi = \frac{\partial}{\partial x}$, then the non-zero components of $\mathcal{A}$ are calculated as follows:

$$\mathcal{D}_{33} = 2r, \quad \mathcal{D}_{44} = 2r\sin^2\theta.$$

From the above components of $\pounds_{\frac{\partial}{\partial x}} g$, it can be concluded that the TCEBW spacetime satisfies the following relation:

$$\pounds_{\frac{\partial}{\partial x}} g + 2\ell_1 S + 2\ell_2 g + 2\ell_3 \eta \otimes \eta = 0,$$

where the 1-form $\eta = \left(\frac{2b\alpha\sqrt{x}}{\sqrt{(b^2+x^2)^2+\alpha^2(b^4-x^4)}}, 0, 0, 0\right)$ and ℓ_1, ℓ_2, ℓ_3 are computed as follows:

$$\left.\begin{aligned} \ell_1 &= \frac{x(b^2+x^2)}{b^2+x^2+\alpha^2(b^2-x^2)}, \\ \ell_2 &= -\frac{2b^2x\alpha^2}{(b^2+x^2)[b^2+x^2+\alpha^2(b^2-x^2)]}, \\ \ell_3 &= -\frac{1}{2}. \end{aligned}\right\} \tag{5}$$

Hence, we can state the following:

Theorem 4.1. *The TCEBW spacetime admits almost η-Ricci-Yamabe soliton with respect to the soliton vector field $\frac{\partial}{\partial x}$ and the 1-form $\eta = \left(\frac{2b\alpha\sqrt{x}}{\sqrt{(b^2+x^2)^2+\alpha^2(b^4-x^4)}}, 0, 0, 0\right)$ if $b^2 + x^2 + \alpha^2(b^2 - x^2) \neq 0$ and $(b^2 + x^2)^2 + \alpha^2(b^4 - x^4) \neq 0$, i.e., the TCEBW spacetime satisfies*

$$\frac{1}{2}\pounds_\xi g + \ell_1 S + \left(\lambda - \frac{1}{2}\ell\kappa\right) g + \ell_3 \eta \otimes \eta = 0,$$

for the non-Killing soliton vector field $\xi = \frac{\partial}{\partial x}$, where $\ell = 2$, $\lambda = \ell_2 + \kappa$, and ℓ_1, ℓ_2, ℓ_3 are given in (5).

Theorem 4.2. *If $(x-1)(b^2 + x^2) = \alpha^2(b^2 - x^2)$, then the TCEBW spacetime realizes an almost η-Ricci soliton for the soliton vector field $\frac{\partial}{\partial x}$ with the 1-form $\eta = \left(\frac{2b\alpha\sqrt{x}}{\sqrt{(b^2+x^2)^2+\alpha^2(b^4-x^4)}}, 0, 0, 0\right)$ provided $b^2 + x^2 + \alpha^2(b^2 - x^2) \neq 0$ and $(b^2 + x^2)^2 + \alpha^2(b^4 - x^4) \neq 0$, i.e., the TCEBW spacetime possesses the relation*

$$\frac{1}{2}\pounds_\xi g + S + \ell_2 g + \ell_3 \eta \otimes \eta = 0,$$

for the non-Killing vector field $\xi = \frac{\partial}{\partial x}$, where ℓ_2, ℓ_3 is given in (5).

5. Morris-Thorne wormhole metric and topologically charged Ellis-Bronnikov-type wormhole metric

MTW spacetime [42] is a stationary solution of Einstein field equations with a cosmological constant. In spherical coordinates (t, l, v, ϕ), the metric of MTW spacetime is given as follows:

$$ds^2 = g_{ij}dx^i dx^j = -c^2\, dt^2 + dl^2 + (b^2 + l^2)\, dv^2 + (b^2 + l^2)\, \sin^2 v d\phi^2,$$

where c is the speed of light, t is the global time, b is the shape constant and l is the proper radial coordinate respectively. It represents a model of wormhole and theoretically a wormhole can be traversed. We note that the TCEBW spacetime is also a model of a wormhole. Hence we compare the curvature restricted geometric properties of MTW and TCEBW spacetime, described as follows:

Similarities:

(i) both are Ricci generalized pseudosymmetric,

(ii) conformal 2-forms of both the spacetimes are recurrent,

(iii) both the spacetimes are pseudosymmetric due to conformal curvature,

(iv) Ricci tensor of both the spacetimes are R, P, K, W and C compatible.

Dissimilarities:

(i) MTW spacetime defies to be generalized Roter type, whereas TCEBW spacetime is generalized Roter type,

(ii) MTW spacetime is Ricci simple, but TCEBW spacetime agrees to be 2-quasi Einstein instead of Ricci simple,

(iii) MTW spacetime is Ein(2), whereas TCEBW spacetime is Ein(3).

6. Conclusions

The paper is concerned about the investigation of geometrical properties of TCEBW spacetime. It is shown that the TCEBW spacetime admits several pseudosymmetric type structures (see, Theorem 3.2), such as, pseudosymmetry due to conformal curvature as well as conharmonic curvature, Ricci generalized pseudosymmetry and Ricci generalized projectively pseudosymmetry, even though it is neither semisymmetric nor pseudosymmtric. Also, we have found that the TCEBW spacetime is an Einstein manifold of degree 3, generalized Roter type and 2-quasi-Einstein (see, Theorem 3.2), but its Ricci tensor is neither Codazzi type nor cyclic parallel. In Theorem 4.1 and 4.2, it is shown that the TCEBW spacetime realizes almost η-Ricci soliton as well as almost η-Ricci-Yamabe soliton. Finally, a comparative study between MTW and TCEBW spacetime has been exhibited in Section 5.

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Conflict of Interest

There is no conflict of interests regarding publication of this paper.

Data Availability Statement

No new data are generated or analysed during this study.

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