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Kachurovskij spectra nonlinear block operator matrices

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Abstract. In this paper we extend the results obtained by X. Dong and D. Wu in [1] to 3×3 Lipschitz continuous nonlinear operator matrices. In this work, the Kachurovskij spectrum of 3×3 Lipschitz continuous nonlinear operator matrices are studied. Firstly, some connections between the Kachurovskij spectrum of certain 3×3 Lipschitz continuous nonlinear operator matrices and that of their entries are established, and the relationship between the Kachurovskij spectrum of 3×3 Lipschitz continuous nonlinear operator matrices and that of their Schur complement is presented by means of Schur decomposition.

1. Introduction

The spectrum for Lipschitz continuous operators which was defined by Kachurovskij in 1969, as well as a spectrum for linearly bounded operators introduced by Dörfner in 1997. In [1] X. Dong and D. Wu study the Kachurovskij spectrum of 2×2 Lipschitz continuous nonlinear operator matrices. The authors give some connections between the Kachurovskij spectrum of certain 2×2 nonlinear operator matrices and that of their entries.

In this paper, the Kachurovskij spectrum of 3×3 Lipschitz continuous nonlinear operator matrices are studied. Firstly, some connections between the Kachurovskij spectrum of certain 3×3 Lipschitz continuous nonlinear operator matrices and that of their entries are established, and the relationship between the Kachurovskij spectrum of 3×3 Lipschitz continuous nonlinear operator matrices and that of their Schur complement is presented by means of Schur decomposition.

Let X be an infinite dimensional complex Hilbert space. Let $\mathcal{C}(X)$ denote the set of all continuous (in general, nonlinear) operators from X into X, and let $\mathcal{L}(X)$ denote the set of all bounded linear operators from X into X. For $F \in \mathcal{C}(X)$,

$$[F]_{Lip} := \sup_{x \neq y} \frac{\|F(x) - F(y)\|}{\|x - y\|} \tag{1}$$

$$[F]_{lip} := \inf_{x \neq y} \frac{\|F(x) - F(y)\|}{\|x - y\|} \tag{2}$$

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If $[F]_{Lip} < \infty$, we write $F \in \mathfrak{Lip}(\mathfrak{X})$, and call F Lipschitz continuous. Let $\mathfrak{Lip}(\mathfrak{X})$ denote the set of all Lipschitz continuous operators from X into X, which map 0 into 0. Note that if F is a bounded linear operator, we have $[F]_{Lip} = \|F\|$. In addition, we define subset of $\mathbb{C}$ by means of the lower characteristic (2):

$$\sigma_{lip}(F) := \{\lambda \in \mathbb{C} \text{ such that } [F-\lambda]_{lip} = 0\}.$$

If $[F-\lambda]_{lip} > 0$, then F is injective and closed.

Definition 1.1. *Given $F \in \mathfrak{Lip}(\mathfrak{X})$, we call the set*

$$\rho_K(F) := \{\lambda \in \mathbb{C} \text{ such that } F-\lambda \text{ is bijective and } (F-\lambda)^{-1} \in \mathfrak{Lip}(\mathfrak{X})\}.$$

The Kachurovskij resolvent set and its complement

$$\sigma_K(F) = \mathbb{C} \backslash \rho_K(F)$$

the Kachurovskij spectrum of F. ◊

$\lambda \in \rho_K(F)$ if, and only if, $F-\lambda$ is a lipeomorphism, i.e., $F-\lambda$ is bijective, and satisfies $[F-\lambda]_{Lip} < \infty$ and $[F-\lambda]_{lip} > 0$.
In the case of a bounded linear operator F, $\sigma_{lip}(F)$ is the approximate point spectrum of F and $\sigma_K(F)$ is the usual spectrum of F.

Definition 1.2. *Let X be a Hilbert space, $F : X \longrightarrow X$ be continuous, the numerical range $W_Z(F)$ of F is denoted by*

$$W_Z(F) := \left\{ \frac{\langle F(x) - F(y), x-y \rangle}{\|x-y\|^2}, \;\; x, \; y \in X, \;\; x \neq y \right\}.$$

◊

Obviously, this definition coincides with the numerical range of Toeplitz [2] in the linear case.

Lemma 1.3. *Let X be a Banach space and $F : X \longrightarrow X$ a lipeomorphism. Suppose that $G \in \mathfrak{Lip}(\mathfrak{X})$ satisfies $[G]_{Lip} < [F]_{lip} F$. Then, $F+G$ is also a lipeomorphism and*

$$[(F+G)^{-1}]_{Lip} \leq \frac{[F^{-1}]_{Lip}}{1 - [G]_{Lip}[F^{-1}]_{Lip}} = \frac{1}{[F]_{lip} - [G]_{Lip}}.$$

◊

Lemma 1.4. *Let X be a Hilbert space, $F \in \mathfrak{Lip}(\mathfrak{X})$ with $F(0) = 0$, and $\lambda \in \mathbb{C}$ with*

$$d_\lambda := dist(\lambda, W_Z(F)) > 0.$$

Then $F - \lambda I$ is a lipeomorphism with

$$[(F-\lambda)^{-1}]_{Lip} \leq \frac{1}{d_\lambda}.$$

◊

2. Main results

First, we study the Kachurovskij spectrum of 3×3 diagonal block operator matrices.

Proposition 2.1. *Let*

$$\mathrm{L}_0 = \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{pmatrix} \in \mathcal{C}(X \times X \times X)$$

with A, E, and K are in $\mathfrak{Lip}(\mathfrak{X})$*.Then,*

(*i*) $\sigma_{lip}(\mathrm{L}_0) = \sigma_{lip}(A) \bigcup \sigma_{lip}(E) \bigcup \sigma_{lip}(K)$.
(*ii*) $\sigma_K(\mathrm{L}_0) = \sigma_K(A) \bigcup \sigma_K(E) \bigcup \sigma_K(K)$. ◊

Proof. (*i*) Let $\lambda \in \sigma_{lip}(A)$, then there exist sequences $(x_n^{(1)})$ and $(y_n^{(1)})$ of X with $x_n^{(1)} \neq y_n^{(1)}$, such that

$$\frac{\|(A-\lambda)x_n^{(1)} - (A-\lambda)y_n^{(1)}\|}{\|x_n^{(1)} - y_n^{(1)}\|} \to 0 \text{ as } n \to \infty.$$

Set $x_n = \begin{pmatrix} x_n^{(1)} \\ 0 \\ 0 \end{pmatrix}$, $y_n = \begin{pmatrix} y_n^{(1)} \\ 0 \\ 0 \end{pmatrix}$, then $x_n \neq y_n$, $n = 1, 2, \cdots$, and

$$\frac{\|(\mathrm{L}_0-\lambda)x_n - (\mathrm{L}_0-\lambda)y_n\|}{\|x_n^{-}y_n^{\|}} = \frac{\|(A-\lambda)x_n^{(1)} - (A-\lambda)y_n^{(1)}\|}{\|x_n^{(1)} - y_n^{(1)}\|} \to 0 \text{ as } n \to \infty$$

i.e., $\lambda \in \sigma_{lip}(\mathrm{L}_0)$. By a similar argument, we can show that

$$\sigma_{lip}(E) \bigcup \sigma_{lip}(K) \subset \sigma_{lip}(\mathrm{L}_0).$$

Conversely, let $\lambda \in \sigma_{lip}(\mathrm{L}_0)$, assume that $\lambda \notin \sigma_{lip}(A) \bigcup \sigma_{lip}(E) \bigcup \sigma_{lip}(K)$. Then, $[A-\lambda]_{lip} > 0$, $[E-\lambda]_{lip} > 0$, and $[K-\lambda]_{lip} > 0$. Thus, for any vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ with $x \neq y$, we have

$$\begin{aligned} &\frac{\|(\mathrm{L}_0-\lambda)x - (\mathrm{L}_0-\lambda)y\|}{\|x-y\|} \\ &= \sqrt{\frac{\|(A-\lambda)x_1 - (A-\lambda)y_1\|^2 + \|(E-\lambda)x_2 - (E-\lambda)y_2\|^2 + \|(K-\lambda)x_3 - (K-\lambda)y_3\|^2}{\|x_1-y_1\|^2 + \|x_2-y_2\|^2 + \|x_3-y_3\|^2}} \\ &\geq \sqrt{\frac{[A-\lambda]_{lip}\|x_1-y_1\|^2 + [E-\lambda]_{lip}\|x_2-y_2\|^2 + [K-\lambda]_{lip}\|x_3-y_3\|^2}{\|x_1-y_1\|^2 + \|x_2-y_2\|^2 + \|x_3-y_3\|^2}} \\ &\geq \min([A-\lambda]_{lip}, [A-\lambda]_{lip}, [K-\lambda]_{lip}) > 0, \end{aligned}$$

and hence $[\mathrm{L}_0-\lambda]_{lip} > 0$, which lead a contradiction. Thus $\lambda \in \sigma_{lip}(A) \bigcup \sigma_{lip}(E) \bigcup \sigma_{lip}(K)$. Therefore, $\sigma_{lip}(\mathrm{L}_0) = \sigma_{lip}(A) \bigcup \sigma_{lip}(E) \bigcup \sigma_{lip}(K)$.

(*ii*) From (*i*), we know that $[\mathrm{L}_0-\lambda]_{lip} > 0$ if and only if $[A-\lambda]_{lip} > 0$, $[A-\lambda]_{lip} > 0$, and $[K-\lambda]_{lip} > 0$. Clearly, L_0 is bijective if and only if $A-\lambda$, $K-\lambda$, and $K-\lambda$ are bijective. Thus, $\rho_K(\mathrm{L}_0) = \rho_K(A) \bigcup \rho_K(E) \bigcup \rho_K(K)$, therefore $\sigma_K(\mathrm{L}_0) = \sigma_K(A) \bigcup \sigma_K(E) \bigcup \sigma_K(K)$. □

Now, we consider upper triangular operator matrices.

Proposition 2.2. *Let*

$$\mathrm{L}_0 = \begin{pmatrix} A & B & C \\ 0 & E & F \\ 0 & 0 & K \end{pmatrix} \in \mathcal{C}(X \times X \times X)$$

with A, B, C, E, F, and K are in $\mathfrak{Lip}(\mathfrak{X})$. *Then,*

$\sigma_{lip}(A) \subset \sigma_{lip}(\mathrm{L}_0) \subset \sigma_{lip}(A) \bigcup \sigma_{lip}(E) \bigcup \sigma_{lip}(K).$ ◇

Proof. It is easy to see that $\sigma_{lip}(A) \subset \sigma_{lip}(\mathrm{L}_0)$, and so we only need to prove that $\sigma_{lip}(\mathrm{L}_0) \subset \sigma_{lip}(A) \bigcup \sigma_{lip}(E) \bigcup \sigma_{lip}(K)$. Let $\lambda \in \sigma_{lip}(\mathrm{L}_0)$, then $[\mathrm{L}_0 - \lambda]_{lip} = 0$. Evidently, the factorization formula

$$\mathrm{L}_0 - \lambda = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & K-\lambda \end{pmatrix} \begin{pmatrix} I & 0 & C \\ 0 & I & F \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & E-\lambda & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & B & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A-\lambda & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{3}$$

holds. Write

$$U = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & K-\lambda \end{pmatrix}$$

$$R = \begin{pmatrix} I & 0 & C \\ 0 & I & F \\ 0 & 0 & I \end{pmatrix}$$

$$V = \begin{pmatrix} I & 0 & 0 \\ 0 & E-\lambda & 0 \\ 0 & 0 & I \end{pmatrix}$$

$$W = \begin{pmatrix} I & B & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

$$Z = \begin{pmatrix} A-\lambda & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

Clearly, R and W are a lipeomorphism, and hence $[R]_{lip} > 0$ and $[W]_{lip} > 0$. Since

$$[\mathrm{L}_0 - \lambda]_{lip} = [URVWZ]_{lip} \geq [U]_{lip}[R]_{lip}[V]_{lip}[W]_{lip}[Z]_{lip},$$

it follows that $[U]_{lip} = 0$ or $[V]_{lip} = 0$ or $[Z]_{lip}, = 0$, which implies that $\lambda \in \sigma_{lip}(A) \bigcup \sigma_{lip}(E) \bigcup \sigma_{lip}(K)$. Consequently, $\sigma_{lip}(\mathrm{L}_0) \subset \sigma_{lip}(A) \bigcup \sigma_{lip}(E) \bigcup \sigma_{lip}(K)$.
□

Corollary 2.3. *Let*

$$\mathrm{L}_0 = \begin{pmatrix} A & B & C \\ 0 & E & F \\ 0 & 0 & K \end{pmatrix} \in \mathcal{C}(X \times X \times X)$$

with A, B, C, E, F, and K are in $\mathfrak{Lip}(\mathfrak{X})$. *If* $\sigma_K(A) \bigcap \sigma_K(E) \bigcap \sigma_K(K) = \emptyset$, *then* $\sigma_K(\mathrm{L}_0) = \sigma_K(A) \bigcup \sigma_K(E) \bigcup \sigma_K(K)$. ◇

Theorem 2.4. *Let*

$$\mathrm{L}_0 = \begin{pmatrix} A & B & C \\ 0 & E & F \\ 0 & 0 & K \end{pmatrix} \in \mathcal{C}(X \times X \times X)$$

with A, B, C, E, F, and K are in $\mathfrak{Lip}(\mathfrak{X})$. *Then,* $\sigma_{lip}(A) \bigcup \sigma_{lip}(E) \bigcup \sigma_{lip}(K) = \sigma_{lip}(\mathrm{L}_0) \bigcup (\sigma_{lip}(E) \bigcap \Delta_1) \bigcup (\sigma_{lip}(K) \bigcap \Delta_2)$, *where* $\Delta_1 = \{\lambda \in \mathbb{C} : [A-\lambda]_{lip} > 0,$ *and* $A - \lambda$ *is not surjective*$\}$ *and* $\Delta_2 = \{\lambda \in \mathbb{C} : [E-\lambda]_{lip} > 0,$ *and* $E - \lambda$ *is not surjective*$\}$. ◇

Proof. It is easy to prove that

$$\sigma_{lip}(K)\bigcup\sigma_{lip}(A)\bigcup\sigma_{lip}(E)\supset\sigma_{lip}(\mathrm{L}_0)\bigcup[\sigma_{lip}(E)\bigcap\Delta_1]\bigcup[\sigma_{lip}(K)\bigcap\Delta_2].$$

Conversely, let $\lambda\in[\sigma_{lip}(K)\bigcup\sigma_{lip}(A)\bigcup\sigma_{lip}(E)]\backslash\sigma_{lip}(\mathrm{L}_0)$; then we have from Proposition 2.2 (i), $\lambda\in[\sigma_{lip}(E)\bigcup\sigma_{lip}(K)]\backslash\sigma_{lip}(A)$, and hence $\lambda\in\Delta_1\bigcup\rho_K(A)$. Assume that $\lambda\in\rho_K(A)$, then $A-\lambda$ is a lipeomorphism. Make the factorisation as in (3), we have that

$$[U]_{lip}=[(\mathrm{L}_0-\lambda)Z^{-1}W^{-1}V^{-1}R^{-1}]_{lip}\geq[\mathrm{L}_0-\lambda]_{lip}[Z^{-1}]_{lip}[W^{-1}]_{lip}[V^{-1}]_{lip}[R^{-1}]_{lip}>0.$$

Thus $[K-\lambda]_{lip}>0$. So, $\lambda\notin\sigma_{lip}(K)$. Make the factorisation as in (3), we have that

$$[V]_{lip}=[R^{-1}U^{-1}(\mathrm{L}_0-\lambda)Z^{-1}W^{-1}]_{lip}\geq[R^{-1}]_{lip}[U^{-1}]_{lip}[\mathrm{L}_0-\lambda]_{lip}[Z^{-1}]_{lip}[W^{-1}]_{lip}>0.$$

Thus $[E-\lambda]_{lip}>0$. So, $\lambda\notin\sigma_{lip}(E)$. Which is a contradiction to $\lambda\in\sigma_{lip}(K)\bigcup\sigma_{lip}(E)$. Hence, $\lambda\in\Delta_1$. Therefore

$$[\sigma_{lip}(A)\bigcup\sigma_{lip}(E)\bigcup\sigma_{lip}(K)]\backslash\sigma_{lip}(\mathrm{L}_0)\subset\sigma_{lip}(E)\bigcap\Delta_1.$$

The same, we can prove $[\sigma_{lip}(A)\bigcup\sigma_{lip}(E)\bigcup\sigma_{lip}(K)]\backslash\sigma_{lip}(\mathrm{L}_0)\subset\sigma_{lip}(K)\bigcap\Delta_2$. □

Corollary 2.5. *Let*

$$\mathrm{L}_0=\begin{pmatrix}A & B & C\\ 0 & E & F\\ 0 & 0 & K\end{pmatrix}\in\mathcal{C}(X\times X\times X)$$

with A, B, C, E, F, *and* K *are in* $\mathfrak{Lip}(\mathfrak{X})$. *Then,*

$$\sigma_{lip}(A)\bigcup\sigma_{lip}(E)\bigcup\sigma_{lip}(K)=\sigma_{lip}(\mathrm{L}_0)\tag{4}$$

if and only if $\sigma_{lip}(E)\bigcap\Delta_1\subset\sigma_{lip}(\mathrm{L}_0)$ *and* $\sigma_{lip}(K)\bigcap\Delta_2\subset\sigma_{lip}(\mathrm{L}_0)$. *Moreover, if* $\sigma_{lip}(E)\bigcap\Delta_1=\sigma_{lip}(K)\bigcap\Delta_2=\emptyset$, *then (19) is hold.* ◊

3. Frobenius-Schur's decomposition

In this section, we are concerned with a 3×3 block operator matrix

$$\mathrm{L}_0:=\begin{pmatrix}A & B & C\\ D & E & F\\ G & H & L\end{pmatrix},\tag{5}$$

where the entries of the matrix are in general unbounded operators. The operator (5) is defined on $(\mathcal{D}(A)\bigcap\mathcal{D}(D)\bigcap\mathcal{D}(G))\times(\mathcal{D}(B)\bigcap\mathcal{D}(E)\bigcap\mathcal{D}(H))\times(\mathcal{D}(C)\bigcap\mathcal{D}(F)\bigcap\mathcal{D}(L))$. The essential work in this section is to impose some conditions on the entries of the operator L_0 in order to establish its closedness. In the product of Banach spaces $X\times Y\times Z$, we consider the operator L_0 defined by (5) where the operator A acts on X and has a domain $\mathcal{D}(A)$, the operator E acts on Y and has a domain $\mathcal{D}(E)$, and the operator L acts on Z and has a domain $\mathcal{D}(L)$. The intertwining operator B is defined on the domain $\mathcal{D}(B)\subset Y$ into X, the operator H is defined on the domain $\mathcal{D}(H)\subset Y$ into Z, the operator C is defined on the domain $\mathcal{D}(C)\subset Z$ into X, the operator F is defined on the domain $\mathcal{D}(F)\subset Z$ into Y, the operator D is defined on the domain $\mathcal{D}(D)\subset X$ into Y, and the operator G is defined on the domain $\mathcal{D}(G)\subset X$ into Z. In what follows, we will consider the following hypotheses(see [5]):

($M1$) The operator A is a closed, densely defined linear operator on X, with a nonempty resolvent set $\rho(A)$.

($M2$) The operator D (resp. G) verifies that $\mathcal{D}(A)\subset\mathcal{D}(D)$ (resp. $\mathcal{D}(A)\subset\mathcal{D}(G)$) and, for some (hence for all) $\mu\in\rho(A)$, the operator $D(A-\mu)^{-1}$ (resp. $G(A-\mu)^{-1}$) is bounded.

Let $F_1(\mu) := D(A-\mu)^{-1}$, and $F_2(\mu) := G(A-\mu)^{-1}$.

• In particular, if D (resp. G) is closable then, from the closed graph theorem, it follows that $F_1(\mu)$ (resp. $F_2(\mu)$) is bounded.

$(M3)$ The operator B (resp. C) is densely defined on Y (resp. Z) and, for some (hence for all) $\mu \in \rho(A)$, the operator $(A-\mu)^{-1}B$ (resp. $(A-\mu)^{-1}C$) is bounded on its domain.

Now, let $G_1(\mu) := \overline{(A-\mu)^{-1}B}$, and $G_2(\mu) := \overline{(A-\mu)^{-1}C}$.

$(M4)$ The lineal $\mathcal{D}(B) \bigcap \mathcal{D}(E)$ is dense in Y and, for some (hence for all) $\mu \in \rho(A)$, the operator $S_1(\mu) := E - D(A-\mu)^{-1}B$ is closed.

$(M5)$ $\mathcal{D}(C) \subset \mathcal{D}(F)$, and the operator $F - D(A-\mu)^{-1}C$ is bounded on its domain, for some $\mu \in \rho(A)$ and therefore, for all $\mu \in \rho(A)$. We will also suppose that there exists μ such that $\mu \in \rho(A) \bigcap \rho(S_1(\mu))$ and we will denote $G_3(\mu)$ by

$$G_3(\mu) := \overline{(S_1(\mu)-\mu)^{-1}(F - D(A-\mu)^{-1}C)}.$$

• To explain this, let $\mu \in \rho(A)$, such that $F - D(A-\mu)^{-1}C$ is bounded on its domain. Then, for an arbitrary $\lambda \in \rho(A)$, we have

$$F - D(A-\lambda)^{-1}C = F - D(A-\mu)^{-1}C + (\mu-\lambda)F_1(\mu)(A-\lambda)^{-1}C.$$

From the assumptions $(M2)$ and $(M3)$, it follows that the operator on the right-hand side is bounded on its domain. Then, the boundedness of the operator $F - D(A-\mu)^{-1}C$ does not depend on $\mu \in \rho(A)$. We will denote $G_4(\mu)$ by $G_4(\mu) := \overline{F - D(A-\mu)^{-1}C}$.

Remark 3.1. *If the operators A and E generate C_0-semigroups, and if the operators D and B are bounded, then there exists $\mu \in \mathbb{C}$, such that $\mu \in \rho(A) \bigcap \rho(S_1(\mu))$. Indeed, it is well known that, if the operators A and E generate C_0-semigroups then, there exist two constants $M > 0$ and $w > 0$, such that $\|(\mu - T)^{-1}\| \leq \frac{M}{Re\mu - w}$, where $T \in \{A, E\}$ for all μ such that $Re\mu > w$. For a fixed $\mu \in \mathbb{C}$ chosen in such a way that $Re\mu > w + \alpha$, where $\alpha > 0$, we consider the following resolvent equation of $S_1(\mu)$*

$$(\lambda - E + D(A-\mu)^{-1}B)\varphi = \psi. \tag{6}$$

Since $\lambda \in \rho(E)$, we deduce that, for $Re\lambda > w + \alpha$, Eq. (6) *may be transformed into*

$$[I + (\lambda - E)^{-1}D(\mu - A)^{-1}B]\varphi = (\lambda - E)^{-1}\psi.$$

The fact that

$$\|(\lambda - E)^{-1}D(\mu - A)^{-1}B\| \leq \frac{M^2\|D\|\|B\|}{\alpha(Re\lambda - w)}$$

allows us to conclude that $\lim\limits_{Re\lambda\to+\infty} \|(\lambda - E)^{-1}D(\mu - A)^{-1}B\| = 0$. Hence, there exists $\beta > w + \alpha$ such that, for $Re\lambda > \beta$, we have $r_\sigma((\lambda - E)^{-1}D(\mu - A)^{-1}B) < 1$, where $r_\sigma(.)$ represents the spectral radius. Hence for μ, such that $Re\mu > \beta$, we have $\mu \in \rho(A)$ and $\mu \in \rho(S_1(\mu))$. Moreover, we can write

$$(\mu - S_1(\mu))^{-1} = \sum_{n\geq 0}[(\mu - E)^{-1}D(\mu - A)^{-1}B]^n(\mu - E)^{-1}. \qquad \diamond$$

$(M6)$ The operator H satisfies the fact that $\mathcal{D}(B) \subset \mathcal{D}(H)$ and, for some (hence for all) $\mu \in \rho(A) \bigcap \rho(S_1(\mu))$, the operator $(H - G(A-\mu)^{-1}B)(S_1(\mu) - \mu)^{-1}$ is bounded. Set

$$F_3(\mu) := (H - G(A-\mu)^{-1}B)(S_1(\mu) - \mu)^{-1}.$$

$(M7)$ For the operator K, we will assume that $\mathcal{D}(C) \subset \mathcal{D}(K)$ and, for some (hence for all) $\mu \in \rho(A) \bigcap \rho(S_1(\mu))$, the operator

$$L - G(A-\mu)^{-1}C - [H - G(A-\mu)^{-1}B](S_1(\mu) - \mu)^{-1}[F - D(A-\mu)^{-1}C]$$

is closable. Let us denote by $S_2(\mu)$ this operator, and by $\overline{S}_2(\mu)$ its closure.

Remark 3.2. (*i*) *From the Hilbert identity, we get for* $\lambda, \mu \in \rho(A)$

$$S_1(\lambda) - S_1(\mu) = (\mu - \lambda)F_1(\mu)(A - \lambda)^{-1}B.$$

Since the operator $F_1(\mu)$ *is bounded and* $(A - \lambda)^{-1}B$ *is bounded on its domain, we deduce that neither the domain of* $S_1(\mu)$ *nor the property of being closable depends on the choice of* μ. *Then,*

$$S_1(\lambda) - S_1(\mu) = (\mu - \lambda)F_1(\mu)G_1(\lambda). \tag{7}$$

(*ii*) *Let* $\lambda \in \rho(A) \bigcap \rho(S_1(\lambda))$ *and* $\mu \in \rho(A) \bigcap \rho(S_1(\mu))$. *Then,*

$$\begin{aligned} S_2(\lambda) - S_2(\mu) &= (\mu - \lambda)F_2(\mu)(A - \lambda)^{-1}C - F_3(\lambda)[F - D(A - \lambda)^{-1}C] + \\ &\quad F_3(\mu)[F - D(A - \mu)^{-1}C] \\ &= (\mu - \lambda)F_2(\mu)(A - \lambda)^{-1}C - F_3(\lambda)[F - D(A - \lambda)^{-1}C] + \\ &\quad F_3(\mu)[F - D(A - \lambda)^{-1}C - (\mu - \lambda)D(A - \mu)^{-1}(A - \lambda)^{-1}C] \\ &= (\mu - \lambda)F_2(\mu)(A - \lambda)^{-1}C + [F_3(\mu) - F_3(\lambda)][F - D(A - \lambda)^{-1}C] + \\ &\quad (\lambda - \mu)F_3(\mu)F_1(\mu)(A - \lambda)^{-1}C. \end{aligned}$$

Since the operators $F_i(.)$, *with* $i = 1, 2, 3$ *are bounded everywhere and since the operators* $(A-\mu)^{-1}C$ *and* $F-D(A-\lambda)^{-1}C$ *are bounded on their domains then, the closedness of the operator* $S_2(\mu)$ *does not depend on the choice of* μ. *Hence,*

$$\overline{S}_2(\lambda) - \overline{S}_2(\mu) = (\mu - \lambda)F_2(\mu)G_2(\lambda) + [F_3(\mu) - F_3(\lambda)]G_4(\lambda) + (\lambda - \mu)F_3(\mu)F_1(\mu)G_2(\lambda). \tag{8}$$

◊

First, we will search the Frobenius-Schur's decomposition of the operator L_0 defined in (5). For this purpose, let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathcal{D}(L_0)$ and $\lambda \in \mathbb{C}$. Then,

$$(L_0 - \lambda)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ if, and only if, } \begin{pmatrix} A-\lambda & B & C \\ D & E-\lambda & F \\ G & H & K-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This leads to the following system

$$\begin{cases} (A - \lambda)x + By + Cz = 0 \\ Dx + (E - \lambda)y + Fz = 0 \\ Gx + Hy + (K - \lambda)z = 0. \end{cases}$$

$$\begin{cases} (A - \lambda)x = -By - Cz \\ Dx + (E - \lambda)y + Fz = 0 \\ Gx + Hy + (K - \lambda)z = 0. \end{cases} \tag{9}$$

Suppose that $\rho(A)$ is nonempty and let $\lambda \in \rho(A)$. Then, the first equation of the system (9) gives $x = -(A - \lambda)^{-1}By - (A - \lambda)^{-1}Cz$. Consequently, the second equation of (9) becomes

$$\begin{cases} D[(A - \lambda)^{-1}By + (A - \lambda)^{-1}Cz] - Fz + (\lambda - E)y = 0 \\ Gx + Hy + (K - \lambda)z = 0. \end{cases} \tag{10}$$

From Eq. (10), we must assume that $\mathcal{D}(A) \subset \mathcal{D}(C)$. Then, Eq. (10) becomes

$$\begin{cases} [E - \lambda - D(A - \lambda)^{-1}B]y = [(A - \lambda)^{-1}C - F]z \\ Gx + Hy + (K - \lambda)z = 0. \end{cases}$$

Let $S_1(\lambda) = E - D(A-\lambda)^{-1}B$. If $\lambda \in \rho(S_1(\lambda))$, then

$$y = (S_1(\lambda) - \lambda)^{-1}[(A-\lambda)^{-1}C - F]z.$$

Hence

$$\{-G(A-\lambda)^{-1}B(S_1(\lambda)-\lambda)^{-1}[(A-\lambda)^{-1}C - F] - G(A-\lambda)^{-1}C + H(S_1(\lambda)-\lambda)^{-1}[(A-\lambda)^{-1}C - F] + (K-\lambda)\}z = 0$$

Let $S_2(\lambda) = K - G(A-\lambda)^{-1}B(S_1(\lambda)-\lambda)^{-1}[(A-\lambda)^{-1}C - F] - G(A-\lambda)^{-1}C + H(S_1(\lambda)-\lambda)^{-1}[(A-\lambda)^{-1}C - F]$.

Now we can search $F_i(\mu)$, $i = 1,2,3$ and $G_i(\mu)$, $i = 1,2,3$ such that the operator

$$\begin{pmatrix} I & 0 & 0 \\ F_1(\mu) & I & 0 \\ F_2(\mu) & F_3(\mu) & I \end{pmatrix}\begin{pmatrix} A-\mu & 0 & 0 \\ 0 & S_1(\mu)-\mu & 0 \\ 0 & 0 & S_2(\mu)-\mu \end{pmatrix}\begin{pmatrix} I & G_1(\mu) & G_2(\mu) \\ 0 & I & G_3(\mu) \\ 0 & 0 & I \end{pmatrix}$$

is equal to $\begin{pmatrix} A-\mu & B & C \\ D & E-\mu & F \\ G & H & L-\mu \end{pmatrix}$.

It follows that for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathcal{D}(\mathcal{L}_0)$

$$\begin{pmatrix} A-\mu & 0 & 0 \\ F_1(\mu)(A-\mu) & S_1(\mu)-\mu & 0 \\ F_2(\mu)(A-\mu) & F_3(\mu)(S_1(\mu)-\mu) & S_2(\mu)-\mu \end{pmatrix}\begin{pmatrix} I & G_1(\mu) & G_2(\mu) \\ 0 & I & G_3(\mu) \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} A-\mu & B & C \\ D & E-\mu & F \\ G & H & E-\mu \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{11}$$

From the last matrix equality (11), we can choose $F_i(\mu)$, $i = 1,2,3$ and $G_i(\mu)$, $i = 1,2,3$, for a necessary condition as follows:

$$(A-\mu)x + (A-\mu)G_1(\mu)y + (A-\mu)G_2(\mu)z = (A-\mu)x + By + Cz$$

then for $\mu \in \rho(A)$ we have

$$x + G_1(\mu)y + G_2(\mu)z = x + (A-\mu)^{-1}By + (A-\mu)^{-1}Cz.$$

Take

$$G_1(\mu) := (A-\mu)^{-1}B \tag{12}$$

and

$$G_2(\mu) := (A-\mu)^{-1}C. \tag{13}$$

The second equation of (11) gives:

$$F_1(\mu)(A-\mu)x + (F_1(\mu)(A-\mu)G_1(\mu) + S_1(\mu) - \mu)\,y + (F_1(\mu)(A-\mu)G_2(\mu) + (S_1(\mu)-\mu)G_3(\mu))z$$

must be equal to

$$Dx + (E-\mu)y + Fz.$$

Take

$$F_1(\mu) := D(A-\mu)^{-1}. \tag{14}$$

From the third equation of (11) we have

$$F_2(\mu)(A-\mu)x + (F_2(\mu)(A-\mu)G_1(\mu) + F_3(\mu)(S_1(\mu)-\mu))y + (F_2(\mu)(A-\mu)G_2(\mu) + F_3(\mu)(S_1(\mu)-\mu) + S_2(\mu)-\mu)z = Gx + Hy + (L-\mu)z$$

Take

$$F_2(\mu) := G(A-\mu)^{-1}. \tag{15}$$

For the action on y, we choose

$$GG_1(\mu) + F_3(\mu)(S_1(\mu)-\mu) - H = 0$$

therefore for $\mu \in \rho(A) \cap \rho(S_1(\mu))$, take

$$F_3(\mu) = [H - G(A-\mu)^{-1}B](S_1(\mu)-\mu)^{-1}$$

i.e.,

$$F_3(\mu) = \Theta(\mu)(S_1(\mu)-\mu)^{-1}. \tag{16}$$

Now for the action on z take,

$$[F_2(\mu)(A-\mu)G_2(\mu) + F_3(\mu)(S_1(\mu)-\mu)G_3(\mu) + S_2(\mu) - \mu - L + \mu] = 0$$

then

$$G(A-\mu)^{-1}C + \Theta(\mu)G_3(\mu) = L - S_2(\mu).$$

From the expression of $S_2(\mu)$ we can choose

$$G_3(\mu) = (S_1(\mu)-\mu)^{-1}(F - D(A-\mu)^{-1}C). \tag{17}$$

We shall now verify the sufficient condition.

We denote by T_μ the operator defined for every $\mu \in \rho(A) \cap \rho(S_1(\mu))$ by

$$T_\mu := \begin{pmatrix} I & 0 & 0 \\ F_1(\mu) & I & 0 \\ F_2(\mu) & F_3(\mu) & I \end{pmatrix} \begin{pmatrix} A-\mu & 0 & 0 \\ 0 & S_1(\mu)-\mu & 0 \\ 0 & 0 & S_2(\mu)-\mu \end{pmatrix} \begin{pmatrix} I & G_1(\mu) & G_2(\mu) \\ 0 & I & G_3(\mu) \\ 0 & 0 & I \end{pmatrix}$$

where $F_i(\mu), i = 1,2,3$ and $G_i(\mu), i = 1,2,3$ are the operators defined in (12)-(17).

Let be $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathcal{D}(\mathcal{L}_0)$. The first row in the product of T_μ gives:

$$(A-\mu)x + (A-\mu)G_1(\mu)y + (A-\mu)G_2(\mu)z \quad = \quad (A-\mu)x + By + Cz$$

The second row of T_μ gives:

$$\begin{aligned} &F_1(\mu)(A-\mu)x + [F_1(A-\mu)G_1(\mu) + S_1(\mu) - \mu]y + [F_1(\mu)(A-\mu)G_2(\mu) + (S_1(\mu)-\mu)G_3(\mu)]z \\ &\quad = Dx + (E - S_1(\mu) + (S_1(\mu)-\mu))y + Fz \\ &\quad = Dx + (E-\mu)y + Fz. \end{aligned}$$

We can show also that the left side of the third row of T_μ, i.e.,

$$F_2(\mu)(A-\mu)x + [F_2(\mu)(A-\mu)G_1(\mu) + F_1(S_1(\mu)-\mu)]y + [F_2(\mu)(A-\mu)G_2(\mu) \\ +F_3(\mu)(S_1(\mu)-\mu)G_3(\mu) + S_1(\mu)-\mu]z$$

is equal to $Gx + Hy + (L-\mu)z$. It follows that $\mathcal{L}_0 - \mu$ is an extension of the operator T_μ, i.e., $\mathcal{L}_0 - \mu \subset T_\mu$. Now it remains to prove that $\mathcal{D}(T_\mu) \subset \mathcal{D}(\mathcal{L}_0)$. Observe that

$$\mathcal{D}(T_\mu) = \left\{ \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} I & -G_1 & G_1G_3 - G_2 \\ 0 & I & -G_3 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{matrix} x \in \mathcal{D}(A) \\ y \in \mathcal{D}(S_1(\mu)) \\ z \in \mathcal{D}(S_2(\mu)) \end{matrix} \right\}.$$

Let be $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \in \mathcal{D}(\mathcal{T}_\mu)$ then

$$\begin{cases} x' = x - G_1(\mu)y + [G_1(\mu)G_3(\mu) - G_2(\mu)]\, z \\ y' = y - G_3(\mu)z \\ z' = z. \end{cases}$$

Observe that $z \in Y_2 \subset \mathcal{D}(C) \cap \mathcal{D}(F) \cap \mathcal{D}(L)$, $y' = y - G_3(\mu)z \in \mathcal{N}(S(\mu)-\mu) \subset Y_1$, $Y_1 \subset \mathcal{D}(B) \cap \mathcal{D}(E)$ and $x' = x - G_1(\mu)y + (G_1(\mu)G_3(\mu) - G_2(\mu))z \in \mathcal{D}(A)$.

Now, we are able to establish the closedness of the operator L_0.

Theorem 3.3. *Let the hypotheses* $(M1)$-$(M6)$ *be satisfied. Then, the operator* L_0 *is closable if, and only if,* $S_2(\mu)$ *is closable on* Z*, for some* $\mu \in \rho(A) \bigcap \rho(S_1(\mu))$*. Moreover, the closure* L *of* L_0 *is given by*

$$\mathrm{L} = \mu - \begin{pmatrix} I & 0 & 0 \\ F_1(\mu) & I & 0 \\ F_2(\mu) & F_3(\mu) & I \end{pmatrix} \begin{pmatrix} \mu - A & 0 & 0 \\ 0 & \mu - S_1(\mu) & 0 \\ 0 & 0 & \mu - \overline{S}_2(\mu) \end{pmatrix} \begin{pmatrix} I & G_1(\mu) & G_2(\mu) \\ 0 & I & G_3(\mu) \\ 0 & 0 & I \end{pmatrix} \tag{18}$$

or, spelled out,

$$\begin{cases} \mathrm{L} : \mathcal{D}(\mathrm{L}) \subset X \times Y \times Z \longrightarrow X \times Y \times Z \\ \mathrm{L}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} A[x + G_1(\mu)y + G_2(\mu)z] - \mu[G_1(\mu)y + G_2(\mu)z] \\ D[x + G_1(\mu)y + G_2(\mu)z] + S_1(\mu)[y + G_3(\mu)z] - \mu G_3(\mu)z \\ G[x + G_1(\mu)y + G_2(\mu)z] + [H - G(A-\mu)^{-1}B][y + G_3(\mu)z] + \overline{S}_2(\mu)z \end{pmatrix} \\ \mathcal{D}(\mathrm{L}) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in X \times Y \times Z \text{ such that } \begin{matrix} x + G_1(\mu)\, y + G_2(\mu)\, z \in \mathcal{D}(A), \\ y + G_3(\mu)\, z \in \mathcal{D}(S_1(\mu)) \\ \text{and } z \in \mathcal{D}(\overline{S}_2(\mu)) \end{matrix} \right\}. \end{cases}$$

◇

Theorem 3.4. *Let the hypotheses* $(M1)$-$(M6)$ *be satisfied. If* C, E, L, H, F, *and* B *are in* $\mathfrak{Lip}(\mathfrak{X})$ *and* G *and* D *are in* $\mathcal{L}(X)$*, then for* $\mu \in \rho(A) \bigcap \rho(S_1(\mu))$*, the following cases hold*

(i) $\mu \in \sigma_{lip}(\mathrm{L})$ *if and only if* $0 \in \sigma_{lip}(S_1(\mu)) \bigcup \sigma_{lip}(S_2(\mu))$.
(ii) $\mu \in \sigma_K(\mathrm{L})$ *if and only if* $0 \in \sigma_K(S_1(\mu)) \bigcup \sigma_K(S_2(\mu))$. ◇

Proof. Since $\mu \in \rho(A)$, then $L - \mu$ has the factorization (18). Denote it by

$$L - \mu = URV,$$

where

$$U = \begin{pmatrix} I & 0 & 0 \\ F_1(\mu) & I & 0 \\ F_2(\mu) & F_3(\mu) & I \end{pmatrix}$$

$$R = \begin{pmatrix} \mu - A & 0 & 0 \\ 0 & \mu - S_1(\mu) & 0 \\ 0 & 0 & \mu - \overline{S}_2(\mu) \end{pmatrix}$$

and

$$V = \begin{pmatrix} I & G_1(\mu) & G_2(\mu) \\ 0 & I & G_3(\mu) \\ 0 & 0 & I \end{pmatrix}.$$

Obviously, U and V are lipeomorphisms on $X \times X \times X$.
(*i*) Let $\mu \in \sigma_{lip}(L)$, then $[L - \mu]_{lip} = 0$. Since $[U]_{lip} > 0$, $[V]_{lip} > 0$, and

$$[L - \mu]_{lip} = [URV]_{lip} = [U]_{lip}[R]_{lip}[V]_{lip},$$

it follows that $[R]_{lip} = 0$, and $[S_1(\mu)]_{lip} = 0$ and $[S_2(\mu)]_{lip} = 0$, i.e., $0 \in \sigma_{lip}(S_1(\mu)) \bigcup \sigma_{lip}(S_2(\mu))$. Conversely, let $0 \in \sigma_{lip}(S_1(\mu)) \bigcup \sigma_{lip}(S_2(\mu))$, then $[R]_{lip} = 0$. Since $[U]_{lip} > 0$, $[V]_{lip} > 0$, and

$$[R]_{lip} = [U^{-1}(L - \mu)V^{-1}]_{lip} \geq [U^{-1}]_{lip}[L - \mu]_{lip}[V^{-1}]_{lip},$$

it follows that $[L - \mu]_{lip} = 0$, i.e., $\mu \in \sigma_{lip}(L)$.
(*ii*) Since U and V are lipeomorphisms, then by the factorization (18), the desired result follows immediately. □

Theorem 3.5. *Let the hypotheses* $(M1)$-$(M6)$ *be satisfied. If* C, E, L, H, F, *and* B *are in* $\mathcal{L}(X)$ *and* G *and* D *are in* $\mathfrak{Lip}(\mathfrak{X})$, *then if* $\rho(K) \neq \emptyset$, *then for* $\mu \in \rho(K)$, *the following cases hold*

(*i*) $\mu \in \sigma_{lip}(L)$ *if and only if* $0 \in \sigma_{lip}(S_3(\mu)) \bigcup \sigma_{lip}(S_4(\mu))$.
(*ii*) $\mu \in \sigma_K(L)$ *if and only if* $0 \in \sigma_K(S_3(\mu)) \bigcup \sigma_K(S_4(\mu))$, *where*

$$S_3(\mu) = A - B(K - \mu)^{-1}D$$

$$S_4(\mu) = L - G(A - \mu)^{-1}C - [H - G(A - \mu)^{-1}B](S_1(\mu) - \mu)^{-1}[F - D(A - \mu)^{-1}C]$$

◊

Proof. The proof is analogue of Theorem 3.5. □

Theorem 3.6. *Let*

$$\mathrm{L}_0 = \begin{pmatrix} 0 & 0 & C \\ 0 & E & 0 \\ G & 0 & 0 \end{pmatrix} \in \mathcal{C}(X \times X \times X)$$

with C, E, *and* G *are in* $\mathfrak{Lip}(\mathfrak{X})$. *If* $C, E \in \mathcal{L}(X)$ *and* $G \in \mathfrak{Lip}(\mathfrak{X})$, *then the following cases holds*
(*i*) $\sigma_{lip}(\mathrm{L}_0)\backslash\{0\} = \{\lambda \in \mathbb{C}$ *such that* $\lambda^3 \in \sigma_{lip}(GCE)\backslash\{0\}\}$.
(*ii*) $\sigma_K(\mathrm{L}_0)\backslash\{0\} = \{\lambda \in \mathbb{C}$ *such that* $\lambda^3 \in \sigma_K(GCE)\backslash\{0\}\}$.

Theorem 3.7. *Let*

$$\mathrm{L}_0 = \begin{pmatrix} 0 & 0 & C \\ 0 & E & 0 \\ G & 0 & 0 \end{pmatrix} \in \mathcal{C}(X \times X \times X)$$

with C, E, *and* G *are in* $\mathfrak{Lip}(\mathfrak{X})$. *If* $C, G \in \mathcal{L}(X)$ *and* $E \in \mathfrak{Lip}(\mathfrak{X})$, *then the following cases holds*
(*i*) $\sigma_{lip}(\mathrm{L}_0)\backslash\{0\} = \{\lambda \in \mathbb{C}$ *such that* $\lambda^3 \in \sigma_{lip}(CGE)\backslash\{0\}\}$.
(*ii*) $\sigma_K(\mathrm{L}_0)\backslash\{0\} = \{\lambda \in \mathbb{C}$ *such that* $\lambda^3 \in \sigma_K(CGE)\backslash\{0\}\}$. ◊

Theorem 3.8. *Let*

$$F := \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \in \mathcal{C}(X \times X \times X),$$

where $A_{ij} \in \mathfrak{Lip}(\mathfrak{X})$ $1 \leq i, j \leq 3$. *Then,* $\sigma_K(F) \subset G_1 \bigcup G_2 \bigcup G_3$, *where* $G_i = \sigma_K(A_{ii}) \bigcup \{\lambda \in \rho_K(A_{ii})$ *such that* $[A_{ii} - \lambda]_{lip} \leq [A_{ji}]_{lip}\}$. ◊

Proof. Let $\lambda \notin G_1 \bigcup G_2 \bigcup G_3$. Then, $\lambda \notin \rho_K(A_{11}) \bigcup \rho_K(A_{22}) \bigcap \rho_K(A_{33})$ and $[A_{ii} - \lambda]_{lip} > [A_{ji} - \lambda]_{lip}$ for $i, j = 1, 2, 3$. Write

$$T_\lambda = \begin{pmatrix} A_{11} - \lambda & 0 & 0 \\ 0 & A_{22} - \lambda & 0 \\ 0 & 0 & A_{33} - \lambda \end{pmatrix}.$$

Then,

$$F - \lambda I = [(F - \lambda I) T_\lambda^{-1}] T_\lambda.$$

Thus, we have the following factorization

$$\begin{aligned} (F - \lambda I) T_\lambda^{-1} &= \begin{pmatrix} A_{11} - \lambda & A_{12} & A_{13} \\ A_{21} & A_{22} - \lambda & A_{23} \\ A_{31} & A_{32} & A_{33} - \lambda \end{pmatrix} \begin{pmatrix} (A_{11} - \lambda)^{-1} & 0 & 0 \\ 0 & (A_{22} - \lambda)^{-1} & 0 \\ 0 & 0 & (A_{33} - \lambda)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I & A_{12}(A_{22} - \lambda)^{-1} & A_{13}(A_{33} - \lambda)^{-1} \\ A_{21}(A_{11} - \lambda)^{-1} & I & A_{23}(A_{33} - \lambda)^{-1} \\ A_{31}(A_{11} - \lambda)^{-1} & A_{32}(A_{22} - \lambda)^{-1} & I \end{pmatrix} \\ &= I + M(\lambda), \end{aligned}$$

where

$$M(\lambda) = \begin{pmatrix} 0 & A_{12}(A_{22} - \lambda)^{-1} & A_{13}(A_{33} - \lambda)^{-1} \\ A_{21}(A_{11} - \lambda)^{-1} & 0 & A_{23}(A_{33} - \lambda)^{-1} \\ A_{31}(A_{11} - \lambda)^{-1} & A_{32}(A_{22} - \lambda)^{-1} & 0 \end{pmatrix}.$$

Note that $[M(\lambda)]_{lip} < 1$. Then, from Lemma 1.3, we have

$$I + M(\lambda) = (F - \lambda I) T_\lambda^{-1}$$

is a lipeomorphism. Therefore, $\lambda \in \rho_K(F)$. □

Theorem 3.9. *Let*

$$F := \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

where $A_{ii} \in \mathcal{L}(X)$ *and* $A_{ij} \in \mathfrak{Lip}(\mathfrak{X})$ *with* $i \neq j$ $1 \leq i, j \leq 3$. *Then,* $\sigma_K(F) \subset \{\lambda \in \mathbb{C} \ : \ dist(\lambda, W_Z(A_{112}) \bigcup W_Z(A_{22}) \bigcup W_Z(A_{33}) \leq \max([A_{12}]_{lip}, [A_{13}]_{lip}, [A_{21}]_{lip}, [A_{23}]_{lip}, [A_{32}]_{lip})\}$. ◊

Proof. The proof follows from Theorem 3.8 and Lemma 1.4. □

Corollary 3.10. *We assume that the diagonal operator matrices*

$$\begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{pmatrix},$$

is a hyponormal operator. Then, we have

$$dist(0, \sigma(A_{112}) \bigcup \sigma(A_{22}) \bigcup \sigma(A_{33})) \geq \max([A_{12}]_{lip}, [A_{13}]_{lip}, [A_{21}]_{lip}, [A_{23}]_{lip}, [A_{32}]_{lip}).$$

◊

Theorem 3.11. *Let*

$$F = \begin{pmatrix} A & B & C \\ 0 & D & E \\ 0 & 0 & G \end{pmatrix} \in \mathcal{C}(X \times X \times X)$$

with A, B, C, D, E *and* $G \in \mathfrak{Lip}(\mathfrak{X})$.

(i) *If A, D, G, and F are lipeomorphism, then B and C are lipeomorphism.*
(ii) *If F is lipeomorphismand satisfies*

$$\min([A]_{lip}^2, [B]_{lip}^2 + [D]_{lip}^2, [C]_{lip}^2 + [E]_{lip}^2 + [G]_{lip}^2) - [A]_{lip}[B]_{lip} \geq ([B]_{lip} + [C]_{lip} + [G]_{lip})^2,$$

then A, D and G are lipeomorphism. ◇

Proof. (i) It follows immediately from Proposition 2.2 (ii).
(ii) Let

$$T = \begin{pmatrix} A & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & G \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & B & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$K = \begin{pmatrix} 0 & 0 & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & E \\ 0 & 0 & 0 \end{pmatrix}.$$

We have $F = T + S + K + W$. Then, for any vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ in $X \times X \times X$ with $x \neq y$, we have
$\frac{\|F(x)-F(y)\|^2}{\|x-y\|^2} =$

$$\frac{\|A(x_1) - A(y_1) + B(x_2) - B(y_2) + C(x_3) - C(y_3)\|^2}{\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2 + \|x_3 - y_3\|^2} + \frac{\|D(x_2) - D(y_2) + E(x_3) - E(y_3)\|^2 + \|G(x_3) - G(y_3)\|^2}{\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2 + \|x_3 - y_3\|^2}$$

$$\geq \frac{\|A(x_1) - A(y_1)\|^2 - \|B(x_2) - B(y_2)\|^2 + \|C(x_3) - C(y_3)\|^2}{\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2 + \|x_3 - y_3\|^2} + \frac{\|D(x_2) - D(y_2)\|^2 + \|E(x_3) - E(y_3)\|^2 + \|G(x_3) - G(y_3)\|^2}{\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2 + \|x_3 - y_3\|^2}$$

$$\geq \frac{\|A(x_1) - A(y_1)\|^2 + \|B(x_2) - B(y_2)\|^2 + \|C(x_3) - C(y_3)\|^2 + \|D(x_2) - D(y_2)\|^2}{\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2 + \|x_3 - y_3\|^2} + -2\frac{\|E(x_3) - E(y_3)\|^2 + \|G(x_3) - G(y_3)\|^2 + \|A(x_1) - A(y_1)\|\|B(x_2) - B(y_2)\|}{\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2 + \|x_3 - y_3\|^2}$$

$$\geq \frac{[A]_{Lip}^2\|x_1 - y_1\|^2 + ([B]_{Lip}^2 + [D]_{Lip}^2)^2\|x_2 - y_2\|^2 + ([C]_{Lip}^2 + [E]_{Lip}^2 + [B]_{Lip}^2)\|x_3 - y_3\|^2}{\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2 + \|x_3 - y_3\|^2}$$

$$-2\frac{[A]_{Lip}[B]_{Lip}\|x_1-y_1\|\|x_2-y_2\|}{\|x_1-y_1\|^2+\|x_2-y_2\|^2+\|x_3-y_3\|^2}.$$

Hence,

$$\begin{aligned}\frac{\|F(x)-F(y)\|^2}{\|x-y\|^2} &\geq \min([A]^2_{Lip},[B]^2_{Lip}+[D]^2_{Lip},[C]^2_{Lip}+[E]^2_{Lip}+[G]^2_{Lip}-[A]_{Lip}[B]_{Lip})\\ &\geq ([B]_{Lip}+[C]_{Lip}+[G]_{Lip})^2.\end{aligned}$$

So,

$$[F]_{Lip}\geq [B]_{Lip}+[C]_{Lip}+[G]_{Lip}.$$

On the other hand,

$$\begin{aligned}\frac{\|S(x)-S(y)\|^2}{\|x-y\|^2} &= \frac{\|B(x_2)-B(y_2)\|^2}{\|x_1-y_1\|^2+\|x_2-y_2\|^2+\|x_3-y_3\|^2}\\ &\leq \frac{[B]^2_{Lip}\|x_2-y_2\|^2}{\|x_1-y_1\|^2+\|x_2-y_2\|^2+\|x_3-y_3\|^2}\\ &\leq [B]^2_{Lip}.\end{aligned}$$

It follows that

$$[S]_{Lip}\leq [B]_{Lip}.$$

Similarly, we can prove

$$[K]_{Lip}\leq [C]_{Lip}$$

and

$$[W]_{Lip}\leq [E]_{Lip}.$$

Thus,

$$[K+W+S]_{Lip}<[F]_{Lip}.$$

By using Lemma 1.3, we have

$$\begin{pmatrix} A & 0 & 0\\ 0 & D & 0\\ 0 & 0 & G\end{pmatrix}$$

is a lipeomorphism. Then, A, D, and G are lipeomorphism. □

Example 3.12. *Let $X=l^2$, for any $x=(x_1,x_2,\dots,)\in X$ and $A(x_1,x_2,\dots,)=\|x\|e$, $E(x_1,x_2,\dots,)=(\|x\|,x_1,x_2,\dots,)$, $K(x_1,x_2,\dots,)=0$ and $C=(0,x_1,x_2,\dots,)$ where $e=(1,0,0,...)$. Consider the block operator matrix*

$$\mathrm{L}_0=\begin{pmatrix} A & B & C\\ 0 & E & F\\ 0 & 0 & K\end{pmatrix}.$$

Then, by Corollary 2.5, we have that. Hence,

$$\sigma_{lip}(A)\bigcup\sigma_{lip}(E)\bigcup\sigma_{lip}(K)=\sigma_{lip}(\mathrm{L}_0)\tag{19}$$

On the other hand, by calculation, we have

$$\sigma_{lip}(A)=\{\lambda\in\mathbb{C}:|\lambda|\leq 1\}.$$

and

$$\sigma_{lip}(E)=\{\lambda\in\mathbb{C}:1\leq|\lambda|\leq\sqrt{2}\}.$$

In addition, we claim that

$$\sigma_{lip}(\mathrm{L}_0) = \{\lambda \in \mathbb{C} : |\lambda| \leq \sqrt{2}\}.$$

In fact, the equalities $[\mathrm{L}_0]_{lip} = 0$ *and* $[\mathrm{L}_0]_{lip} = \sqrt{2}$ *follows from a straightforward calculation. Thus,*

$$\sigma_{lip}(\mathrm{L}_0) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \sqrt{2}\}.$$

It is clear that $0 \in \sigma_{lip}(\mathrm{L}_0)$ *when* $0 < |\lambda| \leq 1$ *, set* $z_1 = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$, *and* $z_2 = \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix}$, *then* $[\mathrm{L}_0 - \lambda]_{lip} = [A - \lambda]_{lip} = 0$. *When* $1 < |\lambda| \leq \sqrt{2}$, *set* $z_1 = \begin{pmatrix} 0 \\ y_1 \\ 0 \end{pmatrix}$, *and* $z_2 = \begin{pmatrix} 0 \\ y_2 \\ 0 \end{pmatrix}$, *then* $[\mathrm{L}_0 - \lambda]_{lip} = [K - \lambda]_{lip} = 0$*set* $z_1 = \begin{pmatrix} 0 \\ 0 \\ w_1 \end{pmatrix}$, *and* $z_2 = \begin{pmatrix} 0 \\ 0 \\ w_2 \end{pmatrix}$, *then* $[\mathrm{L}_0 - \lambda]_{lip} = [E - \lambda]_{lip} = 0$. *Thus,*

$$\sigma_{lip}(\mathrm{L}_0) = \{\lambda \in \mathbb{C} : |\lambda| \leq \sqrt{2}\}.$$

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