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The Massera-Schäffer inequality related to Birkhoff orthogonality in Banach spaces

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Abstract. In this paper, we shall consider a new constant $MS_B(X)$ which is used to study the Massera-Schäffer inequality related to Birkhoff orthogonality. We use this constant to characterize the Hilbert space and also discuss its relations with some geometric properties of Banach spaces, including uniform non-squareness, uniform convexity and uniform smoothness. Furthermore, we provide a study of $MS_B(X)$ in Radon planes. The equivalent form of this constant in Radon planes is established and used to calculate the value of $MS_B(l_p - l_q)$ $(1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1)$.

1. Introduction

Throughout the paper, let X be a real Banach space with $dimX \geq 2$. The unit ball and the unit sphere of X are denoted by B_X and S_X, respectively.

In 1964, Dunkl and Williams [7] proved that, for any Banach space X, the following norm inequality

$$\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \leq \frac{4\|x-y\|}{\|x\| + \|y\|} \tag{1}$$

holds for all nonzero elements x and y. Actually, the Dunkl-Williams inequality (1) gives the upper bound for the angular distance

$$\alpha[x, y] := \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|$$

between two nonzero elements x and y. The concept of angular distance was first introduced by Clarkson [6]. Further, in [7], Dunkl and Williams also found that if X is a Hilbert space, then the Dunkl-Williams inequality can be improved to the following inequality

$$\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \leq \frac{2\|x-y\|}{\|x\| + \|y\|}, \tag{2}$$

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which holds for all nonzero elements x and y. Soon after, in the same year that the Dunkl-Williams inequality came out, Kirk and Smiley [12] proved that the inequality (2) in fact characterizes the Hilbert space.

According to the above results, Jiménez-Melado et al. [10] pointed out that the smallest number which can replace 4 in Dunkl-Williams inequality actually measures the closeness between this Banach space and Hilbert space. Thus, Jiménez-Melado et al. [10] considered the Dunkl-Williams constant as following:

$$DW(X) = \sup\left\{\frac{\|x\|+\|y\|}{\|x-y\|}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \,:\, x,y\in X\backslash\{0\},\ x\neq y\right\},$$

and also obtained some conclusions about $DW(X)$:

(1) $2 \leq DW(X) \leq 4$ holds for any Banach space X.

(2) $DW(X) = 2$ if and only if X is a Hilbert space.

(3) $DW(X) < 4$ if and only if X is uniformly non-square, that is, there exists $\delta > 0$ such that for any $x, y \in S_X$ we have $\min(\|x-y\|, \|x+y\|) \leq 2-\delta$.

(4) If $DW(X) < (3+2\sqrt{2})^{\frac{1}{3}} + (3-2\sqrt{2})^{\frac{1}{3}}$, then X has normal structure, that is, for every bounded closed convex subset K of X that contains more than one element, there exists a $x_0 \in K$ such that

$$\sup\{\|x_0-y\| \,:\, y\in K\} < \operatorname{diam}(K) := \sup\{\|x-y\| \,:\, x,y\in K\}.$$

For more results about the Dunkl-Williams constant $DW(X)$, we refer the reader to [16–20].

The above results on $DW(X)$ make us think of using other estimates concerning the upper bound of angular distance to define some constants, and then using them to determine what kind of geometric properties a Banach space X has, just like $DW(X)$. Based on this idea and the Massera-Schäffer inequality which was proved by Massera and Schäffer [14], that is,

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \leq \frac{2\|x-y\|}{\max\{\|x\|,\|y\|\}} \tag{3}$$

holds for all nonzero elements x and y, Al-Rashed [3] introduced the following parameter

$$\Psi_\infty(X) = \sup\left\{\frac{\max\{\|x\|,\|y\|\}}{\|x-y\|}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \,:\, x,y\in X\backslash\{0\},\ x\neq y\right\}.$$

However, Baronti and Papini [5] proved that $\Psi_\infty(X) = 2$ holds for any Banach space X (see [5], Page 177), in other words, the Massera-Schäffer inequality is always sharp in any Banach space X, which means that we can't judge what kind of geometric properties a Banach space X has by the value of $\Psi_\infty(X)$. This is not what we expected. After careful consideration of the definition of $\Psi_\infty(X)$, it is not difficult to find that the reason why $\Psi_\infty(X) = 2$ holds for any Banach space X is that there are too many x and y considered in the definition of $\Psi_\infty(X)$. Therefore, one way to achieve our goal is to place some restrictions on the x and y considered in the definition of $\Psi_\infty(X)$.

Let x, y be two elements in a Banach space X. Then x is said to be Birkhoff orthogonal to y and denoted by $x \perp_B y$, if $\|x+\lambda y\| \geq \|x\|$ holds for any $\lambda \in \mathbb{R}$. The Birkhoff orthogonality coincides with the usual orthogonality in Hilbert spaces. Obviously, according to the definition of Birkhoff orthogonality, one can easily know that it is homogeneous, that is, $x \perp_B y$ implies $\alpha x \perp_B \beta y$ for any $\alpha, \beta \in \mathbb{R}$. More studies about the Birkhoff orthogonality can be found in [1, 2, 8, 9].

Now, it is time to introduce the following constant

$$MS_B(X) = \sup\left\{\frac{\max\{\|x\|,\|y\|\}}{\|x-y\|}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \,:\, x,y\in X\backslash\{0\},\ x\perp_B y\right\},$$

which can be regard as discussing the Massera-Schäffer inequality related to Birkhoff orthogonality. Obviously, the constant $MS_B(X)$ is the constant $\Psi_\infty(X)$ placed the restriction of Birkhoff orthogonality. In this paper, we will show that the value of $MS_B(X)$ is no longer a fixed value for all Banach spaces, and also connect it with some geometric properties of Banach spaces.

The paper is arranged as follows:

In Section 2, the fact that $MS_B(X)$ is no longer a fixed value for all Banach spaces is shown by some examples. In Section 3, we give a characterization of the Hilbert space in terms of $MS_B(X)$, and also discuss the relations between $MS_B(X)$ and some geometric properties of Banach spaces, including uniform non-squareness, uniform convexity and uniform smoothness. In Section 4, we study $MS_B(X)$ in Radon planes. An equivalent form of $MS_B(X)$ in Radon planes is given and used to calculate the value of $MS_B(l_p - l_q)$ $(1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1)$. Finally, in Section 5, we summarize the results obtained in this paper.

2. The Massera-Schäffer inequality related to Birkhoff orthogonality

Proposition 2.1. *Let X be a Banach space. Then $\sqrt{2} \leq MS_B(X) \leq 2$.*

Proof. First, notice that there exist $x, y \in S_X$ such that $x \perp_B y$ and $\|x + y\| \geq \sqrt{2}$ (see [2], Page 141). Let $y_n = -\frac{1}{n} y$. Then, we get $x \perp_B y_n$ and

$$MS_B(X) \geq \frac{\max\{\|x\|, \|y_n\|\}}{\|x - y_n\|} \left\| \frac{x}{\|x\|} - \frac{y_n}{\|y_n\|} \right\| = \frac{1}{\left\|x + \frac{1}{n} y\right\|} \left\|x + y\right\| \geq \frac{1}{\left\|x + \frac{1}{n} y\right\|} \sqrt{2}.$$

Let $n \to \infty$, it follows that $MS_B(X) \geq \sqrt{2}$.

On the other hand, due to the Massera-Schäffer inequality (3), it is clear that $MS_B(X) \leq 2$ holds. This completes the proof. □

The following examples show that the bounds given in the above proposition are sharp.

Example 2.2. *Let X be a Hilbert space. Then $MS_B(X) = \sqrt{2}$.*

Proof. According to Proposition 2.1, it is sufficient to show that $MS_B(X) \leq \sqrt{2}$. Notice that the Birkhoff orthogonality is now the usual orthogonality in Hilbert space X, thus, for any $x, y \in X \backslash \{0\}$ with $x \perp_B y$, we have

$$\frac{\max\{\|x\|, \|y\|\}}{\|x - y\|} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = \frac{\max\{\|x\|, \|y\|\}}{\|x - y\|} \sqrt{2} = \frac{\max\{\|x\|, \|y\|\}}{\sqrt{\|x\|^2 + \|y\|^2}} \sqrt{2} \leq \sqrt{2},$$

which implies $MS_B(X) \leq \sqrt{2}$. □

Example 2.3. *Let X be the space $l_\infty - l_p$ $(1 \leq p \leq \infty)$, that is, the space $\mathbb{R}^2$ with the norm defined by*

$$\|(x_1, x_2)\| = \begin{cases} \|(x_1, x_2)\|_\infty & (x_1 x_2 \geq 0), \\ \|(x_1, x_2)\|_p & (x_1 x_2 \leq 0). \end{cases}$$

Then $MS_B(X) = 2$.

Proof. According to Proposition 2.1, it is enough to prove that $MS_B(X) \geq 2$. Take $x = (1, 1)$, $y = (0, -1)$, it is clearly that $x, y \in S_X$ and $x \perp_B y$. Now, let $x_n = \frac{1}{n} x$. Then we have $x_n \perp_B y$ and

$$MS_B(X) \geq \frac{\max\{\|x_n\|, \|y\|\}}{\|x_n - y\|} \left\| \frac{x_n}{\|x_n\|} - \frac{y}{\|y\|} \right\| = \frac{1}{\left\|\frac{1}{n} x - y\right\|} \left\|x - y\right\| = \frac{2}{\left\|\frac{1}{n} x - y\right\|}.$$

Let $n \to \infty$, we obtain $MS_B(X) \geq 2$. □

Example 2.4. *Let X be the space $l_p - l_1$ $(1 \leq p \leq \infty)$, that is, the space $\mathbb{R}^2$ with the norm defined by*

$$\|(x_1, x_2)\| = \begin{cases} \|(x_1, x_2)\|_p & (x_1 x_2 \geq 0), \\ \|(x_1, x_2)\|_1 & (x_1 x_2 \leq 0). \end{cases}$$

Then $MS_B(X) = 2$.

Proof. Put $x = (-1, 0)$, $y = (0, -1)$, obviously, $x, y \in S_X$ and $x \perp_B y$. Now, let $x_n = \frac{1}{n}x$. Then we get $x_n \perp_B y$ and

$$MS_B(X) \geq \frac{\max\{\|x_n\|, \|y\|\}}{\|x_n - y\|} \left\| \frac{x_n}{\|x_n\|} - \frac{y}{\|y\|} \right\| = \frac{1}{\left\|\frac{1}{n}x - y\right\|} \left\|x - y\right\| = \frac{2}{\left\|\frac{1}{n}x - y\right\|}.$$

Let $n \to \infty$, it follows that $MS_B(X) \geq 2$. Further, by Proposition 2.1, we obtain $MS_B(X) = 2$. □

From above examples, we can find that $MS_B(X)$ is no longer a fixed value for all Banach spaces. This confirms what we said in the previous section. As long as we put some restrictions on the x and y considered in the definition of $\Psi_\infty(X)$, it will no longer be a fixed value for all Banach spaces. In addition, Example 2.3 and Example 2.4 also indicate that the restriction we put on $\Psi_\infty(X)$ is not too strong, since the upper bound of $MS_B(X)$ is still the same as that of $\Psi_\infty(X)$, both of which are 2.

3. Some geometric properties related to $MS_B(X)$

This section focuses on the relations between $MS_B(X)$ and some geometric properties of Banach spaces. First, we will show that the property $MS_B(X) = \sqrt{2}$ actually characterizes the Hilbert space. The proof requires the following result.

Lemma 3.1 ([1], Page 166). *Let X be a Banach space. Then the following statements are equivalent.*
(1) X is a Hilbert space.
(2) $u, v \in S_X$, $u \perp_B v \Rightarrow \|u + v\| \leq \sqrt{2}$.

Theorem 3.2. *Let X be a Banach space. Then X is a Hilbert space if and only if* $MS_B(X) = \sqrt{2}$.

Proof. According to Example 2.2, we only need to prove that if $MS_B(X) = \sqrt{2}$, then X is a Hilbert space. Now, for any $x, y \in S_X$ with $x \perp_B y$, let $x_n = \frac{1}{n}x$. Then, we have $x_n \perp_B -y$ and

$$\sqrt{2} = MS_B(X) \geq \frac{\max\{\|x_n\|, \|-y\|\}}{\|x_n - (-y)\|} \left\| \frac{x_n}{\|x_n\|} - \frac{-y}{\|-y\|} \right\| = \frac{1}{\left\|\frac{1}{n}x + y\right\|} \left\|x + y\right\|.$$

Let $n \to \infty$, we obtain $\left\|x + y\right\| \leq \sqrt{2}$. This implies that X is a Hilbert space by Lemma 3.1. □

In the preceding result, we find that, similar to $DW(X)$, the lower bound of $MS_B(X)$ can be used to characterize the Hilbert space. Therefore, we will naturally ask whether the upper bound of $MS_B(X)$ can also be used to characterize the uniform non-squareness as the upper bound of $DW(X)$. Unfortunately, the answer is no, we assert that $MS_B(X) < 2$ is a sufficient but not necessary condition for X to be uniformly non-square. To obtain this result, we need to establish the connection between $MS_B(X)$ and the rectangular constant $\mu(X)$ first.

Recall that the rectangular constant $\mu(X)$ was introduced by Joly [11] and defined as

$$\mu(X) = \sup\left\{ \frac{\|x\| + \|y\|}{\|x + y\|} \ : \ x, y \in X \backslash \{0\},\ x \perp_B y \right\}.$$

For more results about $\mu(X)$ can be found [21].

The following proposition relates $MS_B(X)$ and the rectangular constant $\mu(X)$.

Proposition 3.3. *Let X be a Banach space. Then*

$$\mu(X) \leq 1 + MS_B(X).$$

Proof. First, since the Birkhoff orthogonality is homogeneous, one can easily deduce that

$$\mu(X) = \sup\left\{\frac{\|x\| + \|y\|}{\|x - y\|} \ : \ x, y \in X\backslash\{0\},\ x \perp_B y\right\}. \tag{4}$$

Now, for any $x, y \in X\backslash\{0\}$ with $x \perp_B y$, we shall consider the following two cases:

Case 1: $\max\{\|x\|, \|y\|\} = \|x\|$.

Notice that $\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \geq 1$, since $x \perp_B y$. Thus, we obtain

$$\begin{aligned}
\frac{\|x\| + \|y\|}{\|x - y\|} &\leq \frac{\|x\| + \|y\|}{\|x - y\|}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \\
&= \frac{\|y\|}{\|x - y\|}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| + \frac{\|x\|}{\|x - y\|}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \\
&\leq \frac{\|y\|}{\|x\|}\left\|x - \frac{\|x\|}{\|y\|}y\right\|\|x - y\|^{-1} + MS_B(X) \\
&= \frac{\|y\|}{\|x\|}\left\|x - y + \left(1 - \frac{\|x\|}{\|y\|}\right)y\right\|\|x - y\|^{-1} + MS_B(X) \\
&\leq \frac{\|y\|}{\|x\|}\left(\|x - y\| + \left(\frac{\|x\|}{\|y\|} - 1\right)\|y\|\right)\|x - y\|^{-1} + MS_B(X) \\
&\leq \frac{\|y\|}{\|x\|}\left(\|x - y\| + \left(\frac{\|x\|}{\|y\|} - 1\right)\|x\|\right)\|x - y\|^{-1} + MS_B(X) \\
&\leq \frac{\|y\|}{\|x\|}\left(\|x - y\| + \left(\frac{\|x\|}{\|y\|} - 1\right)\|x - y\|\right)\|x - y\|^{-1} + MS_B(X) \\
&= 1 + MS_B(X).
\end{aligned}$$

Case 2: $\max\{\|x\|, \|y\|\} = \|y\|$.

Similarly, we also have

$$\begin{aligned}
\frac{\|x\| + \|y\|}{\|x - y\|} &\leq \frac{\|x\| + \|y\|}{\|x - y\|}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \\
&= \frac{\|x\|}{\|x - y\|}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| + \frac{\|y\|}{\|x - y\|}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \\
&\leq \left\|x - \frac{\|x\|}{\|y\|}y\right\|\|x - y\|^{-1} + MS_B(X) \\
&= \left\|\frac{\|x\|}{\|y\|}(x - y) + \left(1 - \frac{\|x\|}{\|y\|}\right)x\right\|\|x - y\|^{-1} + MS_B(X) \\
&\leq \left(\frac{\|x\|}{\|y\|}\left\|x - y\right\| + \left(1 - \frac{\|x\|}{\|y\|}\right)\|x\|\right)\|x - y\|^{-1} + MS_B(X) \\
&\leq \left(\frac{\|x\|}{\|y\|}\left\|x - y\right\| + \left(1 - \frac{\|x\|}{\|y\|}\right)\left\|x - y\right\|\right)\|x - y\|^{-1} + MS_B(X) \\
&= 1 + MS_B(X).
\end{aligned}$$

Consequently, by (4), we conclude that $\mu(X) \leq 1 + MS_B(X)$ as desired. □

Corollary 3.4. *Let X be a Banach space. If $MS_B(X) < 2$, then X is uniformly non-square. The converse is not true.*

Proof. This result immediately follows from Proposition 3.3 and the fact that $\mu(X) < 3$ if and only if X is uniformly non-square (see [4], Theorem 3.1).

The converse is not true. Example 2.3 and Example 2.4 are both counterexamples, since $l_\infty - l_p$ $(1 \le p < \infty)$ and $l_p - l_1$ $(1 < p \le \infty)$ are both uniformly non-square. In fact, if not, it means that their unit spheres must be parallelograms, which is obviously impossible. □

Now, a natural and interesting question arises. What conditions must the space X have to ensure $MS_B(X) < 2$. Actually, if X is uniformly convex, then $MS_B(X) < 2$ holds. Recall that the space X is uniformly convex if $\delta_X(\epsilon) > 0$ whenever $0 < \epsilon \le 2$, where $\delta_X : [0,2] \to [0,1]$ is given by

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| \ : \ x, y \in S_X, \ \|x-y\| \ge \epsilon\right\}.$$

Next, we will prove the above assertion whose proof is based on the following lemma.

Lemma 3.5. *([15], Proposition 5.2.8) Let X be a Banach space. Then the following statements are equivalent.*
(1) X is uniformly convex.
(2) If $x_n, y_n \in S_X$ and $\|x_n + y_n\| \to 2$, then $\|x_n - y_n\| \to 0$.

Theorem 3.6. *Let X be a Banach space. If X is uniformly convex, then $MS_B(X) < 2$.*

Proof. Suppose conversely that $MS_B(X) = 2$. Then, there exist $x_n, y_n \in X\backslash\{0\}$ with $x_n \perp_B y_n$ such that

$$\frac{\max\{\|x_n\|, \|y_n\|\}}{\|x_n - y_n\|}\left\|\frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|}\right\| \to 2.$$

Let $A = \{n \in \mathbb{N} \ : \ \max\{\|x_n\|, \|y_n\|\} = \|x_n\|\}$. Now, we discuss the following two cases respectively.

Case 1: A is an infinite set.

Since A is an infinite set, without loss generality, we can assume that $\max\{\|x_n\|, \|y_n\|\} = \|x_n\|$ holds for any $n \in \mathbb{N}$, otherwise, we can take suitable subsequences of $\{x_n\}$ and $\{y_n\}$ respectively. Hence, we have

$$\frac{\|x_n\|}{\|x_n - y_n\|}\left\|\frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|}\right\| \to 2. \tag{5}$$

Let $u_n = \frac{x_n}{\|x_n\|}$ and $v_n = \frac{y_n}{\|x_n\|}$. It is clear that $u_n \in S_X$, $v_n \in B_X$ and $u_n \perp_B v_n$. Further, from (5), we obtain

$$\frac{\left\|u_n - \frac{v_n}{\|v_n\|}\right\|}{\|u_n - v_n\|} \to 2.$$

Since $u_n \in S_X$, $v_n \in B_X$ and $u_n \perp_B v_n$, we have $\|v_n\| \in [0,1]$ and $\|u_n - v_n\| \in [1,2]$. Hence, it may be assumed that

$$\|v_n\| \to a \in [0,1], \ \|u_n - v_n\| \to b \in [1,2].$$

Then

$$\frac{\left\|u_n - \frac{v_n}{\|v_n\|}\right\|}{\|u_n - v_n\|} \le \frac{\|u_n - v_n\|}{\|u_n - v_n\|} + \frac{\left\|v_n - \frac{v_n}{\|v_n\|}\right\|}{\|u_n - v_n\|} = 1 + \frac{1 - \|v_n\|}{\|u_n - v_n\|} \le 2.$$

Let $n \to \infty$, it follows that

$$1 - a = b, \ \left\|u_n - \frac{v_n}{\|v_n\|}\right\| \to 2b.$$

The equality $1 - a = b$ implies that $b = 1$, since $a \in [0,1]$, $b \in [1,2]$. Thus, $\left\|u_n - \frac{v_n}{\|v_n\|}\right\| \to 2$. However, due to $u_n \perp_B v_n$, we obtain $\left\|u_n + \frac{v_n}{\|v_n\|}\right\| \ge \|u_n\| = 1$, which implies that $\left\|u_n + \frac{v_n}{\|v_n\|}\right\| \nrightarrow 0$. This leads to a contradiction with Lemma 3.5.

Case 2: A is a finite set.

Since A is a finite set, we can assume that $\max\{\|x_n\|, \|y_n\|\} = \|y_n\|$ holds for any $n \in \mathbb{N}$, otherwise, we can take suitable subsequences of $\{x_n\}$ and $\{y_n\}$ respectively. Hence, we have

$$\frac{\|y_n\|}{\|x_n - y_n\|}\left\|\frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|}\right\| \to 2. \tag{6}$$

Let $z_n = \frac{x_n}{\|y_n\|}$ and $w_n = \frac{y_n}{\|y_n\|}$. Obviously, we have $z_n \in B_X$, $w_n \in S_X$ and $z_n \perp_B w_n$. Moreover, it follows from (6) that

$$\frac{\left\|\frac{z_n}{\|z_n\|} - w_n\right\|}{\|z_n - w_n\|} \to 2.$$

Since $z_n \in B_X$, $w_n \in S_X$ and $z_n \perp_B w_n$, we have $\|z_n\| \in [0,1]$ and $\|z_n - w_n\| \in [0,2]$. Thus, it may be assumed that

$$\|z_n\| \to c \in [0,1],\ \|z_n - w_n\| \to d \in [0,2].$$

Actually, $d \in (0,2]$. If not, by $z_n \perp_B w_n$, then we have

$$\frac{\left\|\frac{z_n}{\|z_n\|} - w_n\right\|}{\|z_n - w_n\|} \geq \frac{\left\|\frac{z_n}{\|z_n\|}\right\|}{\|z_n - w_n\|} = \frac{1}{\|z_n - w_n\|},$$

which means $\frac{\left\|\frac{z_n}{\|z_n\|} - w_n\right\|}{\|z_n - w_n\|} \to \infty \neq 2$. Moreover, similar to Case 1, we also have

$$\frac{\left\|\frac{z_n}{\|z_n\|} - w_n\right\|}{\|z_n - w_n\|} \leq \frac{\left\|\frac{z_n}{\|z_n\|} - z_n\right\|}{\|z_n - w_n\|} + \frac{\|z_n - w_n\|}{\|z_n - w_n\|} = \frac{1 - \|z_n\|}{\|z_n - w_n\|} + 1 \leq 2.$$

Let $n \to \infty$, we obtain

$$1 - c = d,\ \left\|\frac{z_n}{\|z_n\|} - w_n\right\| \to 2d. \tag{7}$$

Now, there are two subcases we need to consider respectively.

Subcase 2.1: $c = 0$.

Then, by applying (7), we have $d = 1$ and $\left\|\frac{z_n}{\|z_n\|} - w_n\right\| \to 2$. In addition, by $z_n \perp_B w_n$, therefore $\left\|\frac{z_n}{\|z_n\|} + w_n\right\| \geq \left\|\frac{z_n}{\|z_n\|}\right\| = 1$, which leads to $\left\|\frac{z_n}{\|z_n\|} + w_n\right\| \nrightarrow 0$. This contradicts Lemma 3.5.

Subcase 2.2: $c \neq 0$.

Now, we have

$$\begin{aligned}\left|\left\|c\frac{z_n}{\|z_n\|} + d\frac{w_n - z_n}{\|w_n - z_n\|}\right\| - 1\right| &= \left|\left\|c\frac{z_n}{\|z_n\|} + d\frac{w_n - z_n}{\|w_n - z_n\|}\right\| - \|z_n + w_n - z_n\| + \|w_n\| - 1\right| \\ &\leq \left|\left\|c\frac{z_n}{\|z_n\|} + d\frac{w_n - z_n}{\|w_n - z_n\|}\right\| - \|z_n + w_n - z_n\|\right| + 0 \\ &\leq \left\|c\frac{z_n}{\|z_n\|} - z_n\right\| + \left\|d\frac{w_n - z_n}{\|w_n - z_n\|} - (w_n - z_n)\right\| \\ &= |c - \|z_n\|| + |d - \|w_n - z_n\||.\end{aligned}$$

Let $n \to \infty$, it follows that

$$\left\|c\frac{z_n}{\|z_n\|} + d\frac{w_n - z_n}{\|w_n - z_n\|}\right\| \to 1.$$

Further, by applying (7) and $d \neq 0$, it is straightforward to obtain that $c, d \in (0,1)$ and $c + d = 1$, thus we get

$$\left\|\frac{z_n}{\|z_n\|} + \frac{w_n - z_n}{\|w_n - z_n\|}\right\| \to 2.$$

Analogously, according to $z_n \perp_B w_n$, we have

$$\left\| \frac{z_n}{\|z_n\|} - \frac{w_n - z_n}{\|w_n - z_n\|} \right\| \geq \left\| \frac{z_n}{\|z_n\|} + \frac{z_n}{\|w_n - z_n\|} \right\| = \frac{\|z_n\|}{\|z_n\|} + \frac{\|z_n\|}{\|w_n - z_n\|} \geq 1,$$

which also leads to a contradiction with Lemma 3.5. This completes the proof. □

Actually, in addition to the uniform convexity, the uniform smoothness also can imply $MS_B(X) < 2$; see the following result. Recall that the space X is uniformly smooth if $\lim_{t\to 0^+} \frac{\rho_X(t)}{t}=0$, where $\rho_X : (0, +\infty) \to [0, +\infty)$ is defined by the formula

$$\rho_X(t) = \sup \left\{ \frac{1}{2}(\|x + ty\| + \|x - ty\|) - 1 \ : \ x, y \in S_X \right\}.$$

Theorem 3.7. *Let X be a Banach space. If X is uniformly smooth, then $MS_B(X) < 2$.*

Proof. Suppose conversely that $MS_B(X) = 2$. Then, there exist $x_n, y_n \in X\backslash\{0\}$ with $x_n \perp_B y_n$ such that

$$\frac{\max\{\|x_n\|, \|y_n\|\}}{\|x_n - y_n\|} \left\| \frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|} \right\| \to 2.$$

Let $A = \{n \in \mathbb{N} \ : \ \max\{\|x_n\|, \|y_n\|\} = \|x_n\|\}$. Similar to Theorem 3.6, we also consider the following two cases:

Case 1: A is an infinite set.

Since A is an infinite set, without loss generality, we can assume that $\max\{\|x_n\|, \|y_n\|\} = \|x_n\|$ holds for any $n \in \mathbb{N}$, otherwise, we can take suitable subsequences of $\{x_n\}$ and $\{y_n\}$ respectively. Now, let $u_n = \frac{x_n}{\|x_n\|}$ and $v_n = \frac{y_n}{\|x_n\|}$. It is clear that $u_n \in S_X$, $v_n \in B_X$ and $u_n \perp_B v_n$. Further, by the proof of Theorem 3.6, it follows that

$$\left\| u_n - \frac{v_n}{\|v_n\|} \right\| \to 2. \tag{8}$$

Thus, for any $t > 0$, it follows from $u_n \perp_B v_n$ that

$$\begin{aligned} \frac{\rho_X(t)}{t} &\geq \frac{\left\| u_n + t\frac{v_n}{\|v_n\|} \right\| + \left\| u_n - t\frac{v_n}{\|v_n\|} \right\| - 2}{2t} \\ &\geq \frac{\|u_n\| + \left\| u_n - t\frac{v_n}{\|v_n\|} \right\| - 2}{2t} \\ &= \frac{\left\| u_n - t\frac{v_n}{\|v_n\|} \right\| - 1}{2t} \\ &= \frac{(1+t)\left\| \frac{1}{1+t}u_n - \frac{t}{1+t}\frac{v_n}{\|v_n\|} \right\| - 1}{2t}. \end{aligned}$$

Let $n \to \infty$, by (8), we obtain

$$\frac{\rho_X(t)}{t} \geq \frac{1 + t - 1}{2t} = \frac{1}{2},$$

holds for any $t > 0$. This leads to the contradiction

$$0 = \lim_{t\to 0^+} \frac{\rho_X(t)}{t} \geq \frac{1}{2}.$$

Case 2: A is a finite set.

Since A is a finite set, we can assume that $\max\{\|x_n\|, \|y_n\|\} = \|y_n\|$ holds for any $n \in \mathbb{N}$, otherwise, we can take suitable subsequences of $\{x_n\}$ and $\{y_n\}$ respectively. Let $z_n = \frac{x_n}{\|y_n\|}$ and $w_n = \frac{y_n}{\|y_n\|}$. Obviously, we have $z_n \in B_X$, $w_n \in S_X$ and $z_n \perp_B w_n$. Then, from the proof of Theorem 3.6, we have

$$\left\|\frac{z_n}{\|z_n\|} - w_n\right\| \to 2 \text{ or } \left\|\frac{z_n}{\|z_n\|} + \frac{w_n - z_n}{\|w_n - z_n\|}\right\| \to 2. \tag{9}$$

Thus, we consider the following two subcases:

Subase 2.1: If $\left\|\frac{z_n}{\|z_n\|} - w_n\right\| \to 2$ holds.

Analogously, for any $t > 0$, it follows from $z_n \perp_B w_n$ that

$$\begin{aligned}\frac{\rho_X(t)}{t} &\geq \frac{\left\|\frac{z_n}{\|z_n\|} + tw_n\right\| + \left\|\frac{z_n}{\|z_n\|} - tw_n\right\| - 2}{2t}\\ &\geq \frac{\left\|\frac{z_n}{\|z_n\|}\right\| + \left\|\frac{z_n}{\|z_n\|} - tw_n\right\| - 2}{2t}\\ &= \frac{\left\|\frac{z_n}{\|z_n\|} - tw_n\right\| - 1}{2t}\\ &= \frac{(1+t)\left\|\frac{1}{1+t}\frac{z_n}{\|z_n\|} - \frac{t}{1+t}w_n\right\| - 1}{2t}.\end{aligned}$$

Let $n \to \infty$, from (9), we obtain

$$\frac{\rho_X(t)}{t} \geq \frac{1+t-1}{2t} = \frac{1}{2}$$

holds for any $t > 0$. This also leads to the contradiction

$$0 = \lim_{t \to 0^+} \frac{\rho_X(t)}{t} \geq \frac{1}{2}.$$

Subase 2.2: If $\left\|\frac{z_n}{\|z_n\|} + \frac{w_n - z_n}{\|w_n - z_n\|}\right\| \to 2$ holds.

Now, for any $t > 0$, it follows from $z_n \perp_B w_n$ that

$$\begin{aligned}\frac{\rho_X(t)}{t} &\geq \frac{\left\|\frac{z_n}{\|z_n\|} + t\frac{w_n - z_n}{\|w_n - z_n\|}\right\| + \left\|\frac{z_n}{\|z_n\|} - t\frac{w_n - z_n}{\|w_n - z_n\|}\right\| - 2}{2t}\\ &\geq \frac{\left\|\frac{z_n}{\|z_n\|} + t\frac{w_n - z_n}{\|w_n - z_n\|}\right\| + \left\|\frac{z_n}{\|z_n\|} + t\frac{z_n}{\|w_n - z_n\|}\right\| - 2}{2t}\\ &= \frac{\left\|\frac{z_n}{\|z_n\|} + t\frac{w_n - z_n}{\|w_n - z_n\|}\right\| + \left(\frac{1}{\|z_n\|} + \frac{t}{\|w_n - z_n\|}\right)\|z_n\| - 2}{2t}\\ &\geq \frac{\left\|\frac{z_n}{\|z_n\|} + t\frac{w_n - z_n}{\|w_n - z_n\|}\right\| + 1 - 2}{2t}\\ &= \frac{(1+t)\left\|\frac{1}{1+t}\frac{z_n}{\|z_n\|} + \frac{t}{1+t}\frac{w_n - z_n}{\|w_n - z_n\|}\right\| - 1}{2t}.\end{aligned}$$

Let $n \to \infty$, by (9), we obtain

$$\frac{\rho_X(t)}{t} \geq \frac{1+t-1}{2t} = \frac{1}{2}$$

holds for any $t > 0$. Obviously, this leads to the same contradiction as in the preceding two cases. This completes the proof. □

We close this section with the following result, which shows that the number 2 in the two results above cannot be replaced by a smaller number.

Proposition 3.8. *For any $\varepsilon > 0$, there exists a Banach space X which is uniformly convex and uniformly smooth, such that $MS_B(X) > 2 - \varepsilon$.*

Proof. Let $p, q \in (1, +\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Now, we consider the space $l_p - l_q$, which is the space $\mathbb{R}^2$ with the norm defined by

$$\|(x_1, x_2)\| = \begin{cases} \|(x_1, x_2)\|_p & (x_1 x_2 \geq 0), \\ \|(x_1, x_2)\|_q & (x_1 x_2 \leq 0), \end{cases}$$

Then, put $x = (-1, 0)$, $y = (0, -1)$, obviously, $x, y \in S_{l_p - l_q}$ and $x \perp_B y$. Now, let $x_n = \frac{1}{n}x$. Then we get $x_n \perp_B y$ and

$$MS_B(l_p - l_q) \geq \frac{\max\{\|x_n\|, \|y\|\}}{\|x_n - y\|} \left\| \frac{x_n}{\|x_n\|} - \frac{y}{\|y\|} \right\| = \frac{1}{\left\|\frac{1}{n}x - y\right\|} \left\|x - y\right\| = \frac{2^{\frac{1}{q}}}{\left\|\frac{1}{n}x - y\right\|}.$$

Let $n \to \infty$, it follows that $MS_B(l_p - l_q) \geq 2^{\frac{1}{q}}$. Further, let $q \to 1^+$, by Proposition 2.1, we obtain $\lim_{q \to 1^+} MS_B(l_p - l_q) = 2$. Thus, for any $\varepsilon > 0$, there exists a q sufficiently close to 1, such that $MS_B(X) > 2 - \varepsilon$. Moreover, it is clear that $l_p - l_q$ is uniformly convex and uniformly smooth, so we obtain the desired result. □

4. $MS_B(X)$ in Radon planes

The usual orthogonality in Hilbert spaces is always symmetric, that is, $x \perp y$ implies $y \perp x$. However, the Birkhoff orthogonality in Banach spaces is not always symmetric in general, since, in [8], James gave the following conclusion.

Theorem 4.1. *[8] A Banach space X whose dimension is at least three is a Hilbert space if and only if Birkhoff orthogonality is symmetric in X.*

The assumption of the dimension of the space X in the above theorem cannot be omitted. James [8] provided an example of two-dimensional space in which the Birkhoff orthogonality is symmetric, that is, the space $l_p - l_q$ is defined for $1 \leq p, q \leq \infty$ as the space $\mathbb{R}^2$ with the norm defined by

$$\|(x_1, x_2)\| = \begin{cases} \|(x_1, x_2)\|_p & (x_1 x_2 \geq 0), \\ \|(x_1, x_2)\|_q & (x_1 x_2 \leq 0), \end{cases}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 4.2. *[13] A two-dimensional Banach space in which the Birkhoff orthogonality is symmetric is called Radon plane.*

Actually, besides the symmetry of Birkhoff orthogonality, Radon planes have many remarkable properties. For example, the radial projection on the Radon plane X is non-expansive, that is, the map $R : X \to X$, defined by

$$R(x) = \begin{cases} x & \|x\| \leq 1, \\ \frac{x}{\|x\|} & \|x\| > 1, \end{cases}$$

such that

$$\|R(x) - R(y)\| \leq \|x - y\|, \ x, y \in X.$$

However, in higher dimensions only Hilbert space has this property. For a survey on Radon planes, including further results, can be found in [13].

In this section, we will focus on calculating the value of $MS_B(l_p - l_q)$ $(1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1)$. Our main idea is to first establish the equivalent form of $MS_B(X)$ in Radon planes, by the symmetry of Birkhoff orthogonality, and then use this equivalent form to calculate the value of $MS_B(l_p - l_q)$ $(1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1)$. Next, we give the equivalent form of $MS_B(X)$ in Radon planes, one can see that it's much simpler than the original definition. The proof requires the following characterization of the Birkhoff orthogonality obtain by James [9].

Lemma 4.3. *[9] Let X be a Banach space. For $x, y \in X$, $x \perp_B y$ if and only if there exists $f \in S_{X^*}$ such that $f(x) = \|x\|$ and $f(y) = 0$.*

Theorem 4.4. *Let X be a Radon plane. Then the following statements hold.*

(1) $MS_B(X) = \sup\{\|x - y\| \ : \ x, y \in S_X,\ x \perp_B y\}$.

(2) If X is polyhedral space, that is, a space with finitely many extreme points, then

$$MS_B(X) = \sup\{\|x - y\| \ : \ x, y \in \mathrm{ext}(B_X),\ x \perp_B y\}.$$

Proof. (1) Notice that the Birkhoff orthogonality is symmetric in X, thus, for any $x, y \in X\backslash\{0\}$ with $x \perp_B y$, we have $y \perp_B x$. Then

$$\frac{\max\{\|x\|, \|y\|\}}{\|x - y\|} \leq 1.$$

Further,

$$\frac{\max\{\|x\|, \|y\|\}}{\|x - y\|}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \leq \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \leq \sup\{\|x - y\| \ : \ x, y \in S_X,\ x \perp_B y\},$$

which follows that

$$MS_B(X) \leq \sup\{\|x - y\| \ : \ x, y \in S_X,\ x \perp_B y\}.$$

One the other hand, for any $x, y \in S_X$ with $x \perp_B y$, let $x_n = \frac{1}{n}x$. Then we obtain $x_n \perp_B y$ and

$$MS_B(X) \geq \frac{\max\{\|x_n\|, \|y\|\}}{\|x_n - y\|}\left\|\frac{x_n}{\|x_n\|} - \frac{y}{\|y\|}\right\| = \frac{1}{\left\|\frac{1}{n}x - y\right\|}\left\|x - y\right\|.$$

Let $n \to \infty$, we obtain $MS_B(X) \geq \left\|x - y\right\|$, which leads to

$$MS_B(X) \geq \sup\{\|x - y\| \ : \ x, y \in S_X,\ x \perp_B y\}.$$

(2) From (1) it is easy to see that

$$MS_B(X) \geq \sup\{\|x - y\| \ : \ x, y \in \mathrm{ext}(B_X),\ x \perp_B y\}.$$

So, we only need to show the converse inequality. Suppose that $x, y \in S_X$ with $x \perp_B y$. Clearly, we can write $x = tx_1 + (1-t)x_2$ and $y = \lambda y_1 + (1-\lambda)y_2$, for some $x_1, x_2, y_1, y_2 \in \mathrm{ext}(B_X)$ and $0 \leq t, \lambda \leq 1$. Since X is a Radon plane and $x \perp_B y$, one can easily know that $x_i \perp_B y_j$, for all $i, j \in \{1, 2\}$, by using Lemma 4.3. Now,

$$\begin{aligned}
\|x - y\| &= \|x - \lambda y_1 - (1-\lambda)y_2\| \\
&= \|\lambda x + (1-\lambda)x - \lambda y_1 - (1-\lambda)y_2\| \\
&\leq \lambda\|x - y_1\| + (1-\lambda)\|x - y_2\| \\
&\leq \max\{\|x - y_1\|, \|x - y_2\|\}.
\end{aligned}$$

Using similar technique, we can deduce that

$$\|x - y_1\| \leq \max\{\|x_1 - y_1\|, \|x_2 - y_1\|\}$$

and

$$\|x - y_2\| \leq \max\{\|x_1 - y_2\|, \|x_2 - y_2\|\}.$$

This shows that

$$\|x - y\| \leq \max\{\|x_i - y_j\| \ : \ x_i \perp_B y_j,\ i, j \in \{1, 2\}\}.$$

This clearly implies that

$$MS_B(X) = \sup\{\|x - y\| \ : \ x, y \in S_X,\ x \perp_B y\} \leq \sup\{\|x - y\| \ : \ x, y \in \mathrm{ext}(B_X),\ x \perp_B y\}.$$

□

The above conclusion actually provides us with a new way to calculate the value of $MS_B(X)$ for Radon planes. Next, we will use it to calculate the value of $MS_B(l_p - l_q)$ ($1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$). Firstly, according to Theorem 4.4 (1), we need to find the elements $x, y \in S_{l_p - l_q}$ such that $x \perp_B y$. Now, assuming $x \in S_{l_p - l_q}$, our goal is to find the elements $y \in S_{l_p - l_q}$ such that $x \perp_B y$. Notice that the space $l_p - l_q$ is a smooth Banach space, from the geometric meaning of the supporting hyperplane, it can be seen that the supporting hyperplane of $B_{l_p - l_q}$ at point x is actually the tangent of $B_{l_p - l_q}$ at point x. So based on the knowledge of calculus, we can obtain the equation for the tangent at point x. Furthermore, since the kernel of the functional corresponding to this supporting hyperplane is actually the straight line passing through the origin and parallel to the tangent. So, from Lemma 4.3, we know that the elements $y \in S_{l_p - l_q}$ with $x \perp_B y$ is actually the intersection points of $S_{l_p - l_q}$ and the straight line passing through the origin and parallel to the tangent. Obviously, finding the intersection points only requires some basic calculations, so we just give the following results and omit the calculation details. It should be noted that the reason why the following lemma only gives the results when the abscess of x is between 0 and 1 is that it is enough for us to calculate the value of $MS_B(l_p - l_q)$ (see Theorem 4.6).

Lemma 4.5. *Let $1 < p < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and*

$$B(x) := \{y \ : \ y \in S_{l_p - l_q},\ x \perp_B y\},\ x \in S_{l_p - l_q}.$$

Then the following statements hold.

(1) If $x = (a, b),\ 0 \leq a \leq 1,\ b = (1 - a^p)^{\frac{1}{p}}$, then $B(x) = \left\{\pm\left(-b^{\frac{p}{q}}, a^{\frac{p}{q}}\right)\right\}$.

(2) If $x = (c, d),\ 0 \leq c \leq 1,\ d = -(1 - c^q)^{\frac{1}{q}}$, then $B(x) = \left\{\pm\left((-d)^{\frac{q}{p}}, c^{\frac{q}{p}}\right)\right\}$.

Now, all the preparations are ready, it is time to calculate the value of $MS_B(l_p - l_q)$.

Theorem 4.6. *Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the following statements hold.*

(1) If $p < q$, then

$$MS_B(l_p - l_q) = \sup\left\{\left\|\left(a + (1 - a^p)^{\frac{1}{q}}, (1 - a^p)^{\frac{1}{p}} - a^{\frac{p}{q}}\right)\right\|_{l_p - l_q} \ : \ 0 \leq a \leq 1\right\}.$$

(2) If $p > q$, then

$$MS_B(l_p - l_q) = \sup\left\{\left\|\left(a + (1 - a^q)^{\frac{1}{p}}, (1 - a^q)^{\frac{1}{q}} - a^{\frac{q}{p}}\right)\right\|_{l_q - l_p} \ : \ 0 \leq a \leq 1\right\}.$$

Proof. (1) For convenience, we still let

$$B(x) := \{y \ : \ y \in S_{l_p - l_q},\ x \perp_B y\},\ x \in S_{l_p - l_q}.$$

First, due to Theorem 4.4 (1) and the homogeneity of Birkhoff orthogonality, one can easily know that

$$\begin{aligned} MS_B(l_p - l_q) &= \sup\{\|x - y\| \ : \ x, y \in S_{l_p - l_q},\ x \perp_B y\} \\ &= \sup\{\|x - y\| \ : \ x = (a, b) \in S_{l_p - l_q},\ y \in B(x),\ -1 \leq a \leq 1\} \\ &= \sup\{\|x - y\| \ : \ x = (a, b) \in S_{l_p - l_q},\ y \in B(x),\ 0 \leq a \leq 1\}. \end{aligned} \tag{10}$$

Now, take an arbitrary $x = (a, b) \in S_{l_p - l_q}$ with $0 \leq a \leq 1$. Then following Lemma 4.5, we can obtain the coordinates of $y \in B(x)$. Now, putting the value of x and y into (10), we can obtain the desired result by a straightforward computation.

(2) It is evident that the mapping $(a, b) \to (-b, a)$ is an isometric isomorphism from $l_p - l_q$ onto $l_q - l_p$, thus, by (1), we have

$$MS_B(l_p - l_q) = MS_B(l_q - l_p) = \sup\left\{\left\|\left(a + (1 - a^q)^{\frac{1}{p}}, (1 - a^q)^{\frac{1}{q}} - a^{\frac{q}{p}}\right)\right\|_{l_q - l_p} \ : \ 0 \leq a \leq 1\right\}.$$

Remark 4.7. *For the case of $p = \infty$, $q = 1$, the value of $MS_B(l_\infty - l_1)$ is given by Example 2.3 and Example 2.4, that is, $MS_B(l_\infty - l_1) = 2$. Moreover, the value of $MS_B(l_1 - l_\infty)$ is also equal to 2, since the mapping $(a, b) \to (-b, a)$ is an isometric isomorphism from $l_\infty - l_1$ onto $l_1 - l_\infty$, then $MS_B(l_1 - l_\infty) = MS_B(l_\infty - l_1) = 2$.*
□

5. Conclusions

In this paper, we consider a new constants $MS_B(X)$ which is used to study the Massera-Schäffer inequality related to Birkhoff orthogonality. It is of interest to characterize the Hilbert space in terms of it and investigate its relationships with some geometric properties, such as uniform non-squareness, uniform convexity and uniform smoothness. Moreover, we provide a study of $MS_B(X)$ in Radon planes. The symmetry of Birkhoff orthogonality in the Radon planes allows us to establish an equivalent form of $MS_B(X)$ for Radon planes. This equivalent form is used to calculate the value of $MS_B(l_p - l_q)$ $(1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1)$. However, there are still many interesting issues to be discussed. Is this constant related to other geometric properties? How to calculate the values of this constant for some other classical Banach spaces? Therefore, the research on $MS_B(X)$ needs to be continued in the future so that we can further understand $MS_B(X)$.

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