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Weighted generalized invertibility in two semigroups of a ring with involution

Wende Li[a], Jianlong Chen[a,*], Yukun Zhou[a], Xiaofeng Chen[b]

[a] School of Mathematics, Southeast University, Nanjing 210096, China
[b] Aliyun School of Big Data, Changzhou University, Changzhou 213164, China

Abstract. Let R be a ring with an involution and $p \in R$ be a weighted projection. We characterize the relation between the weighted Moore-Penrose invertibility (resp., weighted pseudo core invertibility) of the corresponding elements of the two semigroups pRp and $pRp + 1 - p$. As an application, we obtain the relation between the weighted Moore-Penrose invertibility (resp., weighted pseudo core invertibility) of the corresponding elements of the matrix semigroup $AA^{\dagger}_{M,N}R^{m\times m}AA^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}$ and the matrix semigroup $A^{\dagger}_{M,N}AR^{n\times n}A^{\dagger}_{M,N}A + I_n - A^{\dagger}_{M,N}A$, where $A \in R^{m\times n}$ be weighted Moore-Penrose invertible with weights (M, N).

1. Introduction

Let R be a ring with an involution $*$ and $R^{m\times n}$ denote the set of $m \times n$ matrices over R. An involution $*$ in R is an anti-isomorphism satisfying $(a^*)^* = a$, $(a + b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$ for all $a, b \in R$. An element $a \in R$ is called Hermitian if $a^* = a$.

Let $a \in R$. We recall that a is said to be Drazin invertible [10] if there exist $x \in R$ and a positive integer k such that

$$ax = xa,\ ax^2 = x,\ xa^{k+1} = a^k.$$

Such x (if it exists) is unique and called the Drazin inverse of a, denoted by a^D. When $k = 1$, the Drazin inverse of a is called the group inverse of a, denoted by $a^{\#}$. For more details of Drazin inverses, for example, see[4–9, 16, 29].

The weighted Moore-Penrose inverse is a generalization of the Moore-Penrose inverse which was characterized as the unique solution of four matrix equations by Penrose [22]. The concept of the weighted Moore-Penrose inverse was first introduced to investigate the question of least squares fitting of curves and surfaces by Greville [12]. Chipman [3] generalized Greville's weighted generalized inverse with weight being a Hermitian positive definite matrix to the weighted generalized inverse with weights being two

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* Corresponding author: Jianlong Chen
Email addresses: wendly155@163.com (Wende Li), jlchen@seu.edu.cn (Jianlong Chen), ykzhou187@163.com (Yukun Zhou), xfc189130@163.com (Xiaofeng Chen)

Hermitian positive definite matrices. In 1992, Manjunatha Prasad and Bapat [17] defined the generalized Moore-Penrose inverse with weights being two invertible matrices and gave necessary and sufficient conditions for its existence over an integral domain. The weighted Moore-Penrose inverse of a complex matrix with weights being two invertible Hermitian matrices does not necessarily exist [24]. Sheng and Chen [24] presented the sufficient and necessary conditions for the existence of the weighted Moore-Penrose inverse with weights being two invertible Hermitian matrices. In the following, we give the weighted Moore-Penrose inverse of matrices over a ring with involution. More details of weighted Moore-Penrose inverses can refer to, for example, [2, 20, 25].

Definition 1.1. [17] *Let $M \in R^{m\times m}$ and $N \in R^{n\times n}$ be two invertible Hermitian matrices, $A \in R^{m\times n}$. If there exists $X \in R^{n\times m}$ satisfying the equations*

$$(1)\ AXA = A,\quad (2)\ XAX = X,\quad (3M)\ (MAX)^* = MAX,\quad (4N)\ (NXA)^* = NXA,$$

then A is called weighted Moore-Penrose invertible with weights (M, N). Such X is unique if it exists and called the weighted Moore-Penrose inverse with weights (M, N) of A, denoted by $A^{\dagger}_{M,N}$. More generally, if the equation (1) holds, then A is called regular, and X is called an inner inverse of A. We use A^- to denote an inner inverse of A. If X satisfies the conditions (1) and (3M), then X is called a $\{1, 3M\}$-inverse of A and we use $A^{(1,3M)}$ to denote a $\{1, 3M\}$-inverse of A. Similarly, if X satisfies the conditions (1) and (4N), then X is called a $\{1, 4N\}$-inverse of A and we use $A^{(1,4N)}$ to denote a $\{1, 4N\}$-inverse of A. The symbols $A\{1, 3M\}$ and $A\{1, 4N\}$ denote all $\{1, 3M\}$-inverses of A and $\{1, 4N\}$-inverses of A, respectively. Clearly, when $M = I_m$ and $N = I_n$, $A^{\dagger}_{M,N}$ reduces to the Moore-Penrose inverse $A^{\dagger}$ of A.

In 2007, Koliha [13] generalized the definition of the weighted Moore-Penrose inverse from matrices to rings with involution. Throughout this paper, we assume that $e, f \in R$ are invertible Hermitian elements.

Definition 1.2. [13] *An element $a \in R$ is said to be weighted Moore-Penrose invertible with weights (e, f) if there exists $x \in R$ such that*

$$(1)\ axa = a,\quad (2)\ xax = x,\quad (3e)\ (eax)^* = eax,\quad (4f)\ (fxa)^* = fxa.$$

Such x is called the weighted Moore-Penrose inverse of a with weights (e, f) and it is unique if it exists, denoted by $a^{\dagger}_{e,f}$. Definitions of inner inverses, $\{1, 3e\}$-inverses and $\{1, 4f\}$-inverses and their notations are similar to those defined for matrices. If $e = f = 1$, then $a^{\dagger}_{e,f} = a^{\dagger}$.

Later, the weighted core inverse and weighted pseudo core inverse were introduced and investigated. Mosić et al. [19] introduced and investigated e-core inverses, Zhu and Wang [27] defined and characterized pseudo e-core inverses by three equations in a ring with involution. More results concerning core inverses and pseudo core inverses can be found in [1, 11, 18, 23, 26].

Definition 1.3. [27] *Let $a \in R$. Then a is said to be pseudo e-core invertible if there exist $x \in R$ and a positive integer k such that $xax = x$, $xR = a^kR$, $Rx = R(a^k)^*e$.*

In [27], it was also proved that a is pseudo e-core invertible if and only if there exist $x \in R$ and a positive integer k such that $xa^{k+1} = a^k$, $ax^2 = x$, $(eax)^* = eax$. Such x is called the pseudo e-core inverse of a and is unique if it exists, denoted by $a^{e,\textcircled{D}}$. The smallest positive integer k is called the pseudo e-core index of a, and denoted by $\mathrm{ind}(a)$. When $\mathrm{ind}(a) = 1$, the pseudo e-core inverse of a is called the e-core inverse of a and denoted by $a^{e,\textcircled{\#}}$. If $e = 1$, then the pseudo e-core index of a is called the pseudo core index of a and $a^{e,\textcircled{D}} = a^{\textcircled{D}}$ is called the pseudo core inverse of a. When $\mathrm{ind}(a) = 1$ and $e = 1$, $a^{e,\textcircled{D}} = a^{\textcircled{\#}}$ is called the core inverse of a. More details of pseudo e-core inverses and e-core inverses can refer to [15, 19, 27, 28].

Let $M \in R^{m\times m}$ be invertible Hermitian matrix. We note that $A \in R^{m\times m}$ is pseudo M-core invertible if and only if there exist $X \in R^{m\times m}$ and a positive integer k such that $XA^{k+1} = A^k$, $AX^2 = X$, $(MAX)^* = MAX$. Such X is unique if it exists and called the pseudo M-core inverse of A, which is denoted by $A^{M,\textcircled{D}}$. Similarly, the M-core inverse is defined.

A motivation for this research appeared in [21]. Let $p \in R$ be a projection. Then $pRp+1-p = \{pxp+1-p : x \in R\}$ is a (multiplicative) semigroup. Patrício and Puystjens investigated the relation between the Moore-Penrose invertibility of the corresponding elements of pRp and $pRp+1-p$. As an application, they related the Moore-Penrose invertibility of the corresponding elements of the semigroup $AA^{\dagger}R^{m\times m}AA^{\dagger}+I_m-AA^{\dagger}$ and the semigroup $A^{\dagger}AR^{n\times n}A^{\dagger}A+I_n-A^{\dagger}A$, when $A^{\dagger}$ exists. The relevant results for the Drazin invertibility were also investigated.

The article is organized as follows. In Section 2, we first investigate the relation between the weighted Moore-Penrose invertibility of the corresponding elements of pRp and $pRp+1-p$ when p is a weighted projection. Also, we obtain analogous results for the pseudo e-core invertibility (resp., e-core inverse). In Section 3, by applying Corollary 2.7 and Theorem 2.14 of Section 2, we relate the weighted Moore-Penrose invertibility with weights (M,N) (resp., pseudo M-core invertibility and M-core invertibility) of the corresponding elements between the semigroup $AA^{\dagger}_{M,N}R^{m\times m}AA^{\dagger}_{M,N}+I_m-AA^{\dagger}_{M,N}$ and the semigroup $A^{\dagger}_{M,N}AR^{n\times n}A^{\dagger}_{M,N}A+I_n-A^{\dagger}_{M,N}A$ when $A^{\dagger}_{M,N}$ exists. The results on the weighted Moore-Penrose invertibility generalize the relevant results of Patrício and Puystjens on the Moore-Penrose invertibility in [21].

2. Weighted generalized invertibility in pRp and $pRp+1-p$ of R

Recall that $a \in R$ is said to be $\{1,3\}$-invertible (resp., $\{1,4\}$-invertible) if there exists $x \in R$ such that $axa = a$, $(ax)^* = ax$ (resp., $(xa)^* = xa$). In this section, we mainly investigate the relation between the weighted Moore-Penrose invertibility of the corresponding elements of pRp and $pRp+1-p$ when p is a weighted projection, which will play an important role in the forthcoming section. The relation between the $\{1,3\}$-invertibility (resp., $\{1,4\}$-invertibility) of the corresponding elements of pRp and $pRp+1-p$ is first given when p is a projection.

Lemma 2.1. *Let $p \in R$ be a projection, $x \in R$. Then $pxp+1-p$ is $\{1,3\}$-invertible in R if and only if pxp is $\{1,3\}$-invertible in pRp. In this case,*

$$p(pxp+1-p)^{(1,3)}p \in (pxp)\{1,3\},$$

and

$$(pxp)^{(1,3)}+1-p \in (pxp+1-p)\{1,3\},$$

where $(pxp)^{(1,3)} \in pRp$.

Proof. Assume $(pxp+1-p)^{(1,3)}$ is a $\{1,3\}$-inverse of $pxp+1-p$ in R. Then we have

$$(pxp+1-p)(pxp+1-p)^{(1,3)}(pxp+1-p) = pxp+1-p.$$

Multiplying on the left and right sides by p, we can get

$$(pxp)p(pxp+1-p)^{(1,3)}p(pxp) = pxp.$$

Also,

$$\left((pxp+1-p)(pxp+1-p)^{(1,3)}\right)^* = (pxp+1-p)(pxp+1-p)^{(1,3)},$$

multiplying on the left and right sides by p, we have

$$\left(pxpp(pxp+1-p)^{(1,3)}p\right)^* = pxpp(pxp+1-p)^{(1,3)}p,$$

Hence, $p(pxp+1-p)^{(1,3)}p \in (pxp)\{1,3\}$.

Conversely, assume $(pxp)^{(1,3)}$ is a $\{1,3\}$-inverse of pxp in pRp. Then $pxp(pxp)^{(1,3)}pxp = pxp$ implies

$$(pxp+1-p)\left((pxp)^{(1,3)}+1-p\right)(pxp+1-p) = pxp+1-p$$

since $(pxp)^{(1,3)} \in pRp$. Also,

$$\left(pxp(pxp)^{(1,3)}\right)^* = pxp(pxp)^{(1,3)}.$$

As $(1-p)^* = (1-p)$, $(pxp+1-p)\left((pxp)^{(1,3)}+1-p\right)$ is Hermitian. Therefore, $(pxp)^{(1,3)}+1-p \in (pxp+1-p)\{1,3\}$. □

Proposition 2.2. *Let $p \in R$ be a projection, $x \in R$. Then pxp is $\{1,3\}$-invertible in pRp if and only if pxp is $\{1,3\}$-invertible in R.*

Proof. The necessity is clear since $pRp \subseteq R$. For the sufficiency, we assume that pxp is $\{1,3\}$-invertible with a $\{1,3\}$-inverse y in R. Then $pxpypxp = pxp$ implies $pxp(pyp)pxp = pxp$. Since $(pxpy)^* = pxpy$, we have $pxppyp = (pxpy)^*p = (pxpy)^*p^* = (ppxpy)^* = pxpy$. Then $(pxppyp)^* = pxppyp$. Therefore, pyp is a $\{1,3\}$-inverse of pxp in pRp. □

Following [13], the mapping $*e : R \to R$ defined by $x \mapsto e^{-1}x^*e$ is an involution. Furthermore, $a \in R$ is $\{1,3e\}$-invertible with respect to $*$ if and only if $a \in R$ is $\{1,3\}$-invertible with respect to $*e$. Next, we characterize the case of the $\{1,3e\}$-invertibility.

Corollary 2.3. *Let $p \in R$ be an idempotent with $(ep)^* = ep$, $x \in R$. Then $pxp + 1 - p$ is $\{1,3e\}$-invertible in R if and only if pxp is $\{1,3e\}$-invertible in R. In this case,*

$$p(pxp + 1 - p)^{(1,3e)}p \in (pxp)\{1,3e\},$$

and

$$p(pxp)^{(1,3e)}p + 1 - p \in (pxp + 1 - p)\{1,3e\}.$$

Proof. Since $(ep)^* = ep$, we have $p = e^{-1}p^*e = p^{*e}$. Hence, it is easy to obtain the result by Lemma 2.1 and Proposition 2.2. □

We characterize the $\{1,4\}$-invertibility case without proof, as it can be obtained by a similar way of the $\{1,3\}$-invertibility.

Lemma 2.4. *Let $p \in R$ be a projection, $x \in R$. Then $pxp + 1 - p$ is $\{1,4\}$-invertible in R if and only if pxp is $\{1,4\}$-invertible in pRp. In this case,*

$$p(pxp + 1 - p)^{(1,4)}p \in (pxp)\{1,4\},$$

and

$$(pxp)^{(1,4)} + 1 - p \in (pxp + 1 - p)\{1,4\},$$

where $(pxp)^{(1,4)} \in pRp$.

Corollary 2.5. *Let $p \in R$ be an idempotent with $(fp)^* = fp$, $x \in R$. Then $pxp + 1 - p$ is $\{1,4f\}$-invertible in R if and only if pxp is $\{1,4f\}$-invertible in R. In this case,*

$$p(pxp + 1 - p)^{(1,4f)}p \in (pxp)\{1,4f\},$$

and

$$p(pxp)^{(1,4f)}p + 1 - p \in (pxp + 1 - p)\{1,4f\}.$$

It is known in [28] that $a \in R$ is weighted Moore-Penrose invertible with weights (e,f) if and only if $a \in R$ is $\{1,3e\}$-invertible and $\{1,4f\}$-invertible. Moreover, $a^\dagger_{e,f} = a^{(1,4f)}aa^{(1,3e)}$. Combining Corollaries 2.3 and 2.5, then we can present the analogous results for the weighted Moore-Penrose invertibility with weights (e,f).

Theorem 2.6. *Let $p \in R$ be an idempotent with $(ep)^* = ep$ and $(fp)^* = fp$, $x \in R$. Then $pxp + 1 - p$ is weighted Moore-Penrose invertible with weights (e,f) in R if and only if pxp is weighted Moore-Penrose invertible with weights (e,f) in R. In this case,*

$$(pxp)^\dagger_{e,f} = p(pxp + 1 - p)^\dagger_{e,f}p,$$

and

$$(pxp + 1 - p)^\dagger_{e,f} = (pxp)^\dagger_{e,f} + 1 - p.$$

Proof. Here we only need to prove the expressions of $(pxp+1-p)^{\dagger}_{e,f}$ and $(pxp)^{\dagger}_{e,f}$. Assume that $(pxp)^{\dagger}_{e,f}$ is the weighted Moore-Penrose inverse with weights (e,f) of pxp. Then it is easy to check that $p(pxp)^{\dagger}_{e,f}p$ is also the weighted Moore-Penrose inverse with weights (e,f) of pxp according to the similar proof of Proposition 2.2. By the uniqueness of the weighted Moore-Penrose inverse, we get $(pxp)^{\dagger}_{e,f} = p(pxp)^{\dagger}_{e,f}p$. Then for the expression of $(pxp+1-p)^{\dagger}_{e,f}$, we can obtain that

$$\begin{aligned}(pxp+1-p)^{\dagger}_{e,f} &= (pxp+1-p)^{(1,4f)}(pxp+1-p)(pxp+1-p)^{(1,3e)}\\ &= \left(p(pxp)^{(1,4f)}p+1-p\right)(pxp+1-p)\left(p(pxp)^{(1,3e)}p+1-p\right)\\ &= p(pxp)^{(1,4f)}pxp(pxp)^{(1,3e)}p+1-p\\ &= p(pxp)^{\dagger}_{e,f}p+1-p\\ &= (pxp)^{\dagger}_{e,f}+1-p \in pRp+1-p.\end{aligned}$$

For the expression of $(pxp)^{\dagger}_{e,f}$, we can check that $(pxp)^{\dagger}_{e,f} = p(pxp+1-p)^{\dagger}_{e,f}pxp(pxp+1-p)^{\dagger}_{e,f}p$. Since $(pxp)^{\dagger}_{e,f}+1-p \in pRp+1-p$, it follows that $(pxp)^{\dagger}_{e,f} = p(pxp+1-p)^{\dagger}_{e,f}p$. □

In the following result, we illustrate the relation between the weighted Moore-Penrose invertibility of the corresponding elements of R and pRp.

Proposition 2.7. *Let $p \in R$ be a projection with $(ep)^* = ep$ and $(fp)^* = fp$. Then $pxp+1-p$ is weighted Moore-Penrose invertible with weights (e,f) in R if and only if pxp is weighted Moore-Penrose invertible with weights (pe,pf) in pRp. In this case,*

$$(pxp)^{\dagger}_{pe,pf} = p(pxp+1-p)^{\dagger}_{e,f}p \in pRp,$$

and

$$(pxp+1-p)^{\dagger}_{e,f} = (pxp)^{\dagger}_{pe,pf}+1-p \in pRp+1-p.$$

Take $e=f=1$ in Proposition 2.7, we obtain the characterization of the Moore-Penrose invertibility case given in [21].

Corollary 2.8. [21, Theorem 1] *Let $p \in R$ be a projection, $x \in R$. Then $pxp+1-p$ is Moore-Penrose invertible in R if and only if pxp is Moore-Penrose invertible in pRp. In this case,*

$$(pxp)^{\dagger} = p(pxp+1-p)^{\dagger}p \in pRp,$$

and

$$(pxp+1-p)^{\dagger} = (pxp)^{\dagger}+1-p \in pRp+1-p.$$

In [27], Zhu and Wang presented the following two lemmas, which will be useful in proving our results.

Lemma 2.9. [27, Corollary 3.10] *Let $a \in R$. Then a is e-core invertible if and only if a is group invertible and $\{1,3e\}$-invertible. In this case, $a^{e,\#\!\!\!\bigcirc} = a^{\#}aa^{(1,3e)}$.*

Lemma 2.10. [27, Theorem 3.19] *Let $a \in R$. Then a is pseudo e-core invertible if and only if a^n is e-core invertible for some positive integer n. In this case, $a^{e,\textcircled{D}} = a^{n-1}(a^n)^{e,\#\!\!\!\bigcirc}$ and $(a^n)^{e,\#\!\!\!\bigcirc} = (a^{e,\textcircled{D}})^n$.*

Proposition 2.11. *Let $p \in R$ be an idempotent, $x \in R$. Then pxp is group invertible in R if and only if pxp is group invertible in pRp. In this case, the group inverse of pxp in R is consistent with that in pRp.*

Proof. The sufficiency is clear. For the necessity, assume that y is the group inverse of pxp in R. Then $y = pxpy^2 = y^2pxp \in pRp$. That is, y is also the group inverse of pxp in pRp. Hence, the group inverse of pxp in R is consistent with that in pRp by the uniqueness of the group inverse. □

In [11], it was proved that if $a, b \in R$ are pseudo core invertible with $ab = ba = 0$ and $a^*b = 0$, then $a + b$ is pseudo core invertible with $(a + b)^{\textcircled{D}} = a^{\textcircled{D}} + b^{\textcircled{D}}$. As a new involution $*e$ was shown before, it is easy to check that $a \in R$ is pseudo e-core invertible with respect to $*$ if and only if $a \in R$ is pseudo core invertible with respect to $*e$. Then we can easily obtain the following result involving with the additive property of the pseudo e-core inverse.

Corollary 2.12. *Let $a, b \in R$ be pseudo e-core invertible. If $ab = ba = 0$ and $a^*eb = 0$, then $a + b$ is pseudo e-core invertible with $(a + b)^{e,\textcircled{D}} = a^{e,\textcircled{D}} + b^{e,\textcircled{D}}$.*

When $a, b \in R$ are e-core invertible, we have the relevant result of e-core inverses.

Corollary 2.13. *Let $a, b \in R$ be e-core invertible. If $ab = ba = 0$ and $a^*eb = 0$, then $a + b$ is e-core invertible with $(a + b)^{e,\textcircled{\#}} = a^{e,\textcircled{\#}} + b^{e,\textcircled{\#}}$.*

Finally, we illustrate the relation between the pseudo e-core invertibility (resp., e-core invertibility) of the corresponding elements of two semigroups pRp and $pRp + 1 - p$ of R.

Theorem 2.14. *Let $p \in R$ be an idempotent with $(ep)^* = ep$. Then the following statements hold.*

(1) *$pxp + 1 - p$ is e-core invertible in R if and only if pxp is e-core invertible in R. In this case,*

$$(pxp)^{e,\textcircled{\#}} = p(pxp + 1 - p)^{e,\textcircled{\#}}p,$$

and

$$(pxp + 1 - p)^{e,\textcircled{\#}} = (pxp)^{e,\textcircled{\#}} + 1 - p.$$

(2) *$pxp+1-p$ is pseudo e-core invertible with* $\mathrm{ind}(pxp+1-p) = k$ *in R if and only if pxp is pseudo e-core invertible with* $\mathrm{ind}(pxp) = k$ *in R. In this case,*

$$(pxp)^{e,\textcircled{D}} = p(pxp + 1 - p)^{e,\textcircled{D}}p,$$

and

$$(pxp + 1 - p)^{e,\textcircled{D}} = (pxp)^{e,\textcircled{D}} + 1 - p.$$

Proof. (1). Assume that $pxp + 1 - p$ is e-core invertible in R. Then $pxp + 1 - p$ is group invertible and $\{1, 3e\}$-invertible in R by Lemma 2.9. Following Corollary 2.3, Proposition 2.11 and [21, Theorem 1], we have that pxp is group invertible and $\{1, 3e\}$-invertible in R. Moreover, $(pxp)^\# = p(pxp + 1 - p)^\# p$ and $p(pxp + 1 - p)^{(1,3e)}p \in (pxp)\{1, 3e\}$. Hence, by Lemma 2.9 again, we have that pxp is e-core invertible in R. For the expression, since $(pxp + 1 - p)^\# \in pRp + 1 - p$, we obtain

$$\begin{aligned}
(pxp)^{e,\textcircled{\#}} &= (pxp)^\#(pxp)(pxp)^{(1,3e)}\\
&= p(pxp + 1 - p)^\# p(pxp)p(pxp + 1 - p)^{(1,3e)}p\\
&= p(pxp + 1 - p)^\#(pxp + 1 - p)(pxp + 1 - p)^{(1,3e)}p\\
&\quad - p(pxp + 1 - p)^\#(1 - p)(pxp + 1 - p)^{(1,3e)}p\\
&= p(pxp + 1 - p)^{e,\textcircled{\#}}p.
\end{aligned}$$

Conversely, if $(pxp)^{e,\textcircled{\#}}$ is the e-core inverse of pxp in R, then by Corollary 2.13, we have that $pxp + 1 - p$ is e-core invertible, and

$$\begin{aligned}
(pxp + 1 - p)^{e,\textcircled{\#}} &= (pxp)^{e,\textcircled{\#}} + (1 - p)^{e,\textcircled{\#}}\\
&= (pxp)^{e,\textcircled{\#}} + 1 - p
\end{aligned}$$

since $(pxp)(1 - p) = 0 = (1 - p)(pxp)$ and $(pxp)^*e(1 - p) = (e(1 - p)pxp)^* = 0$.

(2). By Lemma 2.10, it can be derived that $a \in R$ is pseudo e-core invertible with $\mathrm{ind}(a) = k$ if and only if k is the smallest positive integer such that a^k is e-core invertible. If $pxp + 1 - p$ is pseudo e-core invertible with $\mathrm{ind}(pxp + 1 - p) = k$, then k is the smallest positive integer such that $(pxp + 1 - p)^k = (pxp)^k + 1 - p =$

$p\left(x(px)^{k-1}\right)p+1-p$ is e-core invertible, and therefore $p\left(x(px)^{k-1}\right)p=(pxp)^k$ is e-core invertible according to (1). We remark that k is the smallest positive integer such that $(pxp)^k$ is e-core invertible. In fact, if there exists a positive integer $m<k$ such that $(pxp)^m$ is e-core invertible, then $p\left(x(px)^{m-1}\right)p=(pxp)^m$ is e-core invertible. The by (1), we get that $(pxp+1-p)^m$ is e-core invertible, a contradiction. Therefore, pxp is pseudo e-core invertible with $\text{ind}(pxp)=k$. For the expression of $(pxp)^{e,\textcircled{D}}$, by Lemma 2.10 we can obtain that

$$\begin{aligned}(pxp)^{e,\textcircled{D}} &= (pxp)^{k-1}\left((pxp)^k\right)^{e,\textcircled{\#}}\\ &= (pxp)^{k-1}p\left((pxp)^k\right)^{e,\textcircled{\#}}\\ &= \left((pxp)^{k-1}+1-p\right)p\left((pxp)^k+1-p\right)^{e,\textcircled{\#}}p\\ &= p(pxp+1-p)^{k-1}\left((pxp+1-p)^k\right)^{e,\textcircled{\#}}p\\ &= p(pxp+1-p)^{e,\textcircled{D}}p.\end{aligned}$$

Conversely, since $(pxp)(1-p)=0=(1-p)(pxp)$ and $(pxp)^*e(1-p)=(e(1-p)pxp)^*=0$, it follows that

$$\begin{aligned}(pxp+1-p)^{e,\textcircled{D}} &= (pxp)^{e,\textcircled{D}}+(1-p)^{e,\textcircled{D}}\\ &= (pxp)^{e,\textcircled{D}}+1-p\end{aligned}$$

by Corollary 2.12. □

In the following result, we also illustrate the relation between the pseudo e-core invertibility of the corresponding elements of R and pRp.

Corollary 2.15. *Let $p\in R$ be a projection with $(ep)^*=ep$, $x\in R$. Then the following statements hold.*

(1) *$pxp+1-p$ is e-core invertible in R if and only if pxp is pe-core invertible in pRp. In this case,*

$$(pxp)^{pe,\textcircled{\#}}=p(pxp+1-p)^{e,\textcircled{\#}}p\in pRp,$$

and

$$(pxp+1-p)^{e,\textcircled{\#}}=(pxp)^{pe,\textcircled{\#}}+1-p\in pRp+1-p.$$

(2) *$pxp+1-p$ is pseudo e-core invertible with* $\text{ind}(pxp+1-p)=k$ *in R if and only if pxp is pseudo pe-core invertible with* $\text{ind}(pxp)=k$ *in pRp. In this case,*

$$(pxp)^{pe,\textcircled{D}}=p(pxp+1-p)^{e,\textcircled{D}}p\in pRp,$$

and

$$(pxp+1-p)^{e,\textcircled{D}}=(pxp)^{pe,\textcircled{D}}+1-p\in pRp+1-p.$$

Take $e=1$. Then we can obtain analogous results of the core invertibility and the pseudo core invertibility, respectively, as follows.

Corollary 2.16. [14, Theorem 1] *Let $p\in R$ be a projection, $x\in R$. Then the following statements hold.*

(1) *$pxp+1-p$ is core invertible in R if and only if pxp is core invertible in pRp. In this case,*

$$(pxp)^{\textcircled{\#}}=p(pxp+1-p)^{\textcircled{\#}}p\in pRp,$$

and

$$(pxp+1-p)^{\textcircled{\#}}=(pxp)^{\textcircled{\#}}+1-p\in pRp+1-p.$$

(2) *$pxp+1-p$ is pseudo core invertible with* $\text{ind}(pxp+1-p)=k$ *in R if and only if pxp is pseudo core invertible with* $\text{ind}(pxp)=k$ *in pRp. In this case,*

$$(pxp)^{\textcircled{D}}=p(pxp+1-p)^{\textcircled{D}}p\in pRp,$$

and

$$(pxp+1-p)^{\textcircled{D}}=(pxp)^{\textcircled{D}}+1-p\in pRp+1-p.$$

3. Weighted generalized invertibility in two matrix semigroups

Given a ring R with an involution $*$, there is a natural involution $* : R^{m\times n} \to R^{n\times m}$, that is for any $A = (a_{ij}) \in R^{m\times n}$, $A^* \in R^{n\times m}$ is defined as (a^*_{ji}).

Let R be a ring with involution ι and S a ring with involution τ. Then $\varphi : R \to S$ is a ι, τ-invariant homomorphism if φ is a ring homomorphism and $\varphi(x^\iota) = (\varphi(x))^\tau$ for all $x \in R$. If ι and τ coincide, then it is written ι-invariant for short, which is equivalent to say that ι and φ commute [21].

Let $A \in R^{m\times n}$ with $A^\dagger$ existing and $\phi_A : AA^\dagger R^{m\times m} AA^\dagger \to A^\dagger A R^{n\times n} A^\dagger A$ with $\phi_A(AA^\dagger XAA^\dagger) = A^\dagger XA$. If ϕ_A is $*$-invariant, then A is called $*$-invariant. Furthermore, Patrício and Puystjens [21] also illustrate that ϕ_A is $*$-invariant if and only if $A^\dagger YA = A^* Y (A^\dagger)^*$ for all $Y \in R^{m\times m}$.

Let $A \in R^{m\times n}$ with $A^\dagger$ existing and $B \in R^{m\times m}$. Denote the conditions (i) $\Gamma = AA^\dagger BAA^\dagger + I_m - AA^\dagger$ is Moore-Penrose invertible and (ii) $\Omega = A^\dagger BA + I_n - A^\dagger A$ is Moore-Penrose invertible. In [21], Patrício and Puystjens gave an example to illustrate that (i)⇔(ii) does not hold in general. In order to give a sufficient condition for (i)⇔(ii), they introduced the notation and definition of $*$-invariance. Additionally, they also gave an example to explain the $*$-invariance of A is not necessary for (i)⇔(ii). Also, the authors [14] showed the analogous equivalence of pseudo core inverses and core inverses.

In this section, let $M \in R^{m\times m}$ and $N \in R^{n\times n}$ be two invertible Hermitian matrices. In order to give a sufficient condition for the analogous results on the weighted Moore-Penrose invertibility with weights (M, N) and pseudo M-core invertibility, respectively. We first illustrate some more notations and definitions.

Let R equip with an involution $*$ and $A \in R^{m\times n}$ with $A^\dagger_{M,N}$ existing. Suppose that $AA^\dagger_{M,N} R^{m\times m} AA^\dagger_{M,N}$ and $A^\dagger_{M,N} A R^{n\times n} A^\dagger_{M,N} A$ are equipped with the involutions $*M$ and $*N$, respectively. We define

$$\phi_A : AA^\dagger_{M,N} R^{m\times m} AA^\dagger_{M,N} \to A^\dagger_{M,N} A R^{n\times n} A^\dagger_{M,N} A$$

with

$$\phi_A(AA^\dagger_{M,N} X AA^\dagger_{M,N}) = A^\dagger_{M,N} XA \quad \text{for} \quad X \in R^{m\times m}.$$

Then we call ϕ_A is $*M, *N$-invariant if $\phi_A(T^{*M}) = \left(\phi_A(T)\right)^{*N}$ for $T \in AA^\dagger_{M,N} R^{m\times m} AA^\dagger_{M,N}$, that is, $\phi_A\left(M^{-1}T^*M\right) = N^{-1}\left(\phi_A(T)\right)^* N$ for $T \in AA^\dagger_{M,N} R^{m\times m} AA^\dagger_{M,N}$. If ϕ_A is $*M, *N$-invariant, then we call that A is $*M, *N$-invariant.

Let $X \in R^{m\times m}$. Then we have $T = AA^\dagger_{M,N} X AA^\dagger_{M,N} \in AA^\dagger_{M,N} R^{m\times m} AA^\dagger_{M,N}$. It follows that

$$\begin{aligned}
\phi_A\left(M^{-1}T^*M\right) &= \phi_A\left(M^{-1}(AA^\dagger_{M,N} X AA^\dagger_{M,N})^* M\right) \\
&= \phi_A\left((MAA^\dagger_{M,N} X AA^\dagger_{M,N} M^{-1})^*\right) \\
&= \phi_A(AA^\dagger_{M,N} M^{-1} X^* M AA^\dagger_{M,N}) \\
&= A^\dagger_{M,N} M^{-1} X^* MA
\end{aligned}$$

and

$$\begin{aligned}
N^{-1}\left(\phi_A(T)\right)^* N &= N^{-1}\left(\phi_A(AA^\dagger_{M,N} X AA^\dagger_{M,N})\right)^* N \\
&= N^{-1}(A^\dagger_{M,N} XA)^* N \\
&= N^{-1} A^* X^* (A^\dagger_{M,N})^* N.
\end{aligned}$$

Hence, we obtain that ϕ_A is $*M, *N$-invariant if and only if

$$A^\dagger_{M,N} M^{-1} X^* MA = N^{-1} A^* X^* (A^\dagger_{M,N})^* N.$$

Then taking the involution $*$ on the both sides, it follows that

$$A^* M X M^{-1} (A^\dagger_{M,N})^* = N A^\dagger_{M,N} X A N^{-1}. \tag{3.1}$$

Let $\psi_A : A^\dagger_{M,N} A R^{n\times n} A^\dagger_{M,N} A \to AA^\dagger_{M,N} R^{m\times m} AA^\dagger_{M,N}$ be defined by

$$\psi_A(A^\dagger_{M,N} A Y A^\dagger_{M,N} A) = A Y A^\dagger_{M,N} \quad \text{for} \quad Y \in R^{n\times n}.$$

Then it is easy to check that $\phi_A\psi_A = I_{A^{\dagger}_{M,N}AR^{n\times n}A^{\dagger}_{M,N}A}$ and $\psi_A\phi_A = I_{AA^{\dagger}_{M,N}R^{m\times m}AA^{\dagger}_{M,N}}$.

Supposing that ϕ_A is $*M,*N$-invariant. For $Y \in R^{n\times n}$, we have that $G = A^{\dagger}_{M,N}AYA^{\dagger}_{M,N}A \in A^{\dagger}_{M,N}AR^{n\times n}A^{\dagger}_{M,N}A$, then it follows that

$$\begin{aligned} M^{-1}\left(\psi_A(G)\right)^* M &= \psi_A\phi_A\left(M^{-1}\left(\psi_A(G)\right)^* M\right) \\ &= \psi_A\left(N^{-1}\left(\phi_A\psi_A(G)\right)^* N\right) \\ &= \psi_A(N^{-1}G^*N). \end{aligned}$$

Thus, ψ_A is $*N,*M$-invariant. Furthermore, we can obtain that

$$\begin{aligned} M^{-1}\left(\psi_A(G)\right)^* M &= M^{-1}\left(\psi_A(A^{\dagger}_{M,N}AYA^{\dagger}_{M,N}A)\right)^* M \\ &= M^{-1}(AYA^{\dagger}_{M,N})^*M \\ &= M^{-1}(A^{\dagger}_{M,N})^*Y^*A^*M, \end{aligned}$$

and

$$\begin{aligned} \psi_A(N^{-1}G^*N) &= \psi_A\left(N^{-1}(A^{\dagger}_{M,N}AYA^{\dagger}_{M,N}A)^*N\right) \\ &= \psi_A\left((NA^{\dagger}_{M,N}AYA^{\dagger}_{M,N}AN^{-1})^*\right) \\ &= \psi_A(A^{\dagger}_{M,N}AN^{-1}Y^*NA^{\dagger}_{M,N}A) \\ &= AN^{-1}Y^*NA^{\dagger}_{M,N}. \end{aligned}$$

Hence, it follows that $M^{-1}(A^{\dagger}_{M,N})^*Y^*A^*M = AN^{-1}Y^*NA^{\dagger}_{M,N}$. Then taking the involution $*$ on the both sides, we have that

$$MAYA^{\dagger}_{M,N}M^{-1} = (A^{\dagger}_{M,N})^*NYN^{-1}A^*. \tag{3.2}$$

Next, we relate the weighted Moore-Penrose invertibility of the corresponding elements between the semigroup $AA^{\dagger}_{M,N}R^{m\times m}AA^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}$ and the semigroup $A^{\dagger}_{M,N}AR^{n\times n}A^{\dagger}_{M,N}A + I_n - A^{\dagger}_{M,N}A$. For this purpose, we first investigate the weighted $\{1,3\}$-invertibility case and the weighted $\{1,4\}$-invertibility case as follows.

Lemma 3.1. *Let $A \in R^{m\times n}$ be weighted Moore-Penrose invertible with weights (M,N) and $B \in R^{m\times m}$. Consider the following conditions:*
(1) $\Gamma = AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}$ *is {1,3M}-invertible.*
(2) $\Omega = A^{\dagger}_{M,N}BA + I_n - A^{\dagger}_{M,N}A$ *is {1,3N}-invertible.*
If A is $*M,*N$-invariant then (1)⇔(2), in which case

$$A\Omega^{(1,3N)}A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N} \in \Gamma\{1,3M\}$$

and

$$A^{\dagger}_{M,N}\Gamma^{(1,3M)}A + I_n - A^{\dagger}_{M,N}A \in \Omega\{1,3N\}.$$

Proof. (1)⇒(2). If Γ is $\{1,3M\}$-invertible, then by Corollary 2.3, $AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N} = AA^{\dagger}_{M,N}\Gamma AA^{\dagger}_{M,N}$ is $\{1,3M\}$-invertible with a $\{1,3M\}$-inverse Γ_0 in $R^{m\times m}$. As

$$AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}\Gamma_0\, AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N} = AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N},$$

then multiplying on the left side by $A^{\dagger}_{M,N}$ and on the right side by A, we can get

$$(A^{\dagger}_{M,N}BA)A^{\dagger}_{M,N}\Gamma_0\, A(A^{\dagger}_{M,N}BA) = A^{\dagger}_{M,N}BA.$$

Also,

$$(MAA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}\Gamma_0)^* = MAA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}\Gamma_0,$$

then by (3.1), we have

$$\begin{aligned}(NA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}\Gamma_0\, AN^{-1})^* &= (NA^{\dagger}_{M,N}AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}\Gamma_0\, AA^{\dagger}_{M,N}AN^{-1})^*\\ &= (A^*MAA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}\Gamma_0\, AA^{\dagger}_{M,N}M^{-1}(A^{\dagger}_{M,N})^*)^*\\ &= A^{\dagger}_{M,N}AA^{\dagger}_{M,N}M^{-1}MAA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}\Gamma_0\, A\\ &= A^{\dagger}_{M,N}BAA^{\dagger}_{M,N}\Gamma_0\, A,\end{aligned}$$

that is,

$$(NA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}\Gamma_0\, A)^* = NA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}\Gamma_0\, A.$$

Hence, $A^{\dagger}_{M,N}\Gamma_0\, A$ is a $\{1,3N\}$-inverse of $A^{\dagger}_{M,N}BA = A^{\dagger}_{M,N}A\Omega A^{\dagger}_{M,N}A$ in $A^{\dagger}_{M,N}AR^{n\times n}A^{\dagger}_{M,N}A$. Therefore, by Corollary 2.3 again, we have

$$A^{\dagger}_{M,N}\Gamma_0\, A + I_n - A^{\dagger}_{M,N}A \in \Omega\{1,3N\}.$$

As $AA^{\dagger}_{M,N}\Gamma^{(1,3M)}AA^{\dagger}_{M,N} \in (AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N})\{1,3M\}$, it follows that

$$A^{\dagger}_{M,N}\Gamma^{(1,3M)}A + I_n - A^{\dagger}_{M,N}A \in \Omega\{1,3N\}.$$

(2)⇒(1). If Ω is $\{1,3N\}$-invertible, then by Corollary 2.3, $A^{\dagger}_{M,N}BA = A^{\dagger}_{M,N}A\Omega A^{\dagger}_{M,N}A$ is $\{1,3N\}$-invertible with a $\{1,3N\}$-inverse Ω_0 in $R^{n\times n}$. As

$$A^{\dagger}_{M,N}BA\Omega_0\, A^{\dagger}_{M,N}BA = A^{\dagger}_{M,N}BA,$$

then multiplying on the left side by A and on the right side $A^{\dagger}_{M,N}$, we have

$$(AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N})(A\Omega_0\, A^{\dagger}_{M,N})(AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}) = AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}.$$

Also,

$$(NA^{\dagger}_{M,N}BA\Omega_0)^* = NA^{\dagger}_{M,N}BA\Omega_0,$$

then by (3.2), we obtain

$$\begin{aligned}(MAA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}A\Omega_0\, A^{\dagger}_{M,N}M^{-1})^* &= \left((A^{\dagger}_{M,N})^*NA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}A\Omega_0\, N^{-1}A^*\right)^*\\ &= AN^{-1}NA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}A\Omega_0\, A^{\dagger}_{M,N}\\ &= AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}A\Omega_0\, A^{\dagger}_{M,N},\end{aligned}$$

that is,

$$(MAA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}A\Omega_0\, A^{\dagger}_{M,N})^* = MAA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}A\Omega_0\, A^{\dagger}_{M,N}.$$

Hence, $A\Omega_0\, A^{\dagger}_{M,N}$ is a $\{1,3M\}$-inverse of $AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N} = AA^{\dagger}_{M,N}\Gamma AA^{\dagger}_{M,N}$ in $AA^{\dagger}_{M,N}R^{m\times m}AA^{\dagger}_{M,N}$. Therefore, by Corollary 2.3 again, we have

$$A\Omega_0\, A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N} \in \Gamma\{1,3M\}.$$

As $A^{\dagger}_{M,N}A\Omega^{(1,3N)}A^{\dagger}_{M,N}A \in (A^{\dagger}_{M,N}BA)\{1,3N\}$, it follows that $A\Omega^{(1,3N)}A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N} \in \Gamma\{1,3M\}$. □

Lemma 3.2. *Let $A \in R^{m\times n}$ be weighted Moore-Penrose invertible with weights (M,N) and $B \in R^{m\times m}$. Consider the following conditions:*

(1) $\Gamma = AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}$ *is {1,4M}-invertible.*

(2) $\Omega = A^{\dagger}_{M,N}BA + I_n - A^{\dagger}_{M,N}A$ *is {1,4N}-invertible.*

If A is $*M, *N$-invariant then (1)⇔(2), in which case

$$A\Omega^{(1,4N)}A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N} \in \Gamma\{1, 4M\}$$

and

$$A^{\dagger}_{M,N}\Gamma^{(1,4M)}A + I_n - A^{\dagger}_{M,N}A \in \Omega\{1, 4N\}.$$

Then combining Lemmas 3.1 and 3.2, it is easy to obtain analogous results on the weighted Moore-Penrose invertibility with weights (M, N).

Theorem 3.3. *Let $A \in R^{m\times n}$ be weighted Moore-Penrose invertible with weights (M, N) and $B \in R^{m\times m}$. Consider the following conditions:*
(1) $\Gamma = AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}$ *is weighted Moore-Penrose invertible with weights (M, M).*
(2) $\Omega = A^{\dagger}_{M,N}BA + I_n - A^{\dagger}_{M,N}A$ *is weighted Moore-Penrose invertible with weights (N, N).*
If A is $*M, *N$-invariant then (1)⇔(2), in which case

$$\Gamma^{\dagger}_{M,M} = A\Omega^{\dagger}_{N,N}A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}$$

and

$$\Omega^{\dagger}_{N,N} = A^{\dagger}_{M,N}\Gamma^{\dagger}_{M,M}A + I_n - A^{\dagger}_{M,N}A.$$

Proof. It suffices to give the expressions of $\Gamma^{\dagger}_{M,M}$ and $\Omega^{\dagger}_{N,N}$. By Lemmas 3.1 and 3.2, it follows that $\Omega\Omega^{(1,3)} \in A^{\dagger}_{M,N}AR^{n\times n}A^{\dagger}_{M,N}A + I_n - A^{\dagger}_{M,N}A$ and $\Gamma\Gamma^{(1,3)} \in AA^{\dagger}_{M,N}R^{m\times m}AA^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}$. Then $A^{\dagger}_{M,N}A\Omega\Omega^{(1,3)} = \Omega\Omega^{(1,3)}A^{\dagger}_{M,N}A$ and $AA^{\dagger}_{M,N}\Gamma\Gamma^{(1,3)} = \Gamma\Gamma^{(1,3)}AA^{\dagger}_{M,N}$. Hence,

$$\begin{aligned}
\Gamma^{\dagger}_{M,M} =& \Gamma^{(1,4M)}\Gamma\Gamma^{(1,3M)} \\
=& (A\Omega^{(1,4N)}A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N})(AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}) \\
& (A\Omega^{(1,3N)}A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}) \\
=& A\Omega^{(1,4N)}A^{\dagger}_{M,N}AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}A\Omega^{(1,3N)}A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N} \\
=& A\Omega^{(1,4N)}A^{\dagger}_{M,N}A\Omega\Omega^{(1,3)}A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N} \\
=& A\Omega^{(1,4N)}\Omega\Omega^{(1,3)}A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N} \\
=& A\Omega^{\dagger}_{N,N}A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Omega^{\dagger}_{N,N} =& \Omega^{(1,4N)}\Omega\Omega^{(1,3N)} \\
=& (A^{\dagger}_{M,N}\Gamma^{(1,4M)}A + I_n - A^{\dagger}_{M,N}A)(A^{\dagger}_{M,N}BA + I_n - A^{\dagger}_{M,N}A) \\
& (A^{\dagger}_{M,N}\Gamma^{(1,3M)}A + I_n - A^{\dagger}_{M,N}A) \\
=& A^{\dagger}_{M,N}\Gamma^{(1,4M)}AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}\Gamma^{(1,3M)}A + I_n - A^{\dagger}_{M,N}A \\
=& A^{\dagger}_{M,N}\Gamma^{(1,4M)}AA^{\dagger}_{M,N}\Gamma\Gamma^{(1,3M)}A + I_n - A^{\dagger}_{M,N}A \\
=& A^{\dagger}_{M,N}\Gamma^{(1,4M)}\Gamma\Gamma^{(1,3M)}A + I_n - A^{\dagger}_{M,N}A \\
=& A^{\dagger}_{M,N}\Gamma^{\dagger}_{M,M}A + I_n - A^{\dagger}_{M,N}A.
\end{aligned}$$

□

Take $M = I_m$ and $N = I_n$ in Theorem 3.3. Then we have the following result given in [21].

Corollary 3.4. [21, Proposition 6] *Let $A \in R^{m\times n}$ be Moore-Penrose invertible and $B \in R^{m\times m}$. Consider the following conditions:*
(1) $\Gamma = AA^{\dagger}BAA^{\dagger} + I_m - AA^{\dagger}$ *is Moore-Penrose invertible.*

(2) $\Omega = A^{\dagger}BA + I_n - A^{\dagger}A$ *is Moore-Penrose invertible.*
If A is $*$-invariant then (1)⇔(2), in which case

$$\Gamma^{\dagger} = A\Omega^{\dagger}A^{\dagger} + I_m - AA^{\dagger}$$

and

$$\Omega^{\dagger} = A^{\dagger}\Gamma^{\dagger}A + I_n - A^{\dagger}A.$$

Note that [14, Example 1] showed that the equivalence that $\Gamma = AA^{\dagger}BAA^{\dagger} + I_m - AA^{\dagger}$ is core invertible if and only if $\Omega = A^{\dagger}BA + I_n - A^{\dagger}A$ is core invertible does not hold in general when $A \in R^{m\times n}$ be Moore-Penrose invertible and $B \in R^{m\times m}$. Also, the $*$-invariance of A is not necessary for this equivalence is shown in [14, Example 2]. In order to relate the equivalence for the pseudo M-core invertibility of the corresponding elements between the semigroup $AA^{\dagger}_{M,N}R^{m\times m}AA^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}$ and the semigroup $A^{\dagger}_{M,N}AR^{n\times n}A^{\dagger}_{M,N}A + I_n - A^{\dagger}_{M,N}A$ when $A^{\dagger}_{M,N}$ exists, we give a sufficient condition that A is $*M, *N$-invariant.

Theorem 3.5. *Let $A \in R^{m\times n}$ be weighted Moore-Penrose invertible with weights (M, N) and $B \in R^{m\times m}$. Consider the following conditions:*
(1) $\Gamma = AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}$ *is pseudo M-core invertible with* $\mathrm{ind}(\Gamma) = k$ *(M-core invertible if $k = 1$).*
(2) $\Omega = A^{\dagger}_{M,N}BA + I_n - A^{\dagger}_{M,N}A$ *is pseudo N-core invertible with* $\mathrm{ind}(\Omega) = k$ *(N-core invertible if $k = 1$).*
If A is $*M, *N$-invariant then (1)⇔(2), in which case

$$\Gamma^{M,\textcircled{D}} = A\Omega^{N,\textcircled{D}}A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}$$

and

$$\Omega^{N,\textcircled{D}} = A^{\dagger}_{M,N}\Gamma^{M,\textcircled{D}}A + I_n - A^{\dagger}_{M,N}A.$$

Proof. Let us first consider the case $k = 1$, i.e., Γ is M-core invertible if and only if Ω is N-core invertible.

If Γ is M-core invertible, then by Lemma 2.9, it is known that Γ is group invertible and $\{1, 3M\}$-invertible. Following the Lemma 3.1 and [21, Proposition 5], we can obtain that Ω is group invertible and $\{1, 3N\}$-invertible. Moreover, $\Omega^{\#} = A^{\dagger}_{M,N}\Gamma^{\#}A + I_n - A^{\dagger}_{M,N}A$ and $A^{\dagger}_{M,N}\Gamma^{(1,3M)}A + I_n - A^{\dagger}_{M,N}A \in \Omega\{1, 3N\}$. By the Lemma 2.9 again, it is easy to get that Ω is N-core invertible. Since $\Gamma^{\#} \in AA^{\dagger}_{M,N}R^{m\times m}AA^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}$, it follows that $AA^{\dagger}_{M,N}\Gamma^{\#} = \Gamma^{\#}AA^{\dagger}_{M,N}$. For the expression of $\Omega^{N,\textcircled{\#}}$, we have

$$\begin{aligned}
\Omega^{N,\textcircled{\#}} &= \Omega^{\#}\Omega\Omega^{(1,3N)} \\
&= (A^{\dagger}_{M,N}\Gamma^{\#}A + I_n - A^{\dagger}_{M,N}A)(A^{\dagger}_{M,N}BA + I_n - A^{\dagger}_{M,N}A)(A^{\dagger}_{M,N}\Gamma^{(1,3M)}A + I_n - A^{\dagger}_{M,N}A) \\
&= A^{\dagger}_{M,N}\Gamma^{\#}AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N}\Gamma^{(1,3M)}A + I_n - A^{\dagger}_{M,N}A \\
&= A^{\dagger}_{M,N}\Gamma^{\#}AA^{\dagger}_{M,N}\Gamma\Gamma^{(1,3M)}A + I_n - A^{\dagger}_{M,N}A \\
&= A^{\dagger}_{M,N}AA^{\dagger}_{M,N}\Gamma^{\#}\Gamma\Gamma^{(1,3M)}A + I_n - A^{\dagger}_{M,N}A \\
&= A^{\dagger}_{M,N}\Gamma^{M,\textcircled{\#}}A + I_n - A^{\dagger}_{M,N}A.
\end{aligned}$$

The converse is analogous. Since $\Omega^{\#} \in A^{\dagger}_{M,N}AR^{n\times n}A^{\dagger}_{M,N}A + I_n - A^{\dagger}_{M,N}A$, it follows that $A^{\dagger}_{M,N}A\Omega^{\#} = \Omega^{\#}A^{\dagger}_{M,N}A$. For the expression of $\Gamma^{M,\textcircled{\#}}$, we have

$$\begin{aligned}
\Gamma^{M,\textcircled{\#}} &= \Gamma^{\#}\Gamma\Gamma^{(1,3M)} \\
&= (A\Omega^{\#}A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N})(AA^{\dagger}_{M,N}BAA^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N})(A\Omega^{(1,3N)}A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}) \\
&= A\Omega^{\#}A^{\dagger}_{M,N}AA^{\dagger}_{M,N}BA\Omega^{(1,3N)}A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N} \\
&= A\Omega^{\#}A^{\dagger}_{M,N}A\Omega\Omega^{(1,3N)}A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N} \\
&= AA^{\dagger}_{M,N}A\Omega^{\#}\Omega\Omega^{(1,3N)}A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N} \\
&= A\Omega^{N,\textcircled{\#}}A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}
\end{aligned}$$

For the general case, suppose that Γ is pseudo M-core invertible with $\operatorname{ind}(\Gamma) = k$, i.e., $\Gamma^{M,\textcircled{D}}$ exists with $\operatorname{ind}(\Gamma) = k$. Then $(\Gamma^k)^{M,\textcircled{\#}} = \left(AA^{\dagger}_{M,N}(BAA^{\dagger}_{M,N})^k AA^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}\right)^{M,\textcircled{\#}}$ exists by Lemma 2.10. Using the first part of the proof and keeping in mind that B is arbitrary, we can obtain that $\Omega^k = A^{\dagger}_{M,N}(BAA^{\dagger}_{M,N})^k A + I_n - A^{\dagger}_{M,N}A$ is N-core invertible. Thus $\Omega^{N,\textcircled{D}}$ exists with $\operatorname{ind}(\Omega) \le k$ by Lemma 2.10 again. Moreover,

$$\begin{aligned}
\Omega^{N,\textcircled{D}} &= \Omega^{k-1}(\Omega^k)^{N,\textcircled{\#}} \\
&= \Omega^{k-1}\left(A^{\dagger}_{M,N}(BAA^{\dagger}_{M,N})^k A + I_n - A^{\dagger}_{M,N}A\right)^{N,\textcircled{\#}} \\
&= \Omega^{k-1}\left(A^{\dagger}_{M,N}(\Gamma^k)^{M,\textcircled{\#}} A + I_n - A^{\dagger}_{M,N}A\right) \\
&= \left(A^{\dagger}_{M,N}(BAA^{\dagger}_{M,N})^{k-1} A + I_n - A^{\dagger}_{M,N}A\right)\left(A^{\dagger}_{M,N}(\Gamma^k)^{M,\textcircled{\#}} A + I_n - A^{\dagger}_{M,N}A\right) \\
&= A^{\dagger}_{M,N}(BAA^{\dagger}_{M,N})^{k-1} AA^{\dagger}_{M,N}(\Gamma^k)^{M,\textcircled{\#}} A + I_n - A^{\dagger}_{M,N}A \\
&= A^{\dagger}_{M,N}\Gamma^{k-1}(\Gamma^k)^{M,\textcircled{\#}} A + I_n - A^{\dagger}_{M,N}A \\
&= A^{\dagger}_{M,N}\Gamma^{M,\textcircled{D}} A + I_n - A^{\dagger}_{M,N}A.
\end{aligned}$$

The converse is analogous and $\operatorname{ind}(\Gamma) \le \operatorname{ind}(\Omega)$. Hence, $\operatorname{ind}(\Gamma) = \operatorname{ind}(\Omega)$. For the expression of $\Gamma^{M,\textcircled{D}}$, we have

$$\begin{aligned}
\Gamma^{M,\textcircled{D}} &= \Gamma^{k-1}(\Gamma^k)^{M,\textcircled{\#}} \\
&= \Gamma^{k-1}\left(AA^{\dagger}_{M,N}(BAA^{\dagger}_{M,N})^k AA^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}\right)^{M,\textcircled{\#}} \\
&= \Gamma^{k-1}\left(A(\Omega^k)^{N,\textcircled{\#}} A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}\right) \\
&= \left(AA^{\dagger}_{M,N}(BAA^{\dagger}_{M,N})^{k-1} AA^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}\right)\left(A(\Omega^k)^{N,\textcircled{\#}} A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}\right) \\
&= AA^{\dagger}_{M,N}(BAA^{\dagger}_{M,N})^{k-1} A(\Omega^k)^{N,\textcircled{\#}} A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N} \\
&= A\Omega^{k-1}(\Omega^k)^{N,\textcircled{\#}} A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N} \\
&= A\Omega^{N,\textcircled{D}} A^{\dagger}_{M,N} + I_m - AA^{\dagger}_{M,N}.
\end{aligned}$$

□

Take $M = I_m$ and $N = I_n$ in Theorem 3.5. Then we have the following corollary.

Corollary 3.6. [14, Theorem 3] *Let $A \in R^{m\times n}$ be Moore-Penrose invertible and $B \in R^{m\times m}$. Consider the following conditions:*

(1) $\Gamma = AA^{\dagger}BAA^{\dagger} + I_m - AA^{\dagger}$ *is pseudo core invertible with index k (core invertible if $k = 1$).*

(2) $\Omega = A^{\dagger}BA + I_n - A^{\dagger}A$ *is pseudo core invertible with index k (core invertible if $k = 1$).*

If A is $*$-invariant then (1)⇔(2), in which case

$$\Gamma^{\textcircled{D}} = A\Omega^{\textcircled{D}}A^{\dagger} + I_m - AA^{\dagger}$$

and

$$\Omega^{\textcircled{D}} = A^{\dagger}\Gamma^{\textcircled{D}}A + I_n - A^{\dagger}A.$$

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