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Some results on the intersection of g-classes of matrices

Setareh Golshan[a], Ali Armandnejad[b,*], Frank J. Hall[c]

[a]*Department of mathematics, Vali-e-Asr University of Rafsanjan, P.O. box: 7713936417, Rafsanjan, Iran.*
[b]*Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran, and Department of Pure Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran.*
[c]*Department of Mathematics and Statistics, Georgia State University, Atlanta, Ga 30303, USA.*

Abstract. The rich collection of G-matrices originated in a 2012 paper by Fiedler and Hall. Let $\mathbf{M}_n$ be the set of all $n \times n$ real matrices. A nonsingular matrix $A \in \mathbf{M}_n$ is called a G-matrix if there exist nonsingular diagonal matrices D_1 and D_2 such that $A^{-T} = D_1AD_2$, where A^{-T} denotes the transpose of the inverse of A. For fixed nonsingular diagonal matrices D_1 and D_2, let $\mathbb{G}(D_1, D_2) = \{A \in \mathbf{M}_n : A^{-T} = D_1AD_2\}$, which is called a G-class. In more recent papers, G-classes of matrices were studied. The purpose of this present work is to find conditions on D_1, D_2, D_3 and D_4 such that the G-classes $\mathbb{G}(D_1, D_2)$ and $\mathbb{G}(D_3, D_4)$ have finite nonempty intersection or empty intersection. A main focus of this work is the use of the diagonal matrix $D = D_3^{1/2}D_1^{-1/2}$. In the case that all the D_i are $n \times n$ diagonal matrices with positive diagonal entries, complete characterizations of the G-classes are obtained for the intersection questions.

1. Introduction

All matrices in this note have real number entries. Let $\mathbf{M}_n$ be the set of all $n \times n$ real matrices. A matrix $J \in \mathbf{M}_n$ is said to be a signature matrix if J is diagonal and its diagonal entries are ± 1; $\mathcal{S}_n$ is the set of all $n \times n$ signature matrices.

A nonsingular matrix $A \in \mathbf{M}_n$ is called a G-matrix if there exist nonsingular diagonal matrices D_1 and D_2 such that $A^{-T} = D_1AD_2$, where A^{-T} denotes the transpose of the inverse of A, see [2]. For a survey of the basic properties of G-matrices and connections to other classes of matrices, the reader can see [2], [3], [4], [9] and [12] and references therein. For fixed nonsingular diagonal matrices D_1 and D_2, let the class of $n \times n$ G-matrices be

$$\mathbb{G}(D_1, D_2) = \{A \in \mathbf{M}_n : A^{-T} = D_1AD_2\}.$$

We call such a class of matrices a G-class of matrices.

For a fixed signature matrix J, $\Gamma_n(J) = \{A \in \mathbf{M}_n : A^\top JA = J\}$. In fact,

$$\Gamma_n(J) = \mathbb{G}(J, J).$$

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* Corresponding author: Ali Armandnejad
Email addresses: setareh.golshan@gmail.com (Setareh Golshan), armandnejad@vru.ac.ir, armandnejad@uk.ac.ir (Ali Armandnejad), fhall@gsu.edu (Frank J. Hall)

We mention that the matrices in $\Gamma_n(J)$ are precisely the J-orthogonal matrices discussed in [5], [6], [7], [10] and [11]. Also note that when J is I or $-I$, $\Gamma_n(J) = O_n$, the set of all $n \times n$ orthogonal matrices.

We note that the nonsingular diagonal matrices D_1 and D_2 satisfying $A^{-T} = D_1AD_2$ are in general not uniquely determined as we can multiply one of them by a nonzero real number and divide the other by the same number. On the other hand, for nonsingular $n \times n$ diagonal matrices D_1 and D_2, the following known result from [2] shows that if $A^{-T} = D_1AD_2$ then D_1 and D_2 have the same inertia matrix. For the definitions of the inertia and the corresponding inertia matrix of a general Hermitian matrix, the reader can refer to [8, pp281–282]. Simply put, the inertia matrix of a Hermitian matrix A is the diagonal matrix

$$\mathrm{diag}(1, \ldots, 1, -1, \ldots, -1, 0, \ldots, 0),$$

where the number of $1's$, $-1's$, $0's$ is the number of positive, negative, zero eigenvalues, respectively of A.

Proposition 1.1. *Suppose A is a G-matrix and $A^{-T} = D_1AD_2$, where D_1 and D_2 are nonsingular diagonal matrices. Then the inertia of D_1 is equal to the inertia of D_2.*

In [4] we have shown that for every n there exist two $n \times n$ G-classes having finite, nonempty intersection. In this paper we find some conditions on D_1, D_2, D_3 and D_4 such that the G-classes $\mathbb{G}(D_1, D_2)$ and $\mathbb{G}(D_3, D_4)$ have finite intersection. In the continuation of our work, we need the following results.

Theorem 1.2. *[12, Theorem 2.2] Let D_1 and D_2 be nonsingular diagonal matrices with the same inertia matrix J. Then there exist permutation matrices P and Q such that*

$$\mathbb{G}(D_1, D_2) = \{|D_1|^{-1/2}P^TAQ|D_2|^{-1/2} : A \in \Gamma_n(J)\}.$$

This characterization shows that $\mathbb{G}(D_1, D_2)$ is in fact nonempty.

Theorem 1.3. *[12, Theorem 3.1] Assume D_1, D_2, D_3 and D_4 are real nonsingular diagonal matrices, all of which have the same inertia matrix I or $-I$. Then*

$$\mathbb{G}(D_1, D_2) = \mathbb{G}(D_3, D_4)$$

if and only if there exists a positive number d such that $D_3 = dD_1$ and $D_4 = \frac{1}{d}D_2$.

2. The intersection results

In this section we discuss the intersection of G-classes and we first present a key preliminary result.

Lemma 2.1. *Let $D_1 = [d_{1j}]$, $D_2 = [d_{2j}]$, $D_3 = [d_{3j}]$ and $D_4 = [d_{4j}]$, be $n \times n$ diagonal matrices with positive diagonal entries. Let $D = D_3^{\frac{1}{2}}D_1^{\frac{-1}{2}}$ and $D' = D_2^{\frac{-1}{2}}D_4^{\frac{1}{2}}$. If $D^{-1} = D'$ and the diagonal entries of D are not distinct then $\mathbb{G}(D_1, D_2)$ and $\mathbb{G}(D_3, D_4)$ have infinite intersection.*

Proof. Since here the inertia matrix of each D_i is $J = I$, $\Gamma_n(J) = O_n$ and the permutation matrices P, Q are not needed, so that by using Theorem 1.2, there exists a matrix $A \in \mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)$ if and only if there exist $V, W \in O_n$ such that

$$A = D_1^{\frac{-1}{2}}VD_2^{\frac{-1}{2}} = D_3^{\frac{-1}{2}}WD_4^{\frac{-1}{2}},$$

that is to say

$$W = D_3^{\frac{1}{2}}D_1^{\frac{-1}{2}}VD_2^{\frac{-1}{2}}D_4^{\frac{1}{2}}.$$

Thus, to have a matrix A in the intersection of the two G-classes, it is necessary and sufficient to have the existence of orthogonal matrices W, V such that $W = DVD^{-1}$. Let

$$V = \begin{pmatrix} v_{11} & v_{12} & 0 & \ldots & 0 \\ v_{21} & v_{22} & 0 & \ldots & 0 \\ 0 & 0 & \pm1 & \ldots & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & & \ddots & \pm1 \end{pmatrix} \in O_n,$$

and let $W = DVD^{-1}$. Since the diagonal entries of $D = [d_j]$ are not distinct, without loss of generality we can assume that $d_1 = d_2 = d$. It is easy to see that $W = DVD^{-1} = V$. So we have

$$A := D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} = D_3^{\frac{-1}{2}} W D_4^{\frac{-1}{2}}.$$

By Theorem 1.2, $A \in \mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)$. Since we have an infinite number of V, we have an infinite number of A. □

Theorem 2.2. *Let $D_1 = [d_{1j}]$, $D_2 = [d_{2j}]$, $D_3 = [d_{3j}]$ and $D_4 = [d_{4j}]$, be $n \times n$ diagonal matrices with positive diagonal entries. Let $D = D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}}$ and $D' = D_2^{\frac{-1}{2}} D_4^{\frac{1}{2}}$. Assume that $D^{-1} = D'$. Then $\mathbb{G}(D_1, D_2)$ and $\mathbb{G}(D_3, D_4)$ have finite intersection if and only if the diagonal entries of D are distinct. Furthermore in the finite case*

$$\begin{aligned} \mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4) &= \{\mathrm{diag}(\pm \frac{1}{\sqrt{d_{1j} d_{2j}}}), \quad j = 1, \ldots, n\} \\ &= \{\mathrm{diag}(\pm \frac{1}{\sqrt{d_{3j} d_{4j}}}), \quad j = 1, \ldots, n\}. \end{aligned}$$

In this case, the intersection of $\mathbb{G}(D_1, D_2)$ and $\mathbb{G}(D_3, D_4)$ has 2^n matrices.

Proof. The proof of the necessity follows from Lemma 2.1. We now prove the sufficiency. The inertia matrix of each of D_1, D_2, D_3, D_4 is I. Since $D = D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}}$ and the diagonal entries of D are distinct, D is not a multiple of I and hence $D_3 \neq dD_1$ for every $d \in \mathbb{R}$. By using Theorem 1.3

$$\mathbb{G}(D_1, D_2) \neq \mathbb{G}(D_3, D_4).$$

Let $A \in \mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)$. Since the inertia matrix of each D_i is I, by the use of Theorem 1.2, there are $V, W \in \mathcal{O}_n$ such that

$$A = D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} = D_3^{\frac{-1}{2}} W D_4^{\frac{-1}{2}}.$$

This implies that

$$W = D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} D_4^{\frac{1}{2}} \in \mathcal{O}_n.$$

From $W = D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} D_4^{\frac{1}{2}}$, with $W = [w_{ij}]$, $V = [v_{ij}]$ and $D = [\alpha_j]$ it follows that

$$w_{ij} = \frac{\alpha_i}{\alpha_j} v_{ij}.$$

Since $A \in \mathbb{G}(D_1, D_2)$ if and only if $P^T A P \in \mathbb{G}(P^T D_1 P, P^T D_2 P)$ for every $n \times n$ permutation matrix P, we have $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)$ is finite if and only if $\mathbb{G}(P^T D_1 P, P^T D_2 P) \cap \mathbb{G}(P^T D_3 P, P^T D_4 P)$ is finite. Then without loss of generality we can assume that the diagonal entries of D are increasing (in fact there exists a permutation matrix such that $P^T D P$ has increasing diagonal entries). Observe that, for all $i < j$, we have $0 < \frac{\alpha_i}{\alpha_j} < 1$ and consequently when $v_{ij} = 0$, $w_{ij} = 0$, and when $v_{ij} \neq 0$, $w_{ij} < v_{ij}$.
From the diagonal entries of $WW^T = I$, we obtain for $1 \leq i \leq n$,

$$1 = (WW^T)_{ii} = \sum_{j=1}^{n} w_{ij}^2 = \sum_{j=1}^{n} \frac{{\alpha_i}^2}{{\alpha_j}^2} v_{ij}^2. \qquad (*_i)$$

From the entries of $VV^T = I$, we obtain for $1 \leq i \leq n$,

$$1 = (VV^T)_{ii} = \sum_{j=1}^{n} v_{ij}^2 \qquad (**_i)$$

and for each i and t with $1 \le i \ne t \le n$,

$$0 = (VV^T)_{i,t} = \sum_{j=1}^{n} v_{ij}v_{tj}. \qquad (***_{i,t})$$

Now we show that the off diagonal entries of row 1 and column 1 of V are zero. In $(*_1)$, if at least one of $v_{1j} \ne 0$ $(j = 2,\dots,n)$, then the right hand sides of $(*_1)$ and $(**_1)$ are not equal, which is a contradiction. Therefore $v_{1j} = 0$, $(j = 2,\dots,n)$, and so $v_{11} = \pm 1$. Now relations $(***_{1,t})$ $(1 < t \le n)$ imply $v_{21} = v_{31} = \cdots = v_{n1} = 0$.

So far we have:

$$V = \begin{pmatrix} \pm 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \bigstar & \\ 0 & & & \end{pmatrix}.$$

The case where $i = 2$ uses the above structure of V and proceeds similar to the case where $i = 1$. We arrive at

$$V = \begin{pmatrix} \pm 1 & 0 & 0 & \dots & 0 \\ 0 & \pm 1 & 0 & \dots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \bigstar & \\ 0 & 0 & & & \end{pmatrix}.$$

The induction hypothesis is that all the off diagonal entries in V in the first $k-1$ rows and columns are zero, and each diagonal entry is ± 1. Since $v_{k1}, v_{k2}, \dots, v_{k\,k-1}$ are zero, in $(*_k)$, if at least one of $v_{kj} \ne 0$, $(j = k+1,\dots,n)$ then the right hand sides of $(*_k)$ and $(**_k)$ are not equal, which is a contradiction. Therefore $v_{k,k+1} = v_{k,k+2} = \dots = v_{kn} = 0$, and so $v_{kk} = \pm 1$. Now relations $(***_{k,t})$ $(k < t \le n)$ imply $v_{k+1,k} = v_{k+2,k} = \cdots = v_{nk} = 0$. So, the off diagonal entries of row k and column k of V are zero. Thus by induction, $V = \mathrm{diag}(\pm 1)$, so that $W = DVD^{-1} = \mathrm{diag}(\pm 1)$. Hence

$$A = D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} = \mathrm{diag}(\pm \frac{1}{\sqrt{d_{1j}d_{2j}}}) = \mathrm{diag}(\pm \frac{1}{\sqrt{d_{3j}d_{4j}}}), \qquad j = 1,\dots,n.$$

Therefore, the intersection of $\mathbb{G}(D_1, D_2)$ and $\mathbb{G}(D_3, D_4)$ is finite and it has 2^n matrices. □

The following example is a special case of Theorem 2.2 when $n = 3$.

Example 2.3. *Let* $D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & \frac{25}{4} \end{pmatrix}$, $D_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, $D_3 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 25 \end{pmatrix}$ *and* $D_4 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *We have* $D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}} = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \frac{\sqrt{6}}{2} & 0 \\ 0 & 0 & 2 \end{pmatrix}$ *and* $D_2^{\frac{-1}{2}} D_4^{\frac{1}{2}} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{2}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$.

Let $A \in \mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)$. *Since the inertia matrix of each* D_i *is* I, *by the use of Theorem 1.2, there are* $V, W \in \mathcal{O}_3$ *such that*

$$A = D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} = D_3^{\frac{-1}{2}} W D_4^{\frac{-1}{2}}.$$

Since $D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}} = (D_2^{\frac{-1}{2}} D_4^{\frac{1}{2}})^{-1}$, *by the proof of Theorem 2.2 it follows that*

$$V = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

Hence

$$A = D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} = D_3^{\frac{-1}{2}} W D_4^{\frac{-1}{2}} = \mathrm{diag}(\pm 1, \pm\frac{1}{\sqrt{24}}, \pm\frac{1}{5})\}.$$

Since $D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}} = (D_2^{\frac{-1}{2}} D_4^{\frac{1}{2}})^{-1}$ *has distinct diagonal entries, by Theorem 2.2 it follows that*

$$\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4) = \{A : A = \mathrm{diag}(\pm 1, \pm\frac{1}{\sqrt{24}}, \pm\frac{1}{5})\}.$$

When in Theorem 2.2, $D_1 = D_2^{-1}$ and $D_3 = D_4^{-1}$ we can obtain the following corollary.

Corollary 2.4. *Let* D_1, D_2, D_3 *and* D_4 *be* $n \times n$ *diagonal matrices with positive diagonal entries. Suppose that* $D_1 = D_2^{-1}$ *and* $D_3 = D_4^{-1}$. *Then* $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)$ *is nonempty and finite if and only if the diagonal entries of* $D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}}$ *are distinct. Furthermore,* $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4) = \mathcal{S}_n$.

Proof. The proof follows from Theorem 2.2, since with our assumptions on the D_i we have $D^{-1} = D^{'}$. □

The $n \times n$ example in [4] illustrates the above corollary. In the following example, another special case of the above corollary can be seen.

Example 2.5. *Let* $D_1 = \begin{pmatrix} 3 & & & 0 \\ & \frac{1}{4} & & \\ & & 1 & \\ 0 & & & 3 \end{pmatrix}$, $D_2 = \begin{pmatrix} \frac{1}{3} & & & 0 \\ & 4 & & \\ & & 1 & \\ 0 & & & \frac{1}{3} \end{pmatrix}$, $D_3 = \begin{pmatrix} 1 & & & 0 \\ & \frac{1}{4} & & \\ & & 2 & \\ 0 & & & 1 \end{pmatrix}$ *and* $D_4 = \begin{pmatrix} 1 & & & 0 \\ & 4 & & \\ & & \frac{1}{2} & \\ 0 & & & 1 \end{pmatrix}$.

Let $V = \begin{pmatrix} v_{11} & 0 & 0 & v_{14} \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ v_{41} & 0 & 0 & v_{44} \end{pmatrix} \in \mathcal{O}_4$, *and let* $W := D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} D_4^{\frac{1}{2}} = \begin{pmatrix} v_{11} & \frac{1}{\sqrt{3}} v_{12} & \frac{1}{\sqrt{6}} v_{13} & v_{14} \\ \sqrt{3} v_{21} & v_{22} & \frac{1}{\sqrt{2}} v_{23} & \sqrt{3} v_{24} \\ \sqrt{6} v_{31} & \sqrt{2} v_{32} & v_{33} & \sqrt{6} v_{34} \\ v_{41} & \frac{1}{\sqrt{3}} v_{42} & \frac{1}{\sqrt{6}} v_{43} & v_{44} \end{pmatrix} \in$ $\mathcal{O}_4$. *Then* $D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} = D_3^{\frac{-1}{2}} W D_4^{\frac{-1}{2}}$. *Let*

$$A := D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} = D_3^{\frac{-1}{2}} W D_4^{\frac{-1}{2}}.$$

By Theorem 1.2, $A \in \mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)$. *Since we have an infinite number of* V, *we have an infinite number of* A.

Note that indeed $D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}}$ *does not have distinct diagonal entries, so that the result also follows by Lemma 2.1. However, it is interesting to see a closed form expression for an infinite subset of the intersection.*

G-matrices have many nice properties, see [2]. For example, if A is an $n \times n$ G-matrix and P is an $n \times n$ permutation matrix, then PA is a G-matrix. Specifically, it is easy to show that if $A \in \mathbb{G}(D_1, D_2)$ then $PA \in \mathbb{G}(PD_1P^T, D_2)$. Similarly if $B \in \mathbb{G}(PD_1P^T, D_2)$ then $P^T B \in \mathbb{G}(D_1, D_2)$. Hence $P\mathbb{G}(D_1, D_2) = \mathbb{G}(PD_1P^T, D_2)$. Thus, we have the following result.

Proposition 2.6. *With the above notation, we have that*

$$P(\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)) = \mathbb{G}(PD_1P^T, D_2) \cap \mathbb{G}(PD_3P^T, D_4).$$

So, if $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)$ is say a finite diagonal intersection, then for $P \neq I$, $\mathbb{G}(PD_1P^T, D_2) \cap \mathbb{G}(PD_3P^T, D_4)$ is a finite non-diagonal intersection!

Example 2.7. *Let* $D_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$, $D_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix}$, $D_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $D_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$ *and* $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *Then* $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4) = \{\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}\}$ *and* $\mathbb{G}(PD_1P^T, D_2) \cap \mathbb{G}(PD_3P^T, D_4) = \{\begin{pmatrix} 0 & \pm 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}\}$.

Remark 2.8. *We note that regarding the second intersection in the above example, $D^{-1} \neq D'$ (although it is easily seen that $D^{-1} = PD'P^T$). In general, we can have $D^{-1} \neq D'$ and the finite intersection property as well. Also observe in that second intersection that the diagonal entries of D are distinct. This brings up the following question. Is there an example where $D^{-1} \neq D'$ and the diagonal entries of D are not distinct, but we have the finite intersection property? This will be resolved at the end of the paper.*

We turn now to conditions which guarantee that $\mathbb{G}(D_1, D_2)$ and $\mathbb{G}(D_3, D_4)$ have empty intersection. In the next result we are in fact able to give a sufficient condition valid for all n.

Theorem 2.9. *Let D_1, D_2, D_3 and D_4 be $n \times n$ diagonal matrices with positive diagonal entries, and $D = D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}} = [d_j]$ and $D^{'} = D_2^{\frac{-1}{2}} D_4^{\frac{1}{2}} = [d_j^{'}]$. If $d_k^{'} = \min\{d_j^{'}, j = 1, \ldots, n\}$ and $d_k d_k^{'} > 1$, then $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4) = \emptyset$.*

Proof. *Suppose $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4) \neq \emptyset$ and $A \in \mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)$. By using Theorem 1.2 , there are $V, W \in \mathcal{O}_n$ such that*

$$A = D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} = D_3^{\frac{-1}{2}} W D_4^{\frac{-1}{2}},$$

which implies that

$$W = D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} D_4^{\frac{1}{2}} = DVD^{'}.$$

From the diagonal entries of $WW^T = I$ and $VV^T = I$, we obtain,

$$v_{k1}^2 + v_{k2}^2 + \cdots + v_{kn}^2 = 1$$

and

$$d_k^2 d_1^{'2} v_{k1}^2 + d_k^2 d_2^{'2} v_{k2}^2 + \cdots + d_k^2 d_n^{'2} v_{kn}^2 = 1.$$

Then

$$\begin{aligned}(d_k^2 d_1^{'2} - d_k^2 d_k^{'2}) v_{k1}^2 + (d_k^2 d_2^{'2} - d_k^2 d_k^{'2}) v_{k2}^2 \quad &+ \quad \cdots \\ +(d_k^2 d_n^{'2} - d_k^2 d_k^{'2}) v_{kn}^2 \quad &= \quad 1 - d_k^2 d_k^{'2}.\end{aligned}$$

In the above relation all coefficients are positive and hence the left side is positive. But the right side is negative, which is a contradiction. Therefore $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4) = \emptyset$. □

Corollary 2.10. *Let D_1, D_2, D_3 and D_4 be $n \times n$ diagonal matrices with positive diagonal entries. Let $D = D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}} = [d_j]$ and $D^{'} = D_2^{\frac{-1}{2}} D_4^{\frac{1}{2}} = [d_j^{'}]$. If $d_j d_j^{'} > 1$, for $j = 1, \ldots, n$, then $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4) = \emptyset$.*

Proof. *Let $d_k^{'} = \min\{d_j^{'}, j = 1, \ldots, n\}$. By the assumption $d_k d_k^{'} > 1$ and hence by Theorem 2.9, $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4) = \emptyset$.* □

The following example shows that the converse of Theorem 2.9 is not true.

Example 2.11. *Let $D_1 = \begin{pmatrix} 3 & 0 \\ 0 & 16 \end{pmatrix}$, $D_2 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $D_3 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ and $D_4 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{10} \end{pmatrix}$. We have $D = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and $D^{'} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$.*

Assume that $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4) \neq \emptyset$ and $A \in \mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)$. By using Theorem 1.2, there are $V, W \in \mathcal{O}_2$ such that

$$A = D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} = D_3^{\frac{-1}{2}} W D_4^{\frac{-1}{2}},$$

which implies that

$$W = D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} D_4^{\frac{1}{2}}.$$

From the diagonal entries of $WW^T = I$, we obtain,

$$\frac{1}{6}{v_{11}}^2 + \frac{1}{15}{v_{12}}^2 = 1,$$

From $VV^T = I$, we obtain

$${v_{11}}^2 + {v_{12}}^2 = 1.$$

Then

$$\frac{-1}{10}{v_{12}}^2 = \frac{5}{6},$$

which is a contradiction. So $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4) = \emptyset$. However, the assumption in Theorem 2.9 does not hold.

By considering the contrapositive of Theorem 2.9, we obtain the following.

Corollary 2.12. *Let D_1, D_2, D_3 and D_4 be $n \times n$ diagonal matrices with positive diagonal entries. Let $D = D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}} = [d_j]$ and $D' = D_2^{\frac{-1}{2}} D_4^{\frac{1}{2}} = [d_j']$. If $\mathbb{G}(D_1, D_2)$ and $\mathbb{G}(D_3, D_4)$ have nonempty intersection, then $d_k d_k' \leq 1$, where $d_k' = \min\{d_j', j = 1, \ldots, n\}$.*

In particular, this shows a certain reliance on the matrices D and D'!

A final aim in this work is to obtain a necessary and sufficient condition for the intersection of $\mathbb{G}(D_1, D_2)$ and $\mathbb{G}(D_3, D_4)$ to be nonempty. In doing so we will make nice use of some previous results. The condition that we obtain will again be in terms of the matrices D and D'. But first we need some preliminary information.

Definition 2.13. *Let $A = [a_{ij}]$ be an $n \times n$ matrix. If σ is a permutation on n symbols, the set $a_{1\sigma(1)}, a_{2\sigma(2)}, \ldots, a_{n\sigma(n)}$ is called a diagonal of A . Each diagonal contains exactly one element from each row and from each column of A.*

Proposition 2.14. *[1, The Konig-Frobenius Theorem] Let $A = [a_{ij}]$ be an $n \times n$ matrix. The following two statements are equivalent:*

- ***(i)*** *every diagonal of A contains a zero element.*
- ***(ii)*** *A has a $k \times l$ submatrix with all entries zero for some k, l such that $k + l > n$.*

Lemma 2.15. *Let $A = [a_{ij}]$ be an $n \times n$ nonsingular matrix. Then A has a diagonal with all nonzero elements.*

Proof. *Assume if possible that every diagonal of A has at least a zero element. From Proposition 2.14 it follows that A has a $k \times l$ submatrix with all entries zero for some k, l such that $k + l > n$. Then without loss of generality we can assume that A has the following form:*

$$A = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix},$$

where 0 is a $k \times l$ zero block. Observe that C is a $(n - k) \times l$ matrix and $l > (n - k)$. These imply that the columns of C are linearly dependent and hence A is a singular matrix which is a contradiction. □

Theorem 2.16. *Let D_1, D_2, D_3 and D_4 be $n \times n$ diagonal matrices with positive diagonal entries. Let $D = D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}} = [d_j]$ and $D' = D_2^{\frac{-1}{2}} D_4^{\frac{1}{2}} = [d_j']$.Then $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4) \neq \emptyset$, if and only if for some permutation matrix P, $D^{-1} = PD'P^T$.*

Proof. *To prove the sufficiency assume that $D^{-1} = PD'P^T$. Consider $\mathbb{G}(D_1, PD_2P^T) \cap \mathbb{G}(D_3, PD_4P^T)$. Observe that*

$$(PD_2P^T)^{-1/2}(PD_4P^T)^{1/2} = PD'P^T.$$

So, in this case, the condition $D^{-1} = D'$ *is replaced by* $D^{-1} = PD'P^T$. *Hence, using Lemma 2.1 and Theorem 2.2 together, we have that if* $D^{-1} = PD'P^T$, *then* $\mathbb{G}(D_1, PD_2P^T) \cap \mathbb{G}(D_3, PD_4P^T)$ *is nonempty.*

Also, a variation of Proposition 2.6 says that

$$\mathbb{G}(D_1, PD_2P^T) \cap \mathbb{G}(D_3, PD_4P^T) = (\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4))P.$$

Thus, if $D^{-1} = PD'P^T$ *for some permutation matrix P, then* $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4) \neq \emptyset$.

To prove the necessity assume that $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4) \neq \emptyset$. *Let* $A = [a_{ij}] \in \mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)$ *and hence*

$$A^{-T} = D_1AD_2 = D_3AD_4.$$

Then

$$AD_2D_4^{-1} = D_1^{-1}D_3A,$$

and hence

$$A(D')^{-2} = D^2A.$$

So we have

$$\begin{pmatrix} a_{11}(d_1^{'})^{-2} & a_{12}(d_2^{'})^{-2} & \dots & a_{1n}(d_n^{'})^{-2} \\ a_{21}(d_1^{'})^{-2} & a_{22}(d_2^{'})^{-2} & \dots & a_{2n}(d_n^{'})^{-2} \\ \vdots & \vdots & & \vdots \\ a_{n1}(d_1^{'})^{-2} & a_{n2}(d_2^{'})^{-2} & \dots & a_{nn}(d_n^{'})^{-2} \end{pmatrix} = \begin{pmatrix} a_{11}{d_1}^2 & a_{12}{d_1}^2 & \dots & a_{1n}{d_1}^2 \\ a_{21}{d_2}^2 & a_{22}{d_2}^2 & \dots & a_{2n}{d_2}^2 \\ \vdots & \vdots & & \vdots \\ a_{n1}{d_n}^2 & a_{n2}{d_n}^2 & \dots & a_{nn}{d_n}^2 \end{pmatrix}.$$

Since A is nonsingular, by the use of Lemma 2.15, A has a diagonal say $a_{1\sigma(1)}, \dots, a_{n\sigma(n)}$ *with all nonzero elements. Thus we obtain for all* $1 \leq j \leq n$, $d_j^2 = (d^{'}_{\sigma(j)})^{-2}$. *Since the entries of D and* $D^{'}$ *are positive, we have* $d_j = (d^{'}_{\sigma(j)})^{-1}$, *for all* $1 \leq j \leq n$. *If P is the permutation matrix corresponding to* σ, *then* $D^{-1} = PD^{'}P^T$. □

Note that $D^{-1} = PD^{'}P^T$ simply means that the diagonal matrices D^{-1} and $D^{'}$ have the same collection of diagonal entries, including their multiplicities.

Corollary 2.17. *Let* D_1, D_2, D_3 *and* D_4 *be* $n \times n$ *diagonal matrices with positive diagonal entries. Let* $D = D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}}$ *and* $D^{'} = D_2^{\frac{-1}{2}} D_4^{\frac{1}{2}}$. *Then* $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4) = \emptyset$ *if and only if* $D^{-1} \neq PD^{'}P^T$ *for all* $n \times n$ *permutation matrices P.*

We are now able to obtain a further culminating result. Suppose for some permutation matrix P we have that $D^{-1} = PD'P^T$ as in Theorem 2.16. Assume that the diagonal entries of D are not distinct. Then, as in the proof of Lemma 2.1, we arrive at the fact that the intersection of $\mathbb{G}(D_1, PD_2P^T)$ and $\mathbb{G}(D_3, PD_4P^T)$ is infinite. We then employ again the identity

$$\mathbb{G}(D_1, PD_2P^T) \cap \mathbb{G}(D_3, PD_4P^T) = [\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)]P$$

to see that the intersection of $\mathbb{G}(D_1, D_2)$ and $\mathbb{G}(D_3, D_4)$ is infinite.

On the other hand, assume that the diagonal entries of D are distinct. Then, as in the proof of Theorem 2.2, it follows that the intersection of $\mathbb{G}(D_1, PD_2P^T)$ and $\mathbb{G}(D_3, PD_4P^T)$ is finite, with 2^n matrices in the intersection.

We again use the identity

$$\mathbb{G}(D_1, PD_2P^T) \cap \mathbb{G}(D_3, PD_4P^T) = [\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)]P$$

to see that in this case $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)$ is finite, with 2^n matrices in the intersection.

Then, in view of Theorem 2.16, we can say the following.

Theorem 2.18. *Let* D_1, D_2, D_3 *and* D_4 *be* $n \times n$ *diagonal matrices with positive diagonal entries. Let* $D = D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}}$ *and* $D^{'} = D_2^{\frac{-1}{2}} D_4^{\frac{1}{2}}$. *Further suppose that* $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4) \neq \emptyset$. *Then*

(i) *if the diagonal entries of D are not distinct, the intersection of* $\mathbb{G}(D_1, D_2)$ *and* $\mathbb{G}(D_3, D_4)$ *is infinite.*

(ii) *if the diagonal entries of D are distinct,* $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)$ *is finite with* 2^n *matrices in the intersection.*

Thus, when $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4) \neq \emptyset$, $\mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)$ is finite nonempty if and only if the diagonal entries of D are distinct!

References

[1] R. Bhatia, *Matrix Analysis*, (2rd edition), Cambridge university press, 1991.
[2] M. Fiedler, F.J. Hall, *G-matrices,* Linear Algebra Appl. **436** (2012) 731–741.
[3] M. Fiedler, T. L. Markham, *More on G-matrices,* Linear Algebra Appl. **438** (2013) 231–241.
[4] S. Golshan, A. Armandnejad and F. J. Hall, *Two $n \times n$ G-classes of matrices having finite intersection,* Spec. Matrices. **11** (2023) 1–4.
[5] F. J. Hall, Z. Li, C. T. Parnass, M. Rozložník, *Sign patterns of J-orthogonal matrices*, Spec. Matrices. **5** (2017) 225–241.
[6] F. J. Hall, M. Rozložník, *G-matrices, J-orthogonal matrices, and their sign patterns*, Czechoslovak Mathematical Journal. **66** (3) (2016) 653–670.
[7] N. J. Higham, *J-orthogonal matrices: properties and generation*, SIAM Review. **45** (3) (2003) 504–519.
[8] R. A. Horn and C. R. Johnson, *Matrix Analysis*, (2rd edition), Cambridge university press, 2013.
[9] M. Matsuura, *A note on generalized G-matrices*, Linear Algebra Appl. **436** (2012) 3475–3479.
[10] S. M. Motlaghian, A. Armandnejad and F. J. Hall, *Topological properties of J-orthogonal matrices*, Linear and Multilinear Algebra. **66** no. 12, (2018) 2524–2533.
[11] S. M. Motlaghian, A. Armandnejad and F. J. Hall, *Topological properties of J-orthogonal matrices, part II*, Linear and Multilinear Algebra. **69** no. 3, (2021) 438–447.
[12] S. M. Motlaghian, A. Armandnejad and F. J. Hall, *A note on some classes of G-matrices*, Operators and Matrices. **16** (2022) 251–263.