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A study on impulsive evolution equation in Banach space

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Abstract. The goal of this paper is to consider abstract impulsive evolution systems with delay in the framework of ordered Banach spaces. Firstly, we discuss the existence and uniqueness of positive mild solutions for the abstract impulsive evolution systems with delay under order conditions and growth conditions. Secondly, based on monotone iterative technique coupled with fixed point theorem, the existence of minimal positive mild solution is discussed without assuming the existence of upper and lower solutions. At the end, applications to partial differential equations are given.

1. Introduction

In this paper, we are concernded with the existence of positive mild solutions for impulsive evolution equation with delay in an ordered Banach space E

$$\begin{cases} u'(t) + Au(t) = f(t, u(t), u_t), \ t \in J = [0, +\infty), \ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \ k = 1, 2, \dots, \\ u(t) = \phi(t), \ t \in [-r, 0], \end{cases} \tag{1.1}$$

where $A : \mathcal{D}(A) \subset E \to E$ is a linear operator and $-A$ generates a C_0-semigroup $T(t)(t \geq 0)$ on E, the history $u_t : [-r, 0] \to E$ defined by $u_t(s) = u(t+s)$ for $s \in [-r, 0]$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically, $\phi \in \mathcal{B}$ and $\phi(0) \in \mathcal{D}(A)$, $r > 0$ is a constant, $f : \mathbb{R}^+ \times E \times \mathcal{B} \to E$, $I_k : E \to E$ are appropriated functions, $0 < t_1 < t_2 < \cdots$, $\Delta u|_{t=t_k}$ denote the jump of $u(t)$ at $t = t_k$, i.e., $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively.

The dynamics of many evolving processes undergo mutations that involve short-term disturbances, and their duration can be negligible compared to the entire evolution of continuous stationary dynamics. The model involving this disturbance is called the "impulses" phenomenon. The theory of impulses evolution equations is a new and important branch of differential equation theory, which has a wide range of application backgrounds in physics, ecology, chemistry, population dynamics, biological systems,

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and engineering. Therefore, the study of pulse evolution equations has aroused great enthusiasm among scholars and has yielded many important theories, see [4, 14, 16, 17, 19, 21, 24, 29–32, 35].

Researchers focus on the field of evolutionary equations from different perspectives, as such equations can reveal the future or past states of models. As is well known, considering that in addition to the current state, the past state of the model can also make it more realistic. The simplest case of ordinary differential equations (ODE) with constant delays was first proposed, but the author always considers state dependent delays in potential real-world processes, see[10, 11, 25, 27].

More importantly, in many specific system applications, sometimes only positive solutions have significant significance. The research on positive solutions to evolution equation is indeed very active, as seen in [19, 20, 22, 23]. Li (1996) [18] discussed the properties of positive operator semigroups and introduced upper and lower solution methods into bilinear evolution equations in ordered spaces. Shu et al. (2015) [26] studied the existence and uniqueness of positive S- asymptotic ω- periodic mild solutions for a class of bilinear neutral fractional evolution equations with time delays on a positive cone. In [5], Chen et al. (2015) studied the existence and uniqueness of positive temperature solutions for semi linear evolution equations with non local initial conditions on infinite intervals using monotonic iterative methods, rather than assuming upper and lower solutions. In [18, 19], Li et al. (2017, 2021) studied the positive S- asymptotic ω-periodic mild solutions of abstract fractional evolution equations on infinite intervals.

The highlights and advantages of this paper are presented as follows:

(1) We discuss the existence of positive mild solutions of initial value problem (1.1) on infinite interval in the framework of ordered Banach spaces.

(2) Without assuming the existence of upper and lower solutions, we directly obtain the existence of positive mild solutions by using monotone iterative technique in the sense of compact and noncompact semigroups, respectively.

(3) We deal with the existence of positive mild solutions of impulsive evolution equation with delay on infinite intervals by introducing the generalized Arzela-Ascoli theorem and Kuratowski measure of noncompactness, which has better significance and application value than the usual spatial continuation methods.

(4) The topological method that some authors have chosen to study existence of positive solutions is the theory of fixed points, which has been a very powerful and important tool to the study of nonlinear phenomena. Specifically, authors have used contraction mapping principle, Leray-Schauder alternative theorem, Schauder theorem and Krasnoselkii's theorem. However, monotone iterative method in the presence of the lower and upper solutions is the first time that it has been used to study our concerned problem in ordered Banach space. Therefore, our results are novel and meaningful.

The paper is organized as follows. In Section 2 we briefly recall some basic results and some lemmas. Section 3 discusses the existence theorems for the problem (1.1). And in the last section, an example is given to illustrate our abstract results.

2. Preliminaries

Let $(E, \|\cdot\|)$ be an ordered Banach space with partial order "$\leq$" induced by the positive normal cone $P = \{u \in E : u \geq \theta\}$ with normal constant N.

We denote by $PC([-r, +\infty), E)$ the space of piecewise continuous functions $u : [-r, +\infty) \to E$ such that $u(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \cdots\}$.

Let $h \in C\Big(\mathbb{R}^+, [1, +\infty)\Big)$ is a nondecreasing function and $\lim\limits_{t\to+\infty} h(t) = +\infty$. In the sequel, $PC_0(E)$ and $PC_h(E)$ are the spaces defined by

$$PC_0(E) = \{u \in PC(\mathbb{R}^+, E) : \lim_{t\to+\infty} \|u(t)\| = 0\},$$

$$PC_h(E) = \left\{u \in PC(\mathbb{R}^+, E) : \lim_{t\to+\infty} \frac{\|u(t)\|}{h(t)} = 0\right\} \tag{2.1}$$

endowed with the norms $\|u\|_{PC} = \sup_{t\geq 0} \|u(t)\|$ and $\|u\|_h = \sup\limits_{t\geq 0} \frac{\|u(t)\|}{h(t)}$.

In this space $\mathcal{B}$ is consided as a Banach space of piecewise continuous functions $\phi : [-r, 0] \to E$ with the norm $\|\phi\|_{\mathcal{B}} = \sup\limits_{s\in[-r,0]} \|\phi(s)\|$.

Evidently, $PC([-r, +\infty), E)$ and $\mathcal{B}$ are also order Banach spaces with partial "$\leq$" reduced by the positive function cones $K_{PC} = \{u \in PC([-r, +\infty), E) : u(t) \geq \theta, t \in [-r, +\infty)\}$ and $K_{\mathcal{B}} = \{\phi \in \mathcal{B} : \phi(s) \geq \theta,\ s \in [-r, 0]\}$($\theta$ is the zero element of E) respectively.

Also, we have the following compactness criterion.

Lemma 2.1([15]). *A set $B \subset PC_h(E)$ is relatively compact if and only if*

(*a*) *B is equicontinuous;*

(*b*) *$B(t) = \{u(t) : u \in B\}$ is relatively compact in E for every $t \in [0, +\infty)$;*

(*c*) $\lim\limits_{t\to+\infty} \frac{\|u(t)\|}{h(t)} = 0$, *uniformly for $u \in B$.*

Let $A : \mathcal{D}(A) \subset E \to E$ be a linear operator and $-A$ generate a C_0-semigroup $T(t)(t \geq 0)$ on E. By the exponential boundedness of C_0-semigroup $T(t)(t \geq 0)$, there exist constants $M \geq 1$ and $\nu \in \mathbb{R}$, such that

$$\|T(t)\| \leq Me^{\nu t},\ t \geq 0. \tag{2.2}$$

In particular, $\|T(t)\| \leq M$ implies that C_0-semigroup $T(t)(t \geq 0)$ is uniformly bounded.

Let $T(t)(t \geq 0)$ be a C_0-semigroup on E, the infimum of ν satisfying (2.2) is the growth exponent of $T(t)(t \geq 0)$ which means

$$\nu_0 = \inf\{\nu \in \mathbb{R} | \text{ there exists } M \geq 1 \text{ such that } \|T(t)\| \leq Me^{\nu t},\ t \geq 0\}. \tag{2.3}$$

Moreover, if $\nu_0 < 0$, then C_0-semigroup $T(t)(t \geq 0)$ is said to be exponentially stable.

Definition 2.2([20]). *If for each $x \geq \theta, x \in E$ and $t \geq 0$, $T(t)x \geq \theta$, then C_0-semigroup $T(t)(t \geq 0)$ on E is positive.*

Definition 2.3([20]). *If for every $t > 0$, $T(t)$ is a compact operator in E, then C_0-semigroup $T(t)(t \geq 0)$ on E is compact.*

Now, if the cone P is a regeneration cone, then by the properties of positive senigroups [20] , it follows that for sufficiently large $\lambda_0 > -\inf\{Re\lambda : \lambda \in \sigma(A)\}$, $\lambda_0 I + A$ has posotive inverse opertor $(\lambda_0 I + A)^{-1}$. Since $\sigma(A) \neq \emptyset$, the spectral radius

$$r((\lambda_0 I + A)^{-1}) = \frac{1}{dist(-\lambda_0, \sigma(A))} > 0.$$

Then, by the famous Krein-Rutman theorem (see [9, 12]), the operator A has the first eigenvalue $\lambda_1 > 0$, which corresponding positive eigenfunction ϕ_1 and

$$\lambda_1 = \inf\{Re\lambda : \lambda \in \sigma(A)\}.$$

It's easy to see that the exponentially stable C_0-semigroup $T(t)(t \geq 0)$ is uniformly bounded. If $T(t)(t \geq 0)$ is continuous under the uniform operator topology in E for each $t > 0$, then ν_0 can also be expressed by the spectral set $\sigma(A)$ of A, i.e.,

$$\nu_0 = -\inf\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}. \tag{2.4}$$

Thus, it is obvious from (2.4) that $\nu_0 = -\lambda_1$.

In fact, we have known from [33] that if C_0-semigroup $T(t)(t \geq 0)$ is compact, then it is continuous under the uniform operator topology for $t \geq 0$.

Definition 2.4. *A function $u \in PC([-r,+\infty),E)$ is the mild solution of problen (1.1) if $u(t)=\phi(t)$ for $t\in[-r,0]$ and*

$$u(t)=T(t)\phi(t)+\int_0^t T(t-s)f(s,u(s),u_s)ds+\sum_{0<t_k<t}T(t-t_k)I_k(u(t_k)),\ t\geq 0. \tag{2.5}$$

Moreover, if $u(t)\geq\theta$ for all $t\in[-r,+\infty)$, then it is said to be a positive mild solution of problen (1.1).

Lemma 2.5 ([13]). *Let E be a Banach space, $B=\{u_n\}_{n=1}^{\infty}\subset C([0,a],E)$ be a bounded and countable set. Then $\alpha(B(t))$ is Lebesgue integrable on $[0,a]$, and*

$$\alpha\Big(\Big\{\int_0^a u_n(t)dt|n\in\mathbb{N}\Big\}\Big)\leq 2\int_0^a \alpha(B(t))dt.$$

Lemma 2.6 ([26]). *Let E be a Banach space, $B\subset C([0,a],E)$ be bounded and equicontinuous. Then $\alpha(B(t))$ is continuous on $[0,a]$, and $\alpha(B)=\max_{t\in[0,a]}\alpha(B(t))$.*

Finally, denote by $\alpha_h(\cdot)$ the Kuratowski measure of noncompactness of the bounded sets on $C_h(E)$(more details see [3, 6]). Then, we have the following key lemma on $\alpha_h(\cdot)$:

Lemma 2.7([34]). *Let $D\subset PC_h(E)$ be a bounded set. If*

(i) D is a family of locally equicontinuous function, i.e, for any constant $a>0$, the functions in D are equicontinuous in $[0,a]$;

(ii) $\lim_{t\to\infty}\frac{1}{h(t)}\|u(t)\|=0$ uniformly for any $u\in D$,

then $\alpha_h(D)=\sup_{t\geq 0}\alpha(\frac{D(t)}{h(t)})$.

3. Main results

Now, let $\phi\in K_{\mathcal{B}},\ \phi(0)\in K_{\mathcal{B}}\cap\mathcal{D}(A)$, for a given $\phi\in K_{\mathcal{B}}$ and $u\in PC_h(E)$, we define $u_\phi(t):[-r,+\infty)\to E$ by

$$u_\phi(t)=\begin{cases} x(t), & t\in[0,+\infty),\\ \phi(t), & t\in[-r,0]. \end{cases} \tag{3.1}$$

A closed subspace of $PC_h(E)$ is defined by

$$PC_{h,\phi}(E)=\{u\in PC([-r,+\infty)):\ u(t)=\phi(t),t\in[-r,0],u|_{[0,+\infty)}\in PC_h(E)\cap PC_0(E)\}$$

with the norm $\|u\|_{h,\phi}=\max\{\|u\|_h,\ \|\phi\|_{\mathcal{B}}\}$.

For $v,w\in PC([-r,+\infty),E)$ with $v\leq w$, we use $[v,w]$ to denote the order interval $\{u\in PC([-r,+\infty),E)|v\leq u\leq w\}$, and $[v(t),w(t)]$ to denote the order interval $\{u\in E: v(t)\leq u(t)\leq w(t), t\in[0,+\infty)\}$.

Theorem 3.1. *Let E be an ordered Banach space,whose positive cone $P\subset E$ is normal, $A:\mathcal{D}(A)\subset E\to E$ be a closed linear operator and $-A$ generate a positive, compact and exponentially stable C_0-semigroup $T(t)(t\geq 0)$ on E, whose growth exponent $\nu_0<0$. If $f\in C(\mathbb{R}^+\times E\times\mathcal{B},E)$, $I_k\in C(E,E)(k=1,2,\cdots)$ and $\phi\in K_{\mathcal{B}}$, $\phi(0)\in K_{\mathcal{B}}\cap\mathcal{D}(A)$ and the following conditions are established:*

(H1) There exist constants $0<a<\frac{-\nu_0}{M}$, $b,c>0$, and constants $a_k>0$, $b_k>0$, such that

$$\|f(t,h(t)x,h(t)\phi)\|\leq a\|x\|+b\|\phi\|_{\mathcal{B}}+c,\ t\in\mathbb{R}^+,\ x\in E,\ \phi\in\mathcal{B},$$

$$\|I_k(h(t)v)\|\leq a_k\|v\|+b_k,\ t\in\mathbb{R}^+,\ v\in PC(\mathbb{R}^+,E),$$

with

$$a + b + \sum_{0<t_k<t} a_k(-\nu_0) < \frac{(-\nu_0)}{M}; \tag{3.2}$$

($H2$) *For any* $x_1, x_2 \in E$ *and* $\phi_1, \phi_2 \in \mathcal{B}$ *with* $x_2 \geq x_1$, $\phi_2 \geq \phi_1$,

$$f(t, x_2, \phi_2) \geq f(t, x_1, \phi_1),\ t \in \mathbb{R}^+;$$

($H3$) *For any* $y_1, y_2 \in PC(\mathbb{R}^+, E)$ *with* $y_2 \geq y_1$,

$$I_k(y_2) \geq I_k(y_1).$$

Then the problem (1.1) has a minimal positive mild solution $u \in PC([-r, +\infty), P)$.

Proof. Define an operator Q on $PC_{h,\phi}(E)$ by

$$Qu(t) = T(t)\phi(0) + \int_0^t T(t-s)f(s, u(s), u_s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(u(t_k)),\ t \in [0, +\infty). \tag{3.3}$$

Due to $T(t)(t \geq 0)$ is a compact and exponentially stable C_0-semigroup, we known that $Q : PC_{h,\phi}(E) \to PC_{h,\phi}(E)$ is well defined. We note that for any $u \in PC_{h,\phi}(E)$ and $t \geq 0$, we get that

$$\|u(t)\|_{PC} \leq h(t)\|u\|_h \leq h(t)\|u\|_{h,\phi}$$

and

$$\|u_t\|_{\mathcal{B}} = \sup_{s\in[-r,0]} \|u(t+s)\| = \max\{\sup_{t\in[-r,0]} \|u(t)\|, \|u(t)\|_{PC}\}$$

$$\leq \max\{\|\phi\|_{\mathcal{B}},\ h(t)\|u\|_h\} \leq h(t)\|u\|_{h,\phi}.$$

Then, from (3.3), for any $t \in \mathbb{R}^+$, it follows that

$$\begin{aligned}\|Qu(t)\| &\leq \|T(t)\phi(0)\| + \Big\| \int_0^t T(t-s)f(s, u(s), u_s)ds\Big\| + \Big\| \sum_{0<t_k<t} T(t-t_k)I_k(u(t_k))\Big\| \\ &:= I_1 + I_2 + I_3.\end{aligned} \tag{3.4}$$

In view of the boundness of C_0-semigroup $T(t)(t \geq 0)$, we have

$$I_1 = \|T(t)\phi(0)\| \leq M\|\phi\|_{\mathcal{B}}. \tag{3.5}$$

And by ($H1$) and ($H2$), we get

$$\begin{aligned}I_2 &= \Big\| \int_0^t T(t-s)f(s, u(s), u_s)ds\Big\| \\ &\leq M\int_0^t e^{\nu_0 s}\Big(a\frac{\|u(s)\|}{h(s)} + b\frac{\|u_s\|_{\mathcal{B}}}{h(s)} + c\Big)ds \\ &\leq \frac{M}{|\nu_0|}\Big((a+b)\|u\|_{h,\phi} + c\Big),\end{aligned} \tag{3.6}$$

$$\begin{aligned}I_3 &= \Big\| \sum_{0<t_k<t} T(t-t_k)I_k(u(t_k))\Big\| \\ &\leq M\sum_{0<t_k<t} \Big(a_k\frac{\|u\|_C}{h(t)} + b_k\Big) \\ &\leq M\sum_{0<t_k<t} (a_k\|u\|_{h,\phi} + b_k).\end{aligned} \tag{3.7}$$

Thus, by (3.4)-(3.7), we have

$$\begin{aligned} \|Qu(t)\| &\le M\|\phi\|_{\mathcal{B}} + \frac{M}{(-\nu_0)}\Big((a+b)\|u\|_{h,\phi} + c\Big) + M\sum_{0<t_k<t}(a_k\|u\|_{h,\phi} + b_k) \\ &:= \gamma + \beta\|u\|_{h,\phi}, \end{aligned} \tag{3.8}$$

where

$$\gamma = M\|\phi\|_{\mathcal{B}} + \frac{Mc}{(-\nu_0)} + M\sum_{0<t_k<t} b_k,$$

$$\beta = M\Big(\frac{a+b}{(-\nu_0)} + \sum_{0<t_k<t} a_k\Big)$$

are positive with $\beta < 1$. So, we have $\lim_{t\to+\infty}\frac{1}{h(t)}\|Qu(t)\| = 0$ as $\lim_{t\to+\infty} h(t) = +\infty$, i.e. $(Qu) \in PC_h(E)$. In addition, we know that $(Qu)(0) = \phi(0)$ from (3.3). Therefore, $Q : PC_{h,\phi}(E) \to PC_{h,\phi}(E)$ is well defined.

Now, we prove that $Q : PC_{h,\phi}(E) \to PC_{h,\phi}(E)$ is continuous. Let $\{u^{(n)}\} \subset PC_{h,\phi}(E)$ be a sequence such that $u^{(n)} \to u$ in $PC_{h,\phi}(E)$ as $n \to \infty$, then, $u^{(n)}(t) \to u(t)$ in E and $u_t^{(n)} \to u_t$ in $\mathcal{B}$ for every $t \in \mathbb{R}^+$ as $n \to \infty$.

For $t \in \mathbb{R}^+$, by the continuity of f and I_k, we get

$$f(t, u^{(n)}(t), u_t^{(n)}) \to f(t, u(t), u_t), \quad I_k(u^{(n)}) \to I_k(u), \quad n \to \infty.$$

Hence, by Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \|Qu^{(n)}(t) - Qu(t)\| \le& \int_0^t \|T(t-s)\| \cdot \|f(s, u^{(n)}(s), u_s^{(n)}) - f(s, u(s), u_s)\|ds \\ &+ \sum_{0<t_k<t} \|T(t-t_k)\| \cdot \|I_k(u^n(t_k)) - I_k(u(t_k))\| \\ \le& M\int_0^t e^{\nu_0 s}\|f(s, u^{(n)}(s), u_s^{(n)}) - f(s, u(s), u_s)\|ds \\ &+ M\sum_{0<t_k<t} \|I_k(u^n(t_k)) - I_k(u(t_k))\| \\ \le& \frac{M}{|\nu_0|}\|f(s, u^{(n)}(s), u_s^{(n)}) - f(s, u(s), u_s)\| \\ &+ M\sum_{0<t_k<t} \|I_k(u^n(t_k)) - I_k(u(t_k))\|. \end{aligned}$$

Thus, we get that

$$\|Qu^{(n)} - Qu\|_h = \sup_{t\in[0,+\infty)} \frac{1}{h(t)}\|Qu^{(n)}(t) - Qu(t)\| \to 0 \ (n \to \infty),$$

by (3.3), which indicates that $\|Qu^{(n)} - Qu\|_{h,\phi} \to 0$ $(n \to \infty)$, hence, $Q : PC_{h,\phi}(E) \to PC_{h,\phi}(E)$ is continuous. By Definition 2.4 and (3.1), we can deduced that the fixed point of Q on $PC_{h,\phi}(E)$ is equivalent to a mild solution of the problem (1.1).

Next, we show that the operator Q has a positive fixed point on $PC_{h,\phi}(E)$ by means of the monotone iterative technique. Let $u,\ v \in PC_{h,\phi} \cap P$ with $\theta \le u \le v$, we see that $\theta \le u(t) \le v(t)$ and $\theta \le u_t \le v_t$ for $t \in \mathbb{R}^+$. Then, from $(H0) - (H3)$ and the positivity of $T(t)(t \ge 0)$, for all $t \in \mathbb{R}^+$, we have

$$\theta \le Qu(t) \le Qv(t),$$

which implies that Q is a monotonically increasing operator in $PC_{h,\phi} \cap P$.

Let $v^0 = \theta \in PC_{h,\phi} \cap P$ and establish a sequence $\{v^{(n)}\}$ by

$$v^{(n)} = Qv^{(n-1)}, \ n = 1, 2, \cdots . \tag{3.9}$$

By the monotonicity of Q and (3.8), one can see $\{v^{(n)}\} \subset PC_{h,\phi} \cap P$ and

$$\theta = v^{(0)} \le v^{(1)} \le \cdots \le v^{(n)} \le \cdots . \tag{3.10}$$

$$\|v^{(n)}\|_{h,\phi} \le \gamma + \beta \|v^{(n-1)}\|_{h,\phi}. \tag{3.11}$$

Since $\|v^{(0)}\|_{h,\phi} = 0$, by (3.11), we have

$$\|v^{(n)}\|_{h,\phi} \le \gamma + \gamma\beta + \gamma\beta^2 + \cdots + \gamma\beta^{(n-1)} = \gamma \frac{1-\beta^n}{1-\beta} \le \frac{\gamma}{1-\beta}. \tag{3.12}$$

Therefore, the sequence $\{v^{(n)}\}$ is uniformly bounded.

Besides, we need to verify that $\{v^{(n)}\}$ is uniformly convergent.

Step 1. *$\{v^{(n)}\} \subset PC_{h,\phi} \cap P$ is equicontinuous in $\mathbb{R}^+$.*

For any $u \in \{v^{(n)}\}$ and $0 < t_1 < t_2$, by (3.3), we have

$$\begin{aligned}
\|(Qu)(t_2) - (Qu)(t_1)\| &= \Big\| T(t_2)\phi(0) + \int_0^{t_2} T(t_2 - s) f(s, u(s), u_s) ds \\
&\quad + \sum_{0<t_k<t_2} T(t_2 - t_k) I_k(u(t_k)) - T(t_1)\phi(0) \\
&\quad - \int_0^{t_1} T(t_1 - s) f(s, u(s), u_s) ds - \sum_{0<t_k<t_1} T(t_1 - t_k) I_k(u(t_k)) \Big\| \\
&\le \|T(t_2)\phi(0) - T(t_1)\phi(0)\| \\
&\quad + \Big\| \int_0^{t_1} \big(T(t_2 - s) - T(t_1 - s)\big) f(s, u(s), u_s) ds \Big\| \\
&\quad + \Big\| \int_{t_1}^{t_2} T(t_2 - s) f(s, u(s), u_s) ds \Big\| \\
&\quad + \Big\| \sum_{0<t_k<t_1} \big(T(t_2 - t_k) - T(t_1 - t_k) I_k(u(t_k))\big) \Big\| \\
&\quad + \Big\| \sum_{t_1<t_k<t_2} T(t_2 - t_k) I_k(u(t_k)) \Big\| \\
&:= J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

Then, we can get that

$$\|Qu(t_2) - Qu(t_1)\| \le J_1 + J_2 + J_3 + J_4 + J_5. \tag{3.13}$$

Now, we verify that $J_i \to 0 (i = 1, 2, 3, 4, 5)$ independently of $u \in \{v^{(n)}\}$ as $t_2 - t_1 \to 0$. Since the strong continuity of $T(t)(t \ge 0)$, obviously,

$$J_1 \le \|T(t_2) - T(t_1)\| \cdot \|\phi\|_{\mathcal{B}} \to 0, \text{ as } t_2 - t_1 \to 0.$$

For a sufficiently small constant $\varepsilon \to 0^+$, we have

$$
\begin{aligned}
J_2 \leq & \int_0^{t_1} \|T(t_2 - s) - T(t_1 - s)\| \cdot \|f(s, u(s), u_s)\| ds \\
\leq & \int_0^{t_1-\varepsilon} \|T(t_2 - s) - T(t_1 - s)\| \cdot \|f(s, u(s), u_s)\| ds \\
& + \int_{t_1-\varepsilon}^{t_1} \|T(t_2 - s) - T(t_1 - s)\| \cdot \|f(s, u(s), u_s)\| ds \\
\leq & \|T(t_2 - t_1 + \varepsilon) - T(\varepsilon)\| \int_0^{t_1-\varepsilon} \|T(t_1 - s - \varepsilon)\| \cdot \|f(s, u(s), u_s)\| ds \\
& + \int_{t_1-\varepsilon}^{t_1} \Big(\|T(t_2 - s)\| + \|T(t_1 - s)\| \Big) \cdot \|f(s, u(s), u_s)\| ds \\
\leq & \Big(a\|u\|_{h,\phi} + b\|\phi\| + c \Big) \Big(\|T(t_2 - t_1 + \varepsilon) - T(\varepsilon)\| \frac{M}{(-\nu_0)} + 2M\varepsilon \Big) \\
\to & 0, \text{ as } t_2 - t_1 \to 0.
\end{aligned}
$$

$$
\begin{aligned}
J_4 \quad \leq \quad & \sum_{0<t_k<t_1} \Big\| (T(t_2 - t_k) - T(t_1 - t_k)) I_k(u(t_k)) \Big\| \\
\leq \quad & \sum_{0<t_k<t_1} \Big\| T(t_2 - t_1 + \varepsilon) T(t_1 - t_k - \varepsilon) - T(t_1 - t_k - \varepsilon) T(\varepsilon) \Big\| \cdot \|I_k(u(t_k))\| \\
\leq \quad & \sum_{0<t_k<t_1} \|T(t_1 - t_k - \varepsilon)\| \cdot \|T(t_2 - t_1 + \varepsilon) - T(\varepsilon)\| \cdot \|I_k(u(t_k))\| \\
\to \quad & 0, \text{ as } t_2 - t_1 \to 0.
\end{aligned}
$$

Hence, from the conditions $(H1)$, we have

$$
\begin{aligned}
J_3 \quad \leq \quad & \int_{t_1}^{t_2} \|T(t_2 - s)\| \cdot \|f(s, u(s), u_s)\| ds \\
\leq \quad & M(a\|u\|_{h,\phi} + b\|\phi\| + c\|)(t_2 - t_1) \\
\to \quad & 0, \text{ as } t_2 - t_1 \to 0,
\end{aligned}
$$

$$
\begin{aligned}
J_5 \quad \leq \quad & \sum_{t_1<t_k<t_2} \|T(t_2 - t_k) I_k(u(t_k))\| \leq M \sum_{t_1<t_k<t_2} \Big(a_k \|u\|_{h,\phi} + b_k \Big) \\
\to \quad & 0, \text{ as } t_2 - t_1 \to 0,
\end{aligned}
$$

Accordingly, $\|Qu(t_2) - Qu(t_1)\| \to 0$ doesn't depend on $u \in \{v^{(n)}\}$ as $t_2 - t_1 \to 0$, which easily implies that $\{v^{(n)}\}$ is equicontinuous in $\mathbb{R}^+$.

Step 2. *$\{v^{(n)}(t)\}$ is precompact on E for $t \in \mathbb{R}^+$.*

Let $t > 0$ be given, for any $\epsilon \in (0, t)$, define an operator

$$\begin{aligned} Q_\epsilon v^{(n)}(t) &= T(t)\phi(0) + \int_0^{t-\epsilon} T(t-s)\Big(f(s, v^{(n)}(s), v_s^{(n)})\Big)ds + \sum_{0<t_k<t-\epsilon} T(t-t_k)I_k(v^{(n)}(t_k)) \\ &= T(t)\phi(0) + T(\epsilon)\int_0^{t-\epsilon} T(t-\epsilon-s)\Big(f(s, v^{(n)}(s), v_s^{(n)})\Big)ds + T(\epsilon)\sum_{0<t_k<t-\epsilon} T(t-\epsilon-t_k)I_k(v^{(n)}(t_k)). \end{aligned}$$

By the compactness of $T(t)(t \geq 0)$, the set $\Big\{Q_\epsilon v^{(n)}(t)\Big\}$ is precompact on E. Moreover, for any $t \in (0, +\infty)$, by $(H1)$, we have

$$\begin{aligned} & \|Qv^{(n)}(t) - Q_\epsilon v^{(n)}(t)\| \\ = \;& \Big\| \int_{t-\epsilon}^t T(t-s)\Big(f(s, v^{(n)}(s), v_s^{(n)})\Big)ds \Big\| + \sum_{t-\epsilon<t_k<t} \|T(t-t_k)I_k(v^{(n)}(t_k))\| \\ \leq \;& M\Big(a\|v^{(n)}\|_{h,\phi} + b\|\phi\| + c\Big)\cdot\epsilon + M\sum_{t-\epsilon<t_k<t}(a_k\|v^{(n)}\|_{h,\phi} + b_k) \\ \to \;& 0, \text{ as } \epsilon \to 0^+. \end{aligned}$$

It is obvious that $\Big\{Qv^{(n)}(t)\Big\}$ is precompact on E for $t \in \mathbb{R}^+$. Thus, $\{v^{(n)}(t)\}$ is precompact on E for every $t \in [0, +\infty)$.

Step 3. $\lim\limits_{t\to+\infty} \frac{1}{h(t)}\|Qu(t)\| = 0$, *uniformly for* $u \in \{v^{(n)}\}$.

Combined (3.8) with (3.12), for any $u \in \{v^{(n)}\}$, we can obtain that

$$\frac{1}{h(t)}\|Qu(t)\| \leq \frac{1}{h(t)}(\gamma + \beta\|u\|_{h,\phi}) \leq \frac{\gamma}{1-\beta}\cdot\frac{1}{h(t)}. \tag{3.14}$$

There is no doubt that $\lim\limits_{t\to+\infty} \frac{1}{h(t)}\|Qu(t)\| = 0$ uniformly for $u \in \{v^{(n)}\}$.

So, according to Lemma 2.1, one can conclude that $\{v^{(n)}\}$ is relatively compact in $PC_{h,\phi} \cap P$. Combing the normality of cone P with the monotonicity of $\{v^{(n)}\}$, it is clear that $\{v^{(n)}\}$ itself is uniformly convergent, i.e., there exist $u \in PC_{h,\phi} \cap P$, such that $u = \lim_{n\to\infty} v^{(n)}$. By (3.9), we have $u = Qu$. Therefore, $u \in PC_{h,\phi} \cap P$ is a fixed point of Q, which means that u defined by (3.1) is a positive mild solution of the problme (1.1).

Finally, we show that u is the minimal positive mild solution of the problme (1.1). Let $\hat{u}_\phi$ be another positive solution the problme (1.1), correspondingly, we have $\hat{u}(t) = Q\hat{u}(t)$ for any $t \in \mathbb{R}^+$. Clearly, $\hat{u}(t) \geq v^{(0)} = 0$. By the monotonicity of Q, we have

$$\hat{u}(t) = (Q\hat{u})(t) \geq (Qv^{(0)})(t) = v^{(1)}(t),$$

hence, $\hat{u} \geq v^{(1)}$. Analogously, we have $\hat{u} \geq v^{(n)}$, $n = 1, 2, \cdots$. Taking limit as $n \to \infty$, we get that $\hat{u} \geq u$, which means $u \in PC(\mathbb{R}^+, P)$ is the minimal positive mild solution of the problme (1.1). □

Next, we replace the compactness of $T(t)(t \geq 0)$, by using the noncompact measure conditions and establish a result of the existence of positive solutions.

Theorem 3.2. *Let E be an ordered Banach space,whose positive cone $P \subset E$ is normal, $A : \mathcal{D}(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive, equicontinuous and exponentially stable C_0-semigroup $T(t)(t \geq 0)$ on E, whose growth exponent $v_0 < 0$. If $f \in C(\mathbb{R}^+ \times E \times \mathcal{B}, E)$, $I_k \in C(E, E)(k = 1, 2, \cdots)$ and $\phi \in K_\mathcal{B}$, $\phi(0) \in K_\mathcal{B} \cap \mathcal{D}(A)$ and the following conditions are established:*

($H4$) *For any* $t \in \mathbb{R}^+$, $x_2 \geq x_1$, $\|x_i\| \leq R$ *and* $\phi_2 \geq \phi_1$, $\|\phi_i\|_{\mathcal{B}} \leq R$,

$$f(t, x_2, \phi_2) \geq f(t, x_1, \phi_1);$$

($H5$) *For any* v_1, $v_2 \in C(\mathbb{R}^+, E)$ *with* $v_2 \geq v_1$, $\|v_i\| \leq R$,

$$I_k(v_2) \geq I_k(v_1);$$

($H6$) *There exist constants* L_f, $L_k > 0$, *such that for any* $t \in \mathbb{R}^+$ *and the monotone increasing sequence* $\{u^{(n)}\} \subset \bar{B}(\theta, R)$,

$$\alpha\Big(\{f(t, u^{(n)}(t), u_t^{(n)})\}\Big) \leq L_f\Big(\alpha\Big(\{u^{(n)}(t)\}\Big) + \sup_{s\in[-r,0]} \alpha\Big(\{u_t^{(n)}(s)\}\Big)\Big),$$

$$\alpha\Big(\{I_k(u^{(n)}(t))\}\Big) \leq L_k\alpha\Big(\{u^{(n)}(t)\}\Big),$$

Then the problem (1.1) has a minimal positive mild solution $u \in PC([-r, +\infty), P)$.

Proof. For $R > 0$, and given $\phi \in K_{\mathcal{B}}$, $\|\phi\|_{\mathcal{B}} \leq R$, define

$$\overline{\Omega}_R = \Big\{u \in PC_{h,\phi}(E) \cap P : \|u(t)\| \leq R, \ t \in \mathbb{R}^+\},$$

and the operator Q on $\overline{\Omega}_R$ by (3.3). From hypothesises ($H4$), ($H5$) and the positivity of $T(t)(t \geq 0)$, it follows that $Q : \overline{\Omega}_R \to E$ is well defined. Hence, if u is a fixed point of Q on $\overline{\Omega}_R$, then u_ϕ is undoubtedly a mild solution of the problem (1.1). Let

$$R_f = \max_{\|u(t)\|_{PC}, \|u_t\|_{\mathcal{B}} \leq R} \|f(t, u(t), u_t)\|; \quad R_I = \max_{\|u\| \leq R} \|I_k(u)\|, \quad t \in \mathbb{R}^+,$$

Step 1. *There exists* $R_0 > 0$ *such that* $Q : \overline{\Omega}_{R_0} \to \overline{\Omega}_{R_0}$, *and for any* $u \in \overline{\Omega}_{R_0}$,

$$\lim_{t\to+\infty} \frac{1}{h(t)} \|Qu(t)\| = 0.$$

In fact, if this property is false, then for any $R > 0$, there is always $u \in \overline{\Omega}_R$ such that $\|Qu\|_h > R$. Then, we get $\sup\limits_{t\geq0} \frac{1}{h(t)} \|Qu(t)\| > R$. For $u \in \overline{\Omega}_R$, $t \in [0, +\infty)$, by (3.4), we can deduce that

$$\|Qu(t)\| \leq M\|\phi\|_{\mathcal{B}} + \frac{M}{-\nu_0} R_g + M \sum_{0<t_k<t} R_I. \tag{3.15}$$

Therefore, from $h(t) \geq 1$, it follows that

$$R < \sup_{t\geq0} \frac{1}{h(t)} \|Qu(t)\| \leq \|Qu(t)\| \leq M\|\phi\|_{\mathcal{B}} + \frac{M}{-\nu_0} R_g + M \sum_{0<t_k<t} R_I.$$

Dividing both sides by R and taking the lower limit as $R \to \infty$, we get the contradiction of $1 < 0$. Thus, combined with $\|Qu(0)\| = \|\phi\|_{\mathcal{B}} \leq R$, we conclude that there is a constant $R_0 > 0$, such that $Q(\overline{\Omega}_{R_0}) \subset \overline{\Omega}_{R_0}$. Further, we can see from (3.15) and $\lim\limits_{t\to+\infty} h(t) = +\infty$ that for any $u \in \overline{\Omega}_{R_0}$,

$$\lim_{t\to+\infty} \frac{1}{h(t)} \|Qu(t)\| = 0.$$

Step 2. $Q(\overline{\Omega}_{R_0})$ *is locally equicontinuous.*

For any $u \in \overline{\Omega}_{R_0}$, let $a \in (0, +\infty)$ and $0 < t_1 < t_2 \le a$, by (3.13), we just prove that $J_i \to 0 (i = 1, 2, 3, 4, 5)$ independently of $u \in \overline{\Omega}_{R_0}$ as $t_2 - t_1 \to 0$. Obviously,

$$J_1 = \|T(t_2)\phi(0) - T(t_1)\phi(0)\| \to 0, \text{ as } t_2 - t_1 \to 0.$$

Combining conditions ($H4$) with ($H5$), for a sufficiently small constant $\varepsilon \to 0^+$, one has

$$\begin{aligned} J_2 &\le \int_0^{t_1} \|T(t_2 - s) - T(t_1 - s)\| \cdot \|f(s, u(s), u_s)\| ds \\ &\le R_f \cdot (\|T(t_2 - t_1 + \varepsilon) - T(\varepsilon)\| \frac{M}{(-\nu_0)} + 2M\varepsilon\Big) \\ &\to 0, \text{ as } t_2 - t_1 \to 0, \\ J_4 &\le \sum_{0<t_k<t_1} \|T(t_2 - t_k) - T(t_1 - t_k))\| \cdot \|I_k(u(t_k))\| \\ &\le R_I \sum_{0<t_k<t_1} \|T(t_1 - t_k - \varepsilon)\| \cdot \|T(t_2 - t_1 + \varepsilon) - T(\varepsilon)\| \\ &\to 0, \text{ as } t_2 - t_1 \to 0. \end{aligned}$$

For J_3 and J_5, we have

$$\begin{aligned} J_3 &\le \int_{t_1}^{t_2} \|T(t_2 - s)\| \cdot \|f(s, u(s), u_s)\| ds \\ &\le MR_f \cdot |t_2 - t_1| \\ &\to 0, \text{ as } t_2 - t_1 \to 0, \\ J_5 &\le \sum_{t_1<t_k<t_2} \|T(t_2 - t_k)\| \cdot \|I_k(u(t_k))\| \\ &\le R_I \cdot \sum_{t_1<t_k<t_2} \|T(t_2 - t_k)\| \\ &\to 0, \text{ as } t_2 - t_1 \to 0. \end{aligned}$$

As a result, $\|Qu(t_2) - Qu(t_1)\| \to 0$ independently of $u \in \overline{\Omega}_{R_0}$ as $t_2 - t_1 \to 0$, which easily implies that $Q(\overline{\Omega}_{R_0})$ is equicontinuous in $[0, a]$. Hence, $Q : \overline{\Omega}_{R_0} \to \overline{\Omega}_{R_0}$ is locally equicontinuous.

Step 3. We show that the operator Q *has a positive fixed point on* $\overline{\Omega}_{R_0}$.

From ($H4$), ($H5$) and the proof of Theorem 3.1, we know that Q is a monotonically increasing operator in $\overline{\Omega}_{R_0}$.

Let $v^0 = \theta \in \overline{\Omega}_{R_0}$ and establish a sequence $\{v^{(n)}\}$ by (3.9), Then by the monotonicity of Q, we can see that $\{v^{(n)}\} \subset \overline{\Omega}_{R_0}$ and (3.10) is valid. Let $B = \{v^{(n)} | n \in \mathbb{N}\}$ and $B_0 = \{v^{(n-1)} | n \in \mathbb{N}\}$. Owing to the boundness of $B_0 \subset \overline{\Omega}_{R_0}$, by (3.3), it is easily to find that $B = QB_0$ is bounded and locally equicontinuous. In view of Lemma 2.7 and $B_0 = B \cup \{v^{(0)}\}$ that $\alpha(B_0(t)) = \alpha(B(t))$ is continuous in $[0, +\infty)$.

Denote $\alpha_{h,\phi}(\cdot)$ the Kuratowski measure of noncompactness of the bounded sets in $C_{h,\phi}(E)$. For $t \in [0, +\infty)$, we have

$$\sup_{s\in[-r,0]} \alpha(\{v_t^{(n)}(s)\}) = \sup_{s\in[-r,0]} \alpha(\{v^{(n)}(t + s)\}) \le \alpha(\{v^{(n)}(t)\}). \tag{3.16}$$

According to the step 1, 2 and by Lemma 2.6, 2.7, the condition $(H6)$ and (3.16), for $t \in [0, t_1)$, we get that

$$\alpha_{h,\phi}(B(t)) = \alpha_h(B(t)) = \alpha\Big(\frac{B(t)}{h(t)}\Big) = \alpha\Big(\frac{QB_0(t)}{h(t)}\Big)$$

$$= \alpha\Big(\Big\{\frac{1}{h(t)}T(t)\phi(0) + \frac{1}{h(t)}\int_0^t T(t-s)f(s, v^{n-1}(s), v_s^{n-1})ds\Big\}\Big)$$

$$\leq \alpha\Big(\Big\{\frac{1}{h(t)}\int_0^t T(t-s)f(s, v^{n-1}(s), v_s^{n-1}))ds\Big\}\Big)$$

$$\leq \frac{2}{h(t)}\int_0^t \|T(t-s)\| \cdot \alpha(\{(f(s, v^{n-1}(s), v_s^{n-1})\})ds$$

$$\leq 4ML_f\int_0^t e^{\nu_0(t-s)}\alpha\Big(\Big\{\frac{B(s)}{h(s)}\Big\}\Big)ds$$

$$\leq 4ML_f\int_0^t \alpha\Big(\Big\{\frac{B(s)}{h(s)}\Big\}\Big)ds$$

$$\leq 4ML_f\int_0^t \alpha_{h,\phi}(B(s))ds.$$

From the Gronwall lemma, for $t \in [0, t_1)$, we have $\alpha_{h,\phi}(B(t)) = 0$. Also $\alpha_{h,\phi}(B(t_1)) = 0$ which implies that $B(t_1)$ and $B_0(t_1)$ are precompact, then $\alpha_{h,\phi}(I_1(B(t_1))) = 0$. For $t \in [t_1, t_2)$, we obtain

$$\alpha_{h,\phi}(B(t)) = \alpha_h(B(t)) = \alpha\Big(\frac{B(t)}{h(t)}\Big) = \alpha\Big(\frac{QB_0(t)}{h(t)}\Big)$$

$$= \alpha\Big(\Big\{\frac{1}{h(t)}\int_0^t T(t-s)f(s, v^{n-1}(s), v_s^{n-1})ds\Big\}\Big)$$

$$\leq \frac{2}{h(t)}\int_0^t \|T(t-s)\| \cdot \alpha(\{(f(s, v^{n-1}(s), v_s^{n-1})\})ds$$

$$\leq 4ML_f\int_0^t \alpha\Big(\Big\{\frac{B(s)}{h(s)}\Big\}\Big)ds$$

$$\leq 4ML_f\int_0^t \alpha_{h,\phi}(B(s))ds.$$

And by Gronwall lemma, for $t \in [t_1, t_2)$, $\alpha_{h,\phi}(B(t)) = 0$, then $\alpha_{h,\phi}(B(t_2)) = 0$, which yields that $\alpha_{h,\phi}(I_2(B(t_2))) = 0$. Continuing such a process interval by interval, we can prove that $\alpha_{h,\phi}(B(t)) = 0$ in $\mathbb{R}^+$. Hence, $\alpha_{h,\phi}(B(t)) \equiv 0$ in $\mathbb{R}^+$, which shows that $\{v^{(n)}(t)\}$ is precompact on $\overline{\Omega}_{R_0}$ for any $t \in \mathbb{R}^+$. Combing the monotonicity and continuity of Q with the normality of the cone P, obviously, $\{v^{(n)}\}$ itself is convergent, i.e., there exists $u \in \overline{\Omega}_{R_0}$, such that

$$v^{(n)} \to u \in \overline{\Omega}_{R_0},\ n \to \infty.$$

Taking limit of both ends of (3.9), we can get $u = Qu$, which implies that $u \in \overline{\Omega}_{R_0}$ is a positive fixed point of Q. Therefore, u defined by (3.1) is a positive mild solution of IVP(1.1). From the proof of Theorem 3.1, it's follows that $u \in PC([-r, +\infty), P)$ is the minimal positive mild solution of the problem (1.1). □

4. Applications

Let $\Omega \subset \mathbb{R}^N(N \geq 1)$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$, $J = [-r, +\infty)$, $0 < t_1 < t_2 < \cdots$, $g : \overline{\Omega} \times J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\varphi_k : \mathbb{R} \to \mathbb{R}(k = 1, 2, \cdots)$ be continuous. Consider the impulsive evolution equation initial value problem

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) = g(x,t,u(x,t),u(x,t+s)),\ x \in \Omega,\ t > 0,\ t \neq t_k, s \in [-r, 0] \\ \Delta u|_{t=t_k} = \varphi_k(x, u(x,t_k)),\ x \in \Omega,\ k = 1, 2, \cdots, \\ u(\cdot,t)|_{\partial\Omega} = 0,\ t \geq 0, \\ u(x,t) = \psi(x,t),\ x \in \Omega,\ t \in [-r, 0]. \end{cases} \tag{4.1}$$

Theorem 4.1. *Let $\lambda_1 > 0$ is the first eigenvalue of Laplace operator $-\Delta$ with boundary condition $u|_{\partial\Omega} = 0$. $g \in C(\overline{\Omega} \times J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ with $g > \theta$, $\varphi_k : \mathbb{R} \to \mathbb{R}(k = 1, 2, \cdots, m)$ are continuous with $\varphi_k \geq 0$. If the following conditions are hold:*

(i) There exist constants $0 < a < \frac{\lambda_1}{M}$, $b, c > 0$, and constants $a_k > 0$, $b_k > 0$, such that

$$\|g(x,t,u(x,t),u(x,t+s))\| \leq a\|u\| + b\|u_t\|_{\mathcal{B}} + c,\ t \in \mathbb{R}^+,\ x \in E,\ u_t \in \mathcal{B}, s \in [-r, 0],$$

$$\|\varphi_k(h(t)v)\| \leq a_k\|v\| + b_k,\ t \in \mathbb{R}^+,\ v \in PC(\mathbb{R}^+, E),$$

(ii) There exist constant $M \geq 1$, such that

$$a + b + \lambda_1 \sum_{0<t_k<t} a_k \leq \frac{\lambda_1}{M}.$$

Then the problem (4.1) has a minimal positive mild solution u.

Proof. Let $E = L^2(\Omega)$, $P = \{u \in L^2(\Omega) : u(x) \geq 0,\ a.e.\ x \in \Omega\}$ is a normal cone in $L^2(\Omega)$, then P is a regular cone of E. Note $\mathcal{B} := C(\Omega \times [-r, 0], E)$ with the normal cone $P_{\mathcal{B}} = \{u \in \mathcal{B} : u(x,t) \in P, t \in [-r, 0],\ a.e.\ x \in \Omega\}$. And define the operator A in E as follows:

$$D(A) = \{u \in H^2(\Omega) \cap H_0^1(\Omega) : u|_{\partial\Omega} = 0\}, \quad Au = -\Delta u,$$

from [26] which implies that $-A$ generates a positive, exponentially stable and analytic C_0-semigroup $T(t)(t \geq 0)$ with growth index $\nu_0 = -\lambda_1$. According to analyticity of $T(t)$ and compactness of resolvent of A, we can obtain that $T(t)$ is also a compact semigroup in E, thus the problem (4.1) can be transformed into problem (1.1). Let $\phi(t) = \phi(\cdot, t)$, $u(t) = u(\cdot, t)$, $u_t(s) = u(\cdot, t+s)$ and define nonlinear mapping $f : J \times E \times \mathcal{B} \to E$ and impulsive functions $I_k : E \to E$:

$$f(t, u(t), u_t) = g(\cdot, t, u(\cdot, t), u(\cdot, t+s)), \quad I_k(u) = \varphi_k(u(\cdot)),$$

then $f : J \times E \times \mathcal{B} \to E$ is continuous in v and partial derivatives $f'_u(t,u,v)$, $f'_v(t,u,v)$is bounded, $I_k : E \to E$ are continuous and differentiable. It's obvious that f and I_k satisfy the conditions (H1), (H2) and (H3). From Theorem 3.1, we can obtain that the problem (1.1) has minimal positive mild solution $u \in PC(J, E)$. □

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