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Some sharp bounds for the generalized Tsallis relative operator entropy

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Abstract. In this paper, we determine some new bounds for the generalized Tsallis relative operator entropy by providing some close and sharp bounds. In particular, we identify some bounds for the Tsallis relative operator entropy. Our main results confirm some results obtained in [16, 23]. Moreover, we reach some inequalities for the generalized relative operator entropy in some sense.

1. Introduction and Preliminaries

A relative operator entropy of positive invertible operators A and B was introduced in the noncommutative information theory by Fujii and Kamei [13] by

$$S(A|B) = A^{\frac{1}{2}} \ln(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

Furuta [10] defined the generalized relative operator entropy (operator Shannon entropy) by

$$S_q(A|B) = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^q \ln(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$$

for $q \in [0,1]$ where A, B are positive invertible operators on a Hilbert space $\mathcal{H}$ and proved parametric extensions of the Shannon inequality and its reverse one in Hilbert space operators, see also [18]. The Tsallis relative operator entropy was introduced by Yanagi et al. [26] and defined by

$$T_\lambda(A|B) := \frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}} - A}{\lambda},$$

which is a generalization of the relative operator entropy $S(A|B)$ in the sense that

$$\lim_{\lambda \to 0} T_\lambda(A|B) = S(A|B).$$

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Moreover, the generalized Tsallis relative operator entropy was introduced and then several operator inequalities were derived by Yanaghi et. al [26]. For two positive invertible operators A, B and $p \in [0,1]$, the p-power operator mean $A\sharp_p B$ was defined by

$$A\sharp_p B := A^{\frac{1}{2}}\Big(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\Big)^{p}A^{\frac{1}{2}}.$$

The notation $A\natural_\mu B$ denotes the extended μ-power operator mean for $\mu \in \mathbb{R}$, see [10] for instance. The extended μ-power operator mean was defined by

$$A\natural_\mu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\mu}A^{\frac{1}{2}}.$$

The generalized Tsallis relative operator entropy was defined by

$$\tilde{T}_{\mu,k,\lambda}(A|B) := \frac{A\natural_{\mu+k\lambda}B - A\natural_{\mu+(k-1)\lambda}B}{\lambda},$$

where A, B are two positive invertible operators, $\lambda, \mu \in \mathbb{R}$, $\lambda \neq 0$, and $k \in \mathbb{Z}$. In particular, for $\lambda \in (0,1]$ we have

$$\tilde{T}_{0,1,\lambda}(A|B) = \frac{A\sharp_\lambda B - A\sharp_0 B}{\lambda} = \frac{A\sharp_\lambda B - A}{\lambda} = T_\lambda(A|B),$$

where $T_\lambda(A|B)$ is the Tsallis relative operator entropy, cf. [15, 27]. For simplicity we shall use the notation $\sharp_\lambda$ for $\lambda \in [0,1]$ and $\lambda \notin [0,1]$.

Some operator inequalities related to the Tsallis relative operator entropy were proved in [27] and then some existing operator inequalities were generalized. Some operator inequalities were presented in [28] for the relative operator entropy and generalized some results obtained in [27]. Moreover, some new lower and upper bounds for the Tsallis relative operator entropy and relative operator entropy were presented. The relation between the relative operator entropy $S(A|B)$ and the Tsallis relative operator entropy $T_\lambda(A|B)$ was verified in [14, 15, 25] as follows:

$$\begin{aligned} A - AB^{-1}A &\leq T_{-\lambda}(A|B) \\ &\leq S(A|B) \\ &\leq T_\lambda(A|B) \leq T_1(A|B) = B - A \end{aligned} \tag{1}$$

for all positive invertible operators A, B and $\lambda \in (0,1]$. Moradi et al. [16] gave tight bounds of the Tsallis relative operator entropy by using the Hermite-Hadamard's inequality. In fact they proved that if A and B are positive invertible operators such that $A \leq B$ and $\lambda \in (0,1]$, then

$$\begin{aligned} A^{\frac{1}{2}}\Big(\frac{A^{\frac{-1}{2}}BA^{\frac{-1}{2}}+I}{2}\Big)^{\lambda-1}&\Big(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}-I\Big)A^{\frac{1}{2}} \\ &\leq T_\lambda(A|B) \leq \frac{1}{2}(A\sharp_\lambda B - A\sharp_{\lambda-1}B + B - A), \end{aligned} \tag{2}$$

which is a considerable refinement of (1), where I is the identity operator. We gave sharp and refined bounds for the Tsallis relative operator entropy by using the improved Hermite-Hadamard inequality in [23]. The relation between the generalized relative operator entropy $S_q(A|B)$ and the generalized Tsallis relative operator entropy $\tilde{T}_{\mu,k,\lambda}(A,B)$ has not considered yet. We considered in [22] some operator inequalities for the generalized relative operator entropy according to the generalized Tsallis relative operator entropy. These operator inequalities generalized the existing operator inequalities in [27, 28]. Moreover, we distinguished the lower and upper bounds for the generalized Tsallis relative operator entropy in [24] and verified the information monotonicity for the generalized Tsallis relative operator entropy and its reverses.

We now describe our motivation for the treated problems. In information theory, it is important to estimate the bounds of average error. In its exponent, the capacity is the maximum of the mutual information and this is a special case of the relative entropy. So, information theorists seek for tight bound of divergences. Entropy also has a special form of relative entropy. So it is important to estimate the entropy. The other motivation comes from finding some properties of entropy, even if some properties refer to its various estimates. We are interested in when entropy is maximal and when it is minimal, so there is the problem of studying the lower bound and the upper bound for entropy.

The study of some new estimations of entropy is very important because we are looking for mathematical expressions that are easier to process, but which come quite close to the estimation of entropy. For example, monotony is one of important characteristics of entropy, so it must be analyzed. This suggests the study of some inequalities that can refine some already known inequalities in the characterization of entropy. This question arises: can these bounds be refined, that is, can we find better ones?

So, several arguments can be made for the study of some improvements in some inequalities used in the estimation of entropy.

Refinements of the lower or upper bounds of entropy can generate a mathematical expression that can give birth to a new type of entropy. In operator theory, it is a mathematical interest to show the ordering of several means.

The purpose of this paper is to find some lower and upper bounds for the generalized Tsallis relative operator entropy. These bounds have several advantages. Among others, they give some bounds for the generalized relative operator entropy. In particular, we confirm the bounds for the Tsallis relative operator entropy presented in [16]. At the final section we find some close and sharp bounds for the generalized Tsallis relative operator entropy. These will also confirm the sharp bounds for the Tsallis relative operator entropy obtained in [23].

For more information on the Tsallis relative entropy and relative operator entropy and their generalized versions the reader is referred to [1, 6, 11, 12, 19, 20] and the references therein.

2. Some new bounds

The first fundamental result for convex functions with a natural geometrical interpretation is the Hermite-Hadamard inequality. It plays a crucial role in the theory of convex functions and has many applications in elementary mathematics. Surveys on various generalizations and developments of the classical Hermite–Hadamard inequality can be found in [3]. The classical Hermite-Hadamard inequality states that

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$. Then,*

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}. \tag{3}$$

Note that the reversed inequalities hold in (3), when f is concave.

Let f be a real valued continuous function defined on the interval $\mathbb{I}$. The value $f(A)$ is defined via the functional calculus for a self–adjoint operator A with spectrum contained in $\mathbb{I}$ as usual. A fully noncommutative perspective of two variables (associated to f), by choosing an appropriate ordering, was introduced in [7] by

$$P_f(A, B) := A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2}$$

for every positive invertible operator A and every self-adjoint operator B on a Hilbert space $\mathcal{H}$, where the spectrum of the operator $A^{-1/2}BA^{-1/2}$ is contained in $\mathbb{I}$. Then, several significant matrix analogues of a classical result for operator convex functions and the necessary and sufficient conditions for joint convexity of a fully noncommutative perspective were proved where restricting to the positive commuting matrices ensures Effros' approach announced in [8].

Lemma 2.2. *[17, 21] Let r, s, and k be real valued and continuous functions on the closed interval $\mathbb{I}$. If $r(t) \le s(t) \le k(t)$ for $t \in \mathbb{I}$, then*

$$P_r(A, B) \le P_s(A, B) \le P_k(A, B),$$

for every positive invertible operator A and every self-adjoint operator B such that the spectrum of the operator $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ lies in $\mathbb{I}$.

We now provide our main results.

Theorem 2.3. *Let A and B be positive invertible operators such that $A \leq B$. If one of the following holds*

(i) $\lambda > 0$ *and* $\alpha(\alpha-1)(\alpha-2) > \beta(\beta-1)(\beta-2) \geq 0$,

(ii) $\lambda < 0$ *and* $\beta(\beta-1)(\beta-2) > \alpha(\alpha-1)(\alpha-2) \geq 0$,

then

$$\begin{aligned}
&\frac{1}{\lambda}\Big(\alpha A\sharp_{\alpha-1}\Big(\frac{A+B}{2}\Big) - \beta A\sharp_{\beta-1}\Big(\frac{A+B}{2}\Big)\Big)(A^{-1}B - I)\\
&\leq \tilde{T}_{\mu,k,\lambda}(A|B) \qquad (4)\\
&\leq \frac{1}{2\lambda}\Big(\alpha(A\sharp_{\alpha}B - A\sharp_{\alpha-1}B) - \beta(A\sharp_{\beta}B - A\sharp_{\beta-1}B) + (\alpha-\beta)(B-A)\Big),
\end{aligned}$$

where $\alpha = \mu + k\lambda$, $\beta = \mu + (k-1)\lambda$, $\mu \in \mathbb{R}$, *and* $k \in \mathbb{Z}$.

Proof. (i) Since $\lambda > 0$, we have $\alpha = \beta + \lambda > \beta$ and by assumption

$$0 \leq \frac{\beta(\beta-1)(\beta-2)}{\alpha(\alpha-1)(\alpha-2)} < 1.$$

Consider the function $f(t) = \frac{\alpha t^{\alpha-1} - \beta t^{\beta-1}}{\lambda}$, $t \geq 0$. Then, $f(t)$ is convex on the interval $[(\frac{\beta(\beta-1)(\beta-2)}{\alpha(\alpha-1)(\alpha-2)})^{\frac{1}{\alpha-\beta}}, +\infty)$. Applying Theorem 2.1 for the function f on $[1, x]$, $x \geq 1$, one can deduce

$$\int_1^x \frac{\alpha t^{\alpha-1} - \beta t^{\beta-1}}{\lambda} dt = \frac{x^\alpha - x^\beta}{\lambda}$$

for every $x \geq 1$. The left hand side of the inequality (3) indicates that

$$\frac{\alpha(\frac{x+1}{2})^{\alpha-1} - \beta(\frac{x+1}{2})^{\beta-1}}{\lambda}(x-1) \leq \frac{x^\alpha - x^\beta}{\lambda}. \tag{5}$$

On the other hand, the right hand side of the inequality (3) signifies that

$$\frac{x^\alpha - x^\beta}{\lambda} \leq \frac{\frac{\alpha x^{\alpha-1} - \beta x^{\beta-1}}{\lambda} + \frac{\alpha-\beta}{\lambda}}{2}(x-1), \tag{6}$$

where $x \geq 1$. Consider

$$\begin{aligned}
l(x) &:= \frac{\alpha(\frac{x+1}{2})^{\alpha-1} - \beta(\frac{x+1}{2})^{\beta-1}}{\lambda}(x-1),\\
m(x) &:= \frac{x^\alpha - x^\beta}{\lambda}, \qquad (7)\\
r(x) &:= \frac{1}{2\lambda}(\alpha(x^\alpha - x^{\alpha-1}) - \beta(x^\beta - x^{\beta-1}) + (\alpha-\beta)(x-1)).
\end{aligned}$$

According to Lemma 2.2 we get

$$P_l(A,B) \leq \tilde{T}_{\mu,k,\lambda}(A|B) \leq P_r(A,B). \tag{8}$$

On the other hand,

$$
\begin{aligned}
&P_l(A,B)\\
&= \frac{1}{\lambda}A^{\frac{1}{2}}\Big(\alpha\Big(\frac{A^{\frac{-1}{2}}BA^{\frac{-1}{2}}+I}{2}\Big)^{\alpha-1}-\beta\Big(\frac{A^{\frac{-1}{2}}BA^{\frac{-1}{2}}+I}{2}\Big)^{\beta-1}\Big)\big(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}-I\big)A^{\frac{1}{2}}\\
&= \frac{1}{\lambda}\Big(\alpha A\sharp_{\alpha-1}\Big(\frac{A+B}{2}\Big)-\beta A\sharp_{\beta-1}\Big(\frac{A+B}{2}\Big)\Big)(A^{-1}B-I)
\end{aligned}
$$

and

$$P_r(A,B) = \frac{1}{2\lambda}\Big(\alpha(A\sharp_\alpha B - A\sharp_{\alpha-1}B) - \beta(A\sharp_\beta B - A\sharp_{\beta-1}B) + (\alpha-\beta)(B-A)\Big).$$

So, the desired results follow from (8).

(ii) Since $\lambda < 0$, we have $\alpha = \beta + \lambda < \beta$ and

$$0 \leq \frac{\alpha(\alpha-1)(\alpha-2)}{\beta(\beta-1)(\beta-2)} < 1.$$

Hence, the function f is convex on $[(\frac{\alpha(\alpha-1)(\alpha-2)}{\beta(\beta-1)(\beta-2)})^{\frac{1}{\beta-\alpha}}, \infty)$. Applying a similar approach as in part (i) and using Theorem 2.1 for the function f on $[1,x]$, $x \geq 1$, one can get the desired results. □

Theorem 2.4. *Let A and B be positive invertible operators such that $A \geq B$. If one of the following holds*

(i) $\lambda > 0$ *and* $\beta(\beta-1)(\beta-2) < \alpha(\alpha-1)(\alpha-2) < 0$,

(ii) $\lambda < 0$ *and* $\alpha(\alpha-1)(\alpha-2) < \beta(\beta-1)(\beta-2) < 0$,

then the reversed inequalities hold in (4).

Proof. (i) Since $\lambda > 0$, we have $\alpha = \beta + \lambda > \beta$ and

$$\gamma_1 := \frac{\beta(\beta-1)(\beta-2)}{\alpha(\alpha-1)(\alpha-2)} > 1.$$

So, the function f defined in the proof of Theorem 2.3 is convex on $[0, \gamma_1^{\frac{1}{\alpha-\beta}}]$. The result follows by a similar approach as in the proof of Theorem 2.3 and using Theorem 2.1 for the function f on $[x,1]$, $x \leq 1$.

(ii) Since $\lambda < 0$, we have $\alpha = \beta + \lambda < \beta$ and

$$\gamma_2 := \frac{\alpha(\alpha-1)(\alpha-2)}{\beta(\beta-1)(\beta-2)} > 1.$$

In this situation, the function f defined in the proof of Theorem 2.3 is convex on the interval $[0, \gamma_2^{\frac{1}{\beta-\alpha}}]$. By a similar approach as in the proof of Theorem 2.3 and using Theorem 2.1 for the function f on $[x,1]$, $0 \leq x \leq 1$, one can obtain the reversed inequalities in (4). □

We now consider some special cases.

Corollary 2.5. *Let A and B be positive invertible operators such that $A \leq B$. If one of the following holds*

(i) $\lambda > 0$, $\alpha(\alpha-1)(\alpha-2) > 0$ *and* $\beta(\beta-1)(\beta-2) = 0$,

(ii) $\lambda < 0$, $\alpha(\alpha-1)(\alpha-2) < 0$ *and* $\beta(\beta-1)(\beta-2) = 0$,

(iii) $\lambda > 0$, $\beta(\beta-1)(\beta-2) < 0$ *and* $\alpha(\alpha-1)(\alpha-2) = 0$,

(iv) $\lambda < 0$, $\beta(\beta-1)(\beta-2) > 0$ *and* $\alpha(\alpha-1)(\alpha-2) = 0$,

then the inequalities (4) *hold.*

Proof. In each case consider the function f as in the proof of Theorem 2.3. Then, f is convex on the interval $[0, +\infty)$. Applying Theorem 2.1 for the function f on $[1, x]$, $x \geq 1$ and using a similar approach as in the proof of Theorem 2.3, one can deduce the result. □

Corollary 2.6. *Let A and B be positive invertible operators such that $A \geq B$. If one of the conditions (i)-(iv) in Corollary 2.5 holds, then the reversed inequalities hold in* (4)*.*

Proof. Like in Corollary 2.5, we know that the function f defined in the proof of Theorem 2.3 is convex on the interval $[0, +\infty)$. In each case using a similar approach as in the proof of Theorem 2.3 and applying Theorem 2.1 for the function f on $[x, 1]$, $0 \leq x \leq 1$, one gets the result. □

Corollary 2.7. *Let A and B be positive invertible operators such that $A \leq B$. If one of the following holds*

(i) $\lambda > 0$, $\alpha(\alpha-1)(\alpha-2) < 0$ *and* $\beta(\beta-1)(\beta-2) = 0$,

(ii) $\lambda < 0$, $\alpha(\alpha-1)(\alpha-2) > 0$ *and* $\beta(\beta-1)(\beta-2) = 0$,

(iii) $\lambda > 0$, $\beta(\beta-1)(\beta-2) > 0$ *and* $\alpha(\alpha-1)(\alpha-2) = 0$,

(iv) $\lambda < 0$, $\beta(\beta-1)(\beta-2) < 0$ *and* $\alpha(\alpha-1)(\alpha-2) = 0$,

then the reversed inequalities hold in (4)*.*

Proof. In each case consider the function f as in the proof of Theorem 2.3. Then, f is concave on the interval $[0, +\infty)$. Applying the reversed inequalities in Theorem 2.1 for the concave function f on $[1, x]$, $x \geq 1$ and using a similar approach as in the proof of Theorem 2.3, we get the result. □

Corollary 2.8. *Let A and B be positive invertible operators such that $A \geq B$. If one of the conditions (i)-(iv) in Corollary 2.7 holds, then the inequalities* (4) *hold.*

Proof. Like in Corollary 2.7, the function f defined in the proof of Theorem 2.3 is concave on the interval $[0, +\infty)$. In each case using a similar approach as in the proof of Theorem 2.3, applying the reversed inequalities in Theorem 2.1 for the function f on $[x, 1]$, $0 \leq x \leq 1$, and multiplying both sides of the inequalities by -1, one can get the result. □

Corollary 2.9. *[16, Theorem 1] Let A and B be positive invertible operators such that $A \leq B$ and $\lambda \in (0, 1]$. Then,*

$$\begin{aligned}\Big(A\sharp_{\lambda-1}\big(\frac{A+B}{2}\big)\Big)(A^{-1}B - I) &\leq T_\lambda(A|B)\\ &\leq \frac{1}{2}(A\sharp_\lambda B - A\sharp_{\lambda-1}B + B - A).\end{aligned} \tag{9}$$

Proof. Consider $\mu = 0$, $k = 1$, and $\lambda \in (0, 1]$. Then part (i) of Theorem 2.3 is fulfilled and we get the result. □

Corollary 2.10. *[16, Corollary 1] Let A and B be positive invertible operators such that $A \geq B$ and $\lambda \in (0, 1]$. Then the reversed inequalities hold in* (9)*.*

Proof. Consider $\mu = 0$, $k = 1$, and $\lambda \in (0, 1]$. Then part (i) of Corollary 2.6 is fulfilled and one can deduce the result. □

We can extend Corollaries 2.9 and 2.10 for $\lambda \notin (0, 1]$.

Corollary 2.11. *Let A and B be positive invertible operators.*

(i) If $A \leq B$ and $\lambda < 0$, then the inequalities (9) *hold.*

(ii) If $A \geq B$ and $\lambda < 0$, then the reversed inequalities hold in (9).

Proof. (i) Consider $\mu = 0, k = 1$, and $\lambda < 0$. Then part (ii) of Corollary 2.5 is fulfilled and one can derive the result.

(ii) Consider $\mu = 0$, $k = 1$, and $\lambda < 0$. Then part (ii) of Corollary 2.6 is fulfilled and one can get the result. □

Corollary 2.12. *Let A and B be positive invertible operators.*

(i) If $A \leq B$ and $\lambda > 2$, then the inequalities (9) *hold.*

(ii) If $A \geq B$ and $\lambda > 2$, then the reversed inequalities hold in (9).

Proof. (i) Consider $\mu = 0, k = 1$, and $\lambda > 2$. So, part (i) of Corollary 2.5 implies the result.

(ii) Consider $\mu = 0, k = 1$, and $\lambda > 2$. So, part (i) of Corollary 2.6 entails the result. □

Corollary 2.13. *Let A and B be positive invertible operators.*

(i) If $A \leq B$ and $1 < \lambda < 2$, then the reversed inequalities hold in (9).

(ii) If $A \geq B$ and $1 < \lambda < 2$, then the inequalities (9) *hold.*

Proof. (i) Consider $\mu = 0, k = 1$, and $1 < \lambda < 2$. The result follows by using part (i) of Corollary 2.7.

(ii) Consider $\mu = 0, k = 1$, and $1 < \lambda < 2$. The result follows by applying part (i) of Corollary 2.8. □

3. Some sharp bounds

In this section, we give some sharp bounds for the generalized Tsallis relative operator entropy. This implies close and narrow bounds for the Tsallis relative entropy. This enables us to reach a decision on the generalized relative operator entropy.

The Hermite-Hadamard inequality includes a basic property of convex functions, see, e.g., [2]. When f is convex an estimation better than (3) is as follows and one can find it in [3, 9, 23].

Theorem 3.1. *Assume that $f : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$. Then there exist real numbers p, q such that*

$$f\left(\frac{a+b}{2}\right) \leq p \leq \frac{1}{b-a}\int_a^b f(t)dt \leq q \leq \frac{f(a)+f(b)}{2}, \tag{10}$$

where

$$p = \frac{1}{2}\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right),$$
$$q = \frac{1}{2}\left(f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right).$$

Theorem 3.2. *Let A and B be positive invertible operators such that $A \leq B$. If one of the following holds*

(i) $\lambda > 0$ and $\alpha(\alpha-1)(\alpha-2) > \beta(\beta-1)(\beta-2) \geq 0$,

(ii) $\lambda < 0$ and $\beta(\beta-1)(\beta-2) > \alpha(\alpha-1)(\alpha-2) \geq 0$,

then

$$\begin{aligned}
&\frac{1}{\lambda}\Big(\alpha A\sharp_{\alpha-1}\Big(\frac{A+B}{2}\Big)-\beta A\sharp_{\beta-1}\Big(\frac{A+B}{2}\Big)\Big)(A^{-1}B-I)\\
&\leq \frac{1}{2\lambda}\Big(\alpha\Big(A\sharp_{\alpha-1}\Big(\frac{3A+B}{4}\Big)+A\sharp_{\alpha-1}\Big(\frac{A+3B}{4}\Big)\Big)\\
&\quad-\beta\Big(A\sharp_{\beta-1}\Big(\frac{3A+B}{4}\Big)+A\sharp_{\beta-1}\Big(\frac{A+3B}{4}\Big)\Big)\Big)(A^{-1}B-I)\\
&\leq \tilde{T}_{\mu,k,\lambda}(A|B) \qquad (11)\\
&\leq \frac{1}{2\lambda}\Big(\alpha A\sharp_{\alpha-1}\Big(\frac{A+B}{2}\Big)-\beta A\sharp_{\beta-1}\Big(\frac{A+B}{2}\Big)\Big)(A^{-1}B-I)\\
&\quad+\frac{1}{4\lambda}(\alpha(A\sharp_{\alpha}B-A\sharp_{\alpha-1}B)-\beta(A\sharp_{\beta}B-A\sharp_{\beta-1}B)+(\alpha-\beta)(B-A))\\
&\leq \frac{1}{2\lambda}\Big(\alpha(A\sharp_{\alpha}B-A\sharp_{\alpha-1}B)-\beta(A\sharp_{\beta}B-A\sharp_{\beta-1}B)+(\alpha-\beta)(B-A)\Big),
\end{aligned}$$

where $\alpha=\mu+k\lambda$, $\beta=\mu+(k-1)\lambda$, $\mu\in\mathbb{R}$, *and* $k\in\mathbb{Z}$.

Proof. (i) As in the proof of Theorem 2.3, the function $f(t)$ is convex on the interval $[(\frac{\beta(\beta-1)(\beta-2)}{\alpha(\alpha-1)(\alpha-2)})^{\frac{1}{\alpha-\beta}},+\infty)$, where

$$0\leq\frac{\beta(\beta-1)(\beta-2)}{\alpha(\alpha-1)(\alpha-2)}<1.$$

In view of Theorem 2.1 for the function f on $[1,x]$, $x>1$, one can find that

$$\frac{1}{x-1}\int_1^x f(t)dt=\frac{1}{x-1}\int_1^x\frac{\alpha t^{\alpha-1}-\beta t^{\beta-1}}{\lambda}dt=\frac{1}{x-1}\frac{x^\alpha-x^\beta}{\lambda}$$

for every $x>1$. The left hand side of the inequality (10) indicates that

$$\begin{aligned}
&f\Big(\frac{x+1}{2}\Big)=\frac{\alpha(\frac{x+1}{2})^{\alpha-1}-\beta(\frac{x+1}{2})^{\beta-1}}{\lambda},\\
&p=\frac{1}{2\lambda}\Big(\alpha(\frac{3+x}{4})^{\alpha-1}-\beta(\frac{3+x}{4})^{\beta-1}+\alpha(\frac{1+3x}{4})^{\alpha-1}-\beta(\frac{1+3x}{4})^{\beta-1}\Big).
\end{aligned}$$

On the other hand, the right hand side of the inequality (10) signifies that

$$\begin{aligned}
&q=\frac{1}{2\lambda}\Big(\alpha(\frac{1+x}{2})^{\alpha-1}-\beta(\frac{1+x}{2})^{\beta-1}+\frac{\alpha x^{\alpha-1}-\beta x^{\beta-1}}{2}+\frac{\alpha-\beta}{2}\Big),\\
&\frac{f(x)+f(1)}{2}=\frac{\frac{\alpha x^{\alpha-1}-\beta x^{\beta-1}}{\lambda}+\frac{\alpha-\beta}{\lambda}}{2},
\end{aligned}$$

where $x \geq 1$. Consider

$$\begin{aligned}
l_1(x) &:= \frac{\alpha(\frac{x+1}{2})^{\alpha-1} - \beta(\frac{x+1}{2})^{\beta-1}}{\lambda}(x-1),\\
l_2(x) &:= p(x-1),\\
m(x) &:= \frac{x^\alpha - x^\beta}{\lambda},\\
r_1(x) &:= q(x-1)\\
&= \frac{1}{2\lambda}\Big(\alpha(\frac{1+x}{2})^{\alpha-1} - \beta(\frac{1+x}{2})^{\beta-1}\Big)(x-1)\\
&\quad + \frac{1}{4\lambda}(\alpha(x^\alpha - x^{\alpha-1}) - \beta(x^\beta - x^{\beta-1}) + (\alpha-\beta)(x-1)),\\
r_2(x) &:= \frac{1}{2\lambda}(\alpha x^{\alpha-1} - \beta x^{\beta-1} + \alpha - \beta)(x-1).
\end{aligned} \tag{12}$$

By applying Lemma 2.2 we get

$$P_{l_1}(A,B) \leq P_{l_2}(A,B) \leq \tilde{T}_{\mu,k,\lambda}(A|B) \leq P_{r_1}(A,B) \leq P_{r_2}(A,B). \tag{13}$$

Note that we have

$$\begin{aligned}
P_{l_1}(A,B) &= \frac{1}{\lambda}A^{\frac{1}{2}}\Big(\alpha\Big(\frac{A^{\frac{-1}{2}}BA^{\frac{-1}{2}} + I}{2}\Big)^{\alpha-1}\\
&\quad - \beta\Big(\frac{A^{\frac{-1}{2}}BA^{\frac{-1}{2}} + I}{2}\Big)^{\beta-1}\Big)\Big(A^{\frac{-1}{2}}BA^{\frac{-1}{2}} - I\Big)A^{\frac{1}{2}},\\
P_{l_2}(A,B) &= \frac{1}{2\lambda}A^{\frac{1}{2}}\Big(\alpha(\frac{3 + A^{\frac{-1}{2}}BA^{\frac{-1}{2}}}{4})^{\alpha-1} - \beta(\frac{3 + A^{\frac{-1}{2}}BA^{\frac{-1}{2}}}{4})^{\beta-1}\\
&\quad + \alpha(\frac{I + 3A^{\frac{-1}{2}}BA^{\frac{-1}{2}}}{4})^{\alpha-1} - \beta(\frac{I + 3A^{\frac{-1}{2}}BA^{\frac{-1}{2}}}{4})^{\beta-1}\Big)\Big(A^{\frac{-1}{2}}BA^{\frac{-1}{2}} - I\Big)A^{\frac{1}{2}},\\
P_{r_1}(A,B) &= \frac{1}{2\lambda}A^{\frac{1}{2}}\Big(\alpha\Big(\frac{I + A^{\frac{-1}{2}}BA^{\frac{-1}{2}}}{2}\Big)^{\alpha-1}\\
&\quad - \beta\Big(\frac{I + A^{\frac{-1}{2}}BA^{\frac{-1}{2}}}{2}\Big)^{\beta-1}\Big)(A^{\frac{-1}{2}}BA^{\frac{-1}{2}} - I)A^{\frac{-1}{2}}\\
&\quad + \frac{1}{4\lambda}(\alpha(A\natural_\alpha B - A\natural_{\alpha-1}B) - \beta(A\natural_\beta B - A\natural_{\beta-1}B) + (\alpha-\beta)(B-A)),\\
P_{r_2}(A,B) &= \frac{1}{2\lambda}\Big(\alpha(A\natural_\alpha B - A\natural_{\alpha-1}B) - \beta(A\natural_\beta B - A\natural_{\beta-1}B) + (\alpha-\beta)(B-A)\Big).
\end{aligned}$$

So, the results follow from (13).

(ii) Since $\lambda < 0$, the function f is convex on $[(\frac{\alpha(\alpha-1)(\alpha-2)}{\beta(\beta-1)(\beta-2)})^{\frac{1}{\beta-\alpha}}, \infty)$, where

$$0 \leq \frac{\alpha(\alpha-1)(\alpha-2)}{\beta(\beta-1)(\beta-2)} < 1.$$

By a similar approach as in part (i) and applying Theorem 2.1 for the function f on $[1, x]$, $x \geq 1$, one can deduce the desired results. □

Theorem 3.3. *Let A and B be positive invertible operators such that $A \geq B$. If one of the following holds*

(i) $\lambda > 0$ *and* $\beta(\beta-1)(\beta-2) < \alpha(\alpha-1)(\alpha-2) < 0$,

(ii) $\lambda < 0$ *and* $\alpha(\alpha-1)(\alpha-2) < \beta(\beta-1)(\beta-2) < 0$,

then the reversed inequalities hold in (11).

Proof. For both parts (i) and (ii) the function f defined in the proof of Theorem 2.3 is convex on $[0, (\frac{\beta(\beta-1)(\beta-2)}{\alpha(\alpha-1)(\alpha-2)})^{\frac{1}{\alpha-\beta}}]$, where

$$\frac{\beta(\beta-1)(\beta-2)}{\alpha(\alpha-1)(\alpha-2)} > 1.$$

Applying a similar approach as in the proof of Theorem 3.2 and using Theorem 2.1 for the function f on $[x, 1]$, $x \leq 1$, one can get the reversed inequalities in (11). □

We consider some refinement of the special cases like Corollaries 2.5, 2.6, 2.7, and 2.8. The approach is completely similar to that of Theorem 3.2 and we provide two cases and their applications and leave the others for interested readers.

Corollary 3.4. *Let A and B be positive invertible operators such that $A \leq B$. If one of the following holds*

(i) $\lambda > 0$, $\alpha(\alpha-1)(\alpha-2) > 0$ *and* $\beta(\beta-1)(\beta-2) = 0$,

(ii) $\lambda < 0$, $\alpha(\alpha-1)(\alpha-2) < 0$ *and* $\beta(\beta-1)(\beta-2) = 0$,

(iii) $\lambda > 0$, $\beta(\beta-1)(\beta-2) < 0$ *and* $\alpha(\alpha-1)(\alpha-2) = 0$,

(iv) $\lambda < 0$, $\beta(\beta-1)(\beta-2) > 0$ *and* $\alpha(\alpha-1)(\alpha-2) = 0$,

then the inequalities (11) *hold.*

Proof. In each case consider the function f as in the proof of Theorem 2.3. Then, f is convex on the interval $[0, +\infty)$. Using a similar approach as in the proof of Theorem 3.2, one can deduce the result. □

Corollary 3.5. *Let A and B be positive invertible operators such that $A \geq B$. If one of the conditions (i)-(iv) in Corollary 3.4 holds, then the reversed inequalities hold in* (11).

Proof. As in Corollary 3.4, the function f defined in the proof of Theorem 2.3 is convex on the interval $[0, +\infty)$. In each case using a similar approach as in the proof of Theorem 3.2 for the function f on $[x, 1]$, $0 \leq x \leq 1$, one gets the result. □

Corollary 3.6. *[23, Theorem 2.4] Let A and B be positive invertible operators such that $A \leq B$ and $\lambda \in (0, 1]$. Then,*

$$\begin{aligned}
&A\sharp_{\lambda-1}\Big(\frac{A+B}{2}\Big)(A^{-1}B - I) \\
&\quad\leq \frac{1}{2}\Big(A\sharp_{\lambda-1}\Big(\frac{3A+B}{4}\Big) + A\sharp_{\lambda-1}\Big(\frac{A+3B}{4}\Big)\Big)(A^{-1}B - I) \\
&\quad\leq T_\lambda(A|B) \\
&\quad\leq \frac{1}{2}\Big(A\sharp_{\lambda-1}\Big(\frac{A+B}{2}\Big)\Big)(A^{-1}B - I) + \frac{1}{4}(A\sharp_\lambda B - A\sharp_{\lambda-1}B + B - A) \\
&\quad\leq \frac{1}{2}(A\sharp_\lambda B - A\sharp_{\lambda-1}B + B - A).
\end{aligned} \tag{14}$$

Proof. Consider $\mu = 0$, $k = 1$, and $\lambda \in (0, 1]$. Then part (i) of Corollary 3.4 entails the result. □

Corollary 3.7. *[23, Corollary 2.7] Let A and B be positive invertible operators such that $A \geq B$ and $\lambda \in (0, 1]$. Then the reversed inequalities hold in* (14).

Proof. Consider $\mu = 0$, $k = 1$, and $\lambda \in (0, 1]$. Then part (i) of Corollary 3.5 is fulfilled and the result follows. □

Note that we can extend Corollaries 3.6 and 3.7 for $\lambda \notin (0,1]$. See Corollaries 2.11, 2.12, and 2.13. As another result, we reach some inequalities for the generalized relative operator entropy.

Corollary 3.8. *Let A and B be positive invertible operators such that $A \leq B$. If one of the following holds*

(i) $\lambda > 0$ *and* $\alpha(\alpha-1)(\alpha-2) > \beta(\beta-1)(\beta-2) \geq 0$*,*

(ii) $\lambda < 0$ *and* $\beta(\beta-1)(\beta-2) > \alpha(\alpha-1)(\alpha-2) \geq 0$*,*

then

$$
\begin{aligned}
&S_\mu\left(A\middle|\frac{A+B}{2}\right)(A^{-1}B-I)\\
&\qquad\leq \frac{1}{2}\left(S_{2\mu-1}\left(A\middle|\frac{3A+B}{4}\right)+S_{2\mu-1}\left(A\middle|\frac{A+3B}{4}\right)\right)(A^{-1}B-I)\\
&\qquad\leq \frac{1}{\mu}S_\mu(A|B)\\
&\qquad\leq \frac{1}{2}\left(S_{2\mu-1}\left(A\middle|\frac{A+B}{2}\right)(A^{-1}B-I)\right.\\
&\qquad\quad\left.+\frac{1}{2}\left(S_\mu(A|B)+S_{\mu-1}(A|B)+B-A\right)\right)\\
&\qquad\leq \frac{1}{2}\left(S_\mu(A|B)+S_{\mu-1}(A|B)+B-A\right)
\end{aligned}
$$

for every $\mu \neq 0$.

Proof. Consider $\mu \neq 0$, $k \in \mathbb{Z}$ in the inequalities (11) and put $\lambda \to 0^+$ in part (i) and $\lambda \to 0^-$ in part (ii), respectively to reach the desired inequalities. □

4. Some Reversed Inequalities

In this section, we use the following reverse of the first Hermite-Hadamard inequality obtained in [4]

$$0 \leq \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}\left[f'_-(b)-f'_+(a)\right](b-a) \tag{15}$$

for a convex function f on $[a,b]$. The constant $\frac{1}{8}$ is best possible in (15).

Now if we consider the convex function

$$f(t) = \frac{\alpha t^{\alpha-1}-\beta t^{\beta-1}}{\lambda}$$

on $[1,x]$, $x>1$ with $\lambda = \alpha-\beta$ and observe that

$$f'(x) = \frac{\alpha(\alpha-1)x^{\alpha-2}-\beta(\beta-1)x^{\beta-2}}{\lambda}$$

and

$$f'(1) = \frac{\alpha(\alpha-1)-\beta(\beta-1)}{\lambda}$$

then by the use of (15) we get

$$0 \leq m(x) - l(x) \leq \frac{1}{8}w(x),$$

where $m(x)$, $l(x)$ are given by (7) while

$$w(x) := \frac{1}{\lambda}\left[\alpha(\alpha-1)\left(x^{\alpha-2}-1\right)-\beta(\beta-1)\left(x^{\beta-2}-1\right)\right]$$

for $x > 1$.

Applying Lemma 2.2 we get the operator inequalities

$$P_l(A,B) \leq \tilde{T}_{\mu,k,\lambda}(A|B) \leq P_l(A,B) + \frac{1}{8}P_w(A,B)$$

for every positive invertible operators A and B for which the upper bound is new in comparing with the upper bound of (8).

Further, we use the following reverse of the second Hermite-Hadamard inequality obtained in [5]

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{8}\left[f'_-(b)-f'_+(a)\right](b-a). \tag{16}$$

Here the constant $\frac{1}{8}$ is also best possible.

Now if we consider the convex function

$$f(t) = \frac{\alpha t^{\alpha-1}-\beta t^{\beta-1}}{\lambda}$$

on $[1,x]$, $x > 1$ with $\lambda = \alpha - \beta$ and make use of (16), then we get

$$0 \leq r(x) - m(x) \leq \frac{1}{8}w(x)$$

where $m(x)$, $l(x)$ are given by (7).

Applying Lemma 2.2 we get the operator inequalities

$$P_r(A,B) - \frac{1}{8}P_w(A,B) \leq \tilde{T}_{\mu,k,\lambda}(A|B) \leq P_r(A,B)$$

for every positive invertible operators A and B for which the lower bound is new in comparing with the lower bound of (8).

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