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A McShane type integral on complete metric measure space

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Abstract.
In this article, we introduce a McShane type integral on a complete metric space, endowed with a Radon measure μ with a family of cells that satisfies the Vitali covering theorem with respect to μ. The Saks-Henstock type lemma in terms of additive functions, some of the fundamental properties of such integrals are investigated. Finally, a relationship of Lebesgue and μ-McShane integral is established.

1. Introduction and preliminaries

The Lebesgue integral is regarded as the official or standard integral in the discipline of mathematics by a large portion of mathematicians. A significant amount of measure theory is needed to understand Lebesgue integral. The abstract concept of measure theory makes Lebesgue integral complicated. In the late 1960s, McShane defined a Riemann-type integral and prove that it is identical to the Lebesgue integral. Being a Riemann-type integral, it is more user-friendly to work than Lebesgue integral. Measures and σ-algebras are also excluded from his integral. McShane integral had undergone several extensions in [5]. Gordon [6] introduced and develop the properties of McShane integral for the case in which the function has values in a Banach space. The main result of that paper is that, in a Banach space every measurable, Pettis integrable function is generalized Riemann integrable. Kurzweil [9] introduced McShane integral of Banach valued function $f : I \to \mathfrak{X}$ on an m-dimensional interval I. A certain type of absolute continuity of the indefinite McShane integral with respect to Lebesgue measure was also derived by him. Fremlin discussed the relationship between the McShane and Talagrand integrals in [4]. He proved the integrability of a weak limit of a sequence of McShane integrable functions. Fremlin et al. [2] discussed if McShane integrable functions are Pettis integrable. It is known that every McShane integrable function is Henstock-Kurzweil integrable but converse may not true (see [5]). One can find detail of Henstock-Kurzweil integrals, and Henstock-Kurzweil type integrals in [8, 11, 12, 14, 15].

The construction of the μ-Henstock-Kurzweil integral of [1, 7] motivated us to constuct μ-McShane integrals for μ-cell functions on a complete metric space $\mathfrak{X}$ that are μ-Vitali cells also.

The paper is organized as follows: in Section 2 the basic concepts and terminology are intro- duced together with some definitions and results. In Section 3 we introduce μ-McShane integral of a cell function

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with respect to a Radon measure μ. Several simple properties of μ-McShane integrals are also discuss in this section. In Section 4, we establish that μ-McShane integral satisfies Saks-Henstock type Lemma. Finally, relationship between Lebesgue and μ-McShane integral are establish in Theorem 4.5 and Theorem 4.6 respectively.

2. Preliminaries

Let $\mathfrak{X} = (\mathfrak{X}, d)$ be a Cauchy metric space. Throughout the paper, the complete metric spaces or the Cauchy metric spaces will be termed as Cauchy spaces.

Let C be an arbitrary collection of subsets of $\mathfrak{X}$. The smallest σ-algebra $\sigma(C)$ containing C, called the σ-algebra generated by C, is the intersection of all σ-algebras in $\mathfrak{X}$ which contain C.

Let M be a σ-algebra of subsets of a set $\mathfrak{X}$. A positive function $\mu : M \to [0, +\infty]$ is called a measure if

1. $\mu(\emptyset) = 0$;
2. $\mu(\bigcup_{j=1}^{\infty} A_j) = \sum\limits_{j=1}^{\infty} \mu(A_j)$ for every sequences $\{A_j\}_j$ of pointwise disjoint sets from M.

Then $(\mathfrak{X}, M, \mu)$ is termed as a measure space. Given an outer measure μ on $\mathfrak{X}$, a subset E of $\mathfrak{X}$ is called μ-measurable in the sense of Carathéodory if $\mu(A) = \mu(A \cap E) + \mu(A \setminus E)$, for each $A \subset \mathfrak{X}$. Suppose U is the Borel σ-algebra of $\mathfrak{X}$. Recall that a measure μ is called a Radon measure if μ is a Borel measure with the followings:

1. $\mu(K) < \infty$ for every compact set $K \subset \mathfrak{X}$.
2. $\mu(V) = \sup\{\mu(K) :\ K \subset V,\ K \text{ is compact}\}$ for every open set $V \subset \mathfrak{X}$;
3. $\mu(A) = \inf\{\mu(V) :\ A \subset V,\ V \text{ is open}\}$ for every $A \subset \mathfrak{X}$.

Let λ be a signed measure defined on the σ-algebra of all μ-measurable subsets of $\mathfrak{X}$. Then λ is called an absolutely continuous with respect to μ if $\mu(E) = 0$ implies $|\lambda|(E) = 0$ for each μ-measurable subset E of $\mathfrak{X}$. It is denoted by $\lambda << \mu$. In the entire work we consider μ a non-atomic Radon measure, $\mathbb{D}$ is a family of non-empty closed subsets of $\mathfrak{X}$. For $E \subset \mathfrak{X}$, we denote the indicator function, diameter, interior and the boundary of E by χ_E, $diam(E)$, E^0 and ∂E, respectively. Throughout the article we denote $d(x, E)$, the distance from x to E. Recall that $\mathfrak{Q}_1,\ \mathfrak{Q}_2 \in \mathbb{D}$ are said to be non overlapping if interiors of $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are disjoint. A finite collection $\left\{\mathfrak{Q}_1,\ \mathfrak{Q}_2, ..., \mathfrak{Q}_m\right\}$ of pairwise non-overlapping elements of $\mathbb{D}$ is a division of $\mathfrak{Q}$ if $\bigcup\limits_{i=1}^{m} \mathfrak{Q}_i = \mathfrak{Q}$. Let $\mathcal{G}$ be a sub family of $\mathbb{D}$, then $\mathcal{G}$ is called a fine cover of $E \subset \mathfrak{X}$ if

$$\inf\left\{\text{diam}(\mathfrak{Q}) :\ \mathfrak{Q} \in \mathcal{G},\ x \in \mathfrak{Q}\right\} = 0$$

for each $x \in E$.

Definition 2.1. *[7, Definition 2.14] We say $\mathbb{D}$ is a μ-Vitali family if for each subset E of $\mathfrak{X}$ and for each subfamily $\mathcal{G}$ of $\mathbb{D}$ that is a fine cover of E, there exists a countable system $\left\{\mathfrak{Q}_1, \mathfrak{Q}_2, .., \mathfrak{Q}_j, ..\right\}$ of pairwise non-overlapping elements of $\mathcal{G}$ such that $\mu(E \setminus \cup \mathfrak{Q}_j) = 0$.*

Definition 2.2. *[8] Let $\mathbb{F}$ be a μ-Vitali family. We say $\mathbb{F}$ is a family of μ-cells if it satisfies the following conditions:*

(a) Given $\mathfrak{Q} \in \mathbb{F}$ and a constant $\delta > 0$, there exist a division $\left\{\mathfrak{Q}_1, \mathfrak{Q}_2, ..., \mathfrak{Q}_m\right\}$ of $\mathfrak{Q}$, such that $diam(\mathfrak{Q}_i) < \delta$, for $i = 1, 2, .., m$;

(b) Given $A,\ \mathfrak{Q} \in \mathbb{F}$ and $A \subset \mathfrak{Q}$, there exists a division $\left\{\mathfrak{Q}_1, \mathfrak{Q}_2, ..., \mathfrak{Q}_m\right\}$ of $\mathfrak{Q}$, such that $A = \mathfrak{Q}_1$;

(c) $\mu(\partial \mathfrak{Q}) = 0$ for each $\mathfrak{Q} \in \mathbb{F}$.

Recalling the Vitali Carathéodory Theorem as follows.

Theorem 2.3. *[13, Theorem 2.25] Let f be a real function defined on a cell $\mathfrak{Q}$. If f is Lebesgue integrable on $\mathfrak{Q}$ with respect to μ and $\epsilon > 0$, then there exists functions g_1 and g_2 on $\mathfrak{Q}$ such that $g_1 \leq f \leq g_2$, g_1 is upper semi-continuous and bounde above, g_2 is lower semi-continuous and bounded below and $(L)\int_{\mathfrak{Q}}(g_2 - g_1)d\mu < \epsilon$.*

3. μ-McShane integral in regards to a Radon measure

In this section, we shall define McShane integral in regards to a Radon measure. We recall that, a gauge on a set $\mathfrak{Q}$ is any positive real function ν defined on $\mathfrak{Q}$.

Definition 3.1. *Let $\mathfrak{Q} \in \mathbb{F}$, let $E \subset \mathfrak{Q}$ and ν be a gauge on $\mathfrak{Q}$. A collection $\mathfrak{P} = \left\{(x_i, \mathfrak{Q}_i)\right\}_{i=1}^{m}$ of finite ordered pairs of points and cells is said to be*

1. *a free partition of $\mathfrak{Q}$ if $\left\{\mathfrak{Q}_1, \mathfrak{Q}_2, ..., \mathfrak{Q}_m\right\}$ is a division of $\mathfrak{Q}$ and $x_i \in \mathfrak{Q}$ for $i = 1, 2, ..., m$;*
2. *a free partial partition on $\mathfrak{Q}$ if $\left\{\mathfrak{Q}_1, \mathfrak{Q}_2, ..., \mathfrak{Q}_m\right\}$ is a subsystem of a division of $\mathfrak{Q}$ and $x_i \in \mathfrak{Q}$ for $i = 1, 2, ..., m$;*
3. *ν-fine if $diam(\mathfrak{Q}_i) < \nu(x_i)$ for $i = 1, 2, ..., m$;*
4. *E-tagged if the points $x_1, x_2, x_3, ..., x_m$ belongs to E.*

The following Cousin's type lemma addresses the existence of ν-fine free partitions of a given cell $\mathfrak{Q}$.

Lemma 3.2. *If ν is a gauge on $\mathfrak{Q}$, then there exists a ν-fine free partition of $\mathfrak{Q}$.*

Proof. It is easy to see that if there exists a partition in such a way that the tag belongs to the respective cell in the tagged pair, then it is also true for the free partition. So, the proof is directly follows from [1, Lemma 3.1].
□

Let $f : \mathfrak{Q} \to \mathbb{R}$ be a given function. If $\mathfrak{P} = \left\{(x_i, \mathfrak{Q}_i)\right\}_{i=1}^{m}$ is any partition of $\mathfrak{Q} \in \mathbb{F}$, we define the Riemann sum as $\mathfrak{S}(f, \mathfrak{P}) = \sum_{i=1}^{m} f(x_i)\mu(\mathfrak{Q}_i)$. We are ready to define μ-McShane integral on $\mathfrak{Q}$ as follows:

Definition 3.3. *A function $f : \mathfrak{Q} \to \mathbb{R}$ is called μ-McShane integrable on a cell $\mathfrak{Q}$ with respect to μ if there exists a real number l such that for each $\epsilon > 0$ there is a gauge ν on $\mathfrak{Q}$ so that $|\mathfrak{S}(f, \mathfrak{P}) - l| < \epsilon$ whenever $\mathfrak{P}$ is a free tagged partition of $\mathfrak{Q}$ that is ν-fine.*

We write $l = \int_{\mathfrak{Q}} f d\mu$. We denote the collection of all μ-McShane integrable functions on $\mathfrak{Q}$ by μ-$M(\mathfrak{Q})$. It is not hard to see the number l is unique. It is clear that every μ-McShane integrable function is μ-Henstock-Kurzweil integrable on a cell $\mathfrak{Q}$ and that the integrals are equal. Few simple properties of μ-McShane integrals are as follows.

Theorem 3.4. *Let $f, g \in \mu$-$M(\mathfrak{Q})$, then $f + g \in \mu$-$M(\mathfrak{Q})$, and $\int_{\mathfrak{Q}}(f + g)d\mu = \int_{\mathfrak{Q}} f d\mu + \int_{\mathfrak{Q}} g d\mu$.*

Theorem 3.5. *If $f \in \mu$-$M(\mathfrak{Q})$ and $k \in \mathbb{R}$, then $kf \in \mu$-$M(\mathfrak{Q})$ and $\int_{\mathfrak{Q}} kf d\mu = k\int_{\mathfrak{Q}} f d\mu$.*

Theorem 3.6. *If $f \in \mu$-$M(\mathfrak{Q})$ and $f(x) \geq 0$ for each $x \in \mathfrak{Q}$, then $\int_{\mathfrak{Q}} f d\mu \geq 0$.*

Corollary 3.7. *Let $f, g \in \mu$-$M(\mathfrak{Q})$. If $f \geq g$ for each $x \in \mathfrak{Q}$, then $\int_{\mathfrak{Q}} f d\mu \geq \int_{\mathfrak{Q}} g d\mu$.*

Theorem 3.8. (The Cauchy Criterion) A function $f : \mathfrak{Q} \to \mathbb{R}$ is μ- McShane integrable on $\mathfrak{Q}$ if and only if for each $\epsilon > 0$ there exists a gauge ν on $\mathfrak{Q}$ such that $|\mathfrak{S}(f, \mathfrak{P}_1) - \mathfrak{S}(f, \mathfrak{P}_2| < \epsilon$ for each pair ν-fine free partitions $\mathfrak{P}_1$ and $\mathfrak{P}_2$ of $\mathfrak{Q}$.

Proof. Let $f : \mathfrak{Q} \to \mathbb{R}$ be a μ-McShane integrable on $\mathfrak{Q}$. Then for a given $\epsilon > 0$, there exists a gauge ν on $\mathfrak{Q}$ such that $|\mathfrak{S}(f, \mathfrak{P}) - \int_{\mathfrak{Q}} f d\mu| < \frac{\epsilon}{2}$ for each ν-fine free partitions $\mathfrak{P}$ of $\mathfrak{Q}$. If $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are two ν-fine partitions of $\mathfrak{Q}$,

$$|\mathfrak{S}(f, \mathfrak{P}_1) - \mathfrak{S}(f, \mathfrak{P}_2)| \le |\mathfrak{S}(f, \mathfrak{P}_1) - \int_{\mathfrak{Q}} f d\mu| + |\mathfrak{S}(f, \mathfrak{P}_2) - \int_{\mathfrak{Q}} f d\mu| < \epsilon.$$

Conversly, for each $n \in \mathbb{N}$ let ν_n be a gauge on $\mathfrak{Q}$ such that

$$|\mathfrak{S}(f, A_n) - \mathfrak{S}(f, B_n)| < \frac{1}{n}$$

whenever for each pair ν_n-fine free partitions A_n and B_n of $\mathfrak{Q}$.

Let $\triangle_n(x) = min\{\nu_1(x), ..., \nu_n(x)\}$ be a gauge on $\mathfrak{Q}$. By Lemma 3.2 there exists a $\triangle_n$-fine free partition $\mathfrak{P}_n$ of $\mathfrak{Q}$, for each $n \in \mathbb{N}$.

Let $\epsilon > 0$ be given and choose a positive natural N such that $\frac{1}{N} < \frac{\epsilon}{2}$. If m and n are positive natural ($n < m$) such that $n \ge N$, then $\mathfrak{P}_n$ and $\mathfrak{P}_m$ are $\triangle_n$-fine partition of $\mathfrak{Q}$;

Hence $|\mathfrak{S}(f, \mathfrak{P}_n) - \mathfrak{S}(f, \mathfrak{P}_m)| < \frac{1}{n} < \frac{\epsilon}{2}$.

Consequently, $\{\mathfrak{S}(f, \mathfrak{P}_n)\}_{n=1}^{\infty}$ is a Cauchy sequence of a real number and hence converges. If $l = \lim_{n\to\infty} \mathfrak{S}(f, \mathfrak{P}_n)$, then $|\mathfrak{S}(f, \mathfrak{P}_n) - l| < \frac{\epsilon}{2}$, for each $n \ge N$. Let $\mathfrak{P}$ be a $\triangle_N$- fine free partition of $\mathfrak{Q}$, then $|\mathfrak{S}(f, \mathfrak{P}) - l| \le |\mathfrak{S}(f, \mathfrak{P}) - \mathfrak{S}(f, \mathfrak{P}_N)| + |\mathfrak{S}(f, \mathfrak{P}_N) - l| < \epsilon$. Thus, f is μ-McShane integrable on $\mathfrak{Q}$ and $l = \int_{\mathfrak{Q}} f d\mu$. □

In the following theorem, we shall prove that μ-McShane integrability of f on a set $\mathfrak{Q}$ implies its μ-McShane integrability on each subcells of $\mathfrak{Q}$.

Theorem 3.9. *If $f \in \mu$-$M(\mathfrak{Q})$, and if A is a subcell of $\mathfrak{Q}$, then $f \in \mu$-$M(\mathfrak{Q})$ and $\int_A f d\mu = \int_{\mathfrak{Q}} f \chi_A d\mu$.*

Proof. Let $\epsilon > 0$ be given. By Theorem 3.8, there exists a gauge ν on $\mathfrak{Q}$ such that

$$\left|\mathfrak{S}(f, \mathfrak{P}_1) - \mathfrak{S}(f, \mathfrak{P}_2\right| < \epsilon$$

for each pair ν-fine free partitions $\mathfrak{P}_1$ and $\mathfrak{P}_2$ of $\mathfrak{Q}$.

Given that there exists a division $\mathfrak{P} = \{\mathfrak{Q}_1, ..., \mathfrak{Q}_m\}$ of $\mathfrak{Q}$ and $A \subset \mathfrak{Q}$, such that $A = \mathfrak{Q}_1$. For each $k \in \{2, ..., m\}$ we fix ν-fine free partitions $\mathfrak{P}_k$ of $\mathfrak{Q}_k$. If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are ν-fine free partitions of A, then $\mathfrak{R}_1 \cup \bigcup_{k=2}^{m} \mathfrak{P}_k$ and $\mathfrak{R}_2 \cup \bigcup_{k=2}^{m} \mathfrak{P}_k$ are ν-fine free partitions of $\mathfrak{Q}$. Thus

$$\begin{aligned}\left|\mathfrak{S}(f, \mathfrak{R}_1) - \mathfrak{S}(f, \mathfrak{R}_2)\right| &= \left|\mathfrak{S}(f, \mathfrak{R}_1) + \sum_{k=2}^{m}(\mathfrak{S}(f, \mathfrak{P}_k) - \mathfrak{S}(f, \mathfrak{R}_2) - \sum_{k=2}^{m} \mathfrak{S}(f, \mathfrak{P}_k)\right| \\ &= \left|\mathfrak{S}\Big(f, \mathfrak{R}_1 \cup \bigcup_{k=2}^{m} \mathfrak{P}_k\Big) - \mathfrak{S}\Big(f, \mathfrak{R}_2 \cup \bigcup_{k=2}^{m} \mathfrak{P}_k\Big)\right| < \epsilon.\end{aligned}$$

Therefore by Theorem 3.8, it follows that $f \in \mu$-$M(A)$. □

Next, we shall prove that if f is μ-McShane integrable, then $|f|$ is also μ-McShane integrable on $\mathfrak{Q}$. This property is not valid for the μ-Henstock-Kurzweil integrals (see [1, 7]). The following lemma is needed in the proof.

Lemma 3.10. *Let $f : \mathfrak{Q} \to \mathbb{R}$ be μ-McShane integrable on $\mathfrak{Q}$. Given $\epsilon > 0$, let ν-be a positive function on $\mathfrak{Q}$. If $\Big\{(x_i, \mathcal{I}_i) : 1 \le i \le N\Big\}$ and $\Big\{(y_j, \mathcal{K}_j) : 1 \le j \le M\Big\}$ are free tagged partitions of $\mathfrak{Q}$ that are ν-fine, then*

$$\sum_{i=1}^{N} \sum_{j=1}^{M} |f(x_i) - f(y_j)| \mu(\mathcal{I}_i \cap \mathcal{K}_j) < \epsilon.$$

Proof. It is easy to see the nondegenerate subcells of the collection $\left\{\mathcal{I}_i \cap \mathcal{K}_j : 1 \leq i \leq N;\ 1 \leq j \leq M\right\}$ form a partitions of $\mathfrak{Q}$. Let us use these subcells of $\mathfrak{Q}$ to form two free tagged partitions $\mathfrak{P}_1$ and $\mathfrak{P}_2$ of $\mathfrak{Q}$ as below: if $f(x_i) \geq f(y_j)$, then $(x_i, \mathcal{I}_i \cap \mathcal{K}_j)$ in $\mathfrak{P}_1$ and $(y_j, \mathcal{I}_i \cap \mathcal{K}_j)$ in $\mathfrak{P}_2$; if $f(x_i) < f(y_j)$, then $(y_j, \mathcal{I}_i \cap \mathcal{K}_j)$ in $\mathfrak{P}_1$ and $(x_i, \mathcal{I}_i \cap \mathcal{K}_j)$ in $\mathfrak{P}_2$.

Next, $\mathfrak{S}(f, \mathfrak{P}_1) - \mathfrak{S}(f, \mathfrak{P}_2) = \sum_{i=1}^{N} \sum_{j=1}^{M} |f(x_i) - f(y_j)| \mu(\mathcal{I}_i \cap \mathcal{K}_j)$. Since $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are free tagged partitions of $\mathfrak{Q}$ that are ν-fine,

$$\begin{aligned}
\mathfrak{S}(f, \mathfrak{P}_1) - \mathfrak{S}(f, \mathfrak{P}_2) &\leq \left|\mathfrak{S}(f, \mathfrak{P}_1) - l\right| + \left|l - \mathfrak{S}(f, \mathfrak{P}_2)\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \text{ (by } \mu\text{-McShane integrability of } f) \\
&= \epsilon.
\end{aligned}$$

□

Theorem 3.11. *Let $f : \mathfrak{Q} \to \mathbb{R}$ is μ-McShane integrable on $\mathfrak{Q}$, then $|f|$ is also μ-McShane integrable on $\mathfrak{Q}$.*

Proof. Let $\epsilon > 0$ and choose a gauge ν on $\mathfrak{Q}$ such that $|\mathfrak{S}(f, \mathfrak{P}) - l| < \epsilon$ whenever $\mathfrak{P}$ is a free tagged partition of $\mathfrak{Q}$. Let $\mathfrak{P}_1 = \left\{(x_i, \mathcal{I}_i) : 1 \leq i \leq N\right\}$ and $\mathfrak{P}_2 = \left\{(y_j, \mathcal{K}_j) : 1 \leq j \leq M\right\}$ be ν-fine free tagged partitions of $\mathfrak{Q}$. Let the nondegenerate subcells of collection $\left\{\mathcal{I}_i \cap \mathcal{K}_j : 1 \leq i \leq N;\ 1 \leq j \leq M\right\}$ to form two tagged partitions $\mathfrak{R}_1 = \left\{(x_i, \mathcal{I}_i \cap \mathcal{K}_j)\right\}$ and $\mathfrak{R}_2 = \left\{(y_j, \mathcal{I}_i \cap \mathcal{K}_j)\right\}$. Clearly $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are free tagged partitions of $\mathfrak{Q}$ that are ν-fine and,

$$\begin{aligned}
\mathfrak{S}(|f|, \mathfrak{R}_1) &= \sum_{i=1}^{N} \sum_{j=1}^{M} |f(x_i)| \mu(\mathcal{I}_i \cap \mathcal{K}_j) \\
&= \sum_{i=1}^{N} |f(x_i)| \mu(\mathcal{I}_i) \\
&= \mathfrak{S}(|f|, \mathfrak{P}_1)
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{S}(|f|, \mathfrak{R}_2) &= \sum_{i=1}^{N} \sum_{j=1}^{M} |f(y_j)| \mu(\mathcal{I}_i \cap \mathcal{K}_j) \\
&= \sum_{j=1}^{M} |f(y_j)| \mu(\mathcal{K}_j) \\
&= \mathfrak{S}(|f|, \mathfrak{P}_2).
\end{aligned}$$

By Lemma 3.10, we obtain,

$$\begin{aligned}
\left|\mathfrak{S}(|f|, \mathfrak{P}_1) - \mathfrak{S}(|f|, \mathfrak{P}_2)\right| &= \left|\mathfrak{S}(|f|, \mathfrak{R}_1) - \mathfrak{S}(|f|, \mathfrak{R}_2)\right| \\
&\leq \sum_{i=1}^{N} \sum_{j=1}^{M} \left||f(x_i)| - |f(y_j)|\right| \mu(\mathcal{I}_i \cap \mathcal{K}_j) \\
&\leq \sum_{i=1}^{N} \sum_{j=1}^{M} |f(x_i) - f(x_j)| \mu(\mathcal{I}_i \cap \mathcal{K}_j) \\
&< \epsilon.
\end{aligned}$$

So, $|f|$ is μ-McShane integrable on $\mathfrak{Q}$. $\square$

4. Characterization of the indefinite μ-McShane integral

In this Section, we prove Saks-Henstock type Lemma of μ-McShane integral. We start with the following definition.

Definition 4.1. *[7, Definition 2.3.1] Let $\pi : \mathbb{F} \to \mathbb{R}$ be a function. We say that π is an additive function of cell, if for each $\mathfrak{Q} \in \mathbb{F}$ and for each division $\left\{\mathfrak{Q}_1, \mathfrak{Q}_2, ..., \mathfrak{Q}_n\right\}$ of $\mathfrak{Q}$, $\pi(\mathfrak{Q}) = \sum_{i=1}^{n} \pi(\mathfrak{Q}_i)$.*

Proposition 4.2. *Let $f : \mathfrak{Q} \to \mathbb{R}$ be a μ-McShane integrable function on $\mathfrak{Q}$. If $\left\{\mathfrak{Q}_1, \mathfrak{Q}_2, ..., \mathfrak{Q}_n\right\}$ is a division of $\mathfrak{Q}$ then $f \in \mu$-$M(\mathfrak{Q}_1) \cap M(\mathfrak{Q}_2) \cap ... \cap M(\mathfrak{Q}_n)$ and $\int_{\mathfrak{Q}} f d\mu = \sum_{i=1}^{n} \int_{\mathfrak{Q}_i} f d\mu$.*

Proof. For $\epsilon > 0$ there exists a gauge ν on $\mathfrak{Q}$ such that $|\mathfrak{S}(f, \mathfrak{P}) - \int_{\mathfrak{Q}} f d\mu| < \epsilon$ for each ν-fine free tagged partition $\mathfrak{P}$ of $\mathfrak{Q}$. By Theorem 3.9, $f \in \mu$-$M(\mathfrak{Q})$ for $i = 1, 2, ..., n$. Then there exist gauge ν_i on $\mathfrak{Q}_i$ for $i = 1, 2, ..., n$ such that $\nu_i(x) < \nu(x)$ for each $x \in \mathfrak{Q}_i$ and such that $|\mathfrak{S}(f, \mathfrak{P}_i) - \int_{\mathfrak{Q}} f d\mu| < \frac{\epsilon}{n}$, for each ν_i-fine free tagged partitions $\mathfrak{P}_i$ of $\mathfrak{Q}$ so, $\mathfrak{P} = \mathfrak{P}_1 \cup \mathfrak{P}_2 \cup \mathfrak{P}_3 ... \cup \mathfrak{P}_n$ is ν-fine free tagged partition of $\mathfrak{Q}$. Consequently, $|\mathfrak{S}(f, \mathfrak{P}) - \sum_{i=1}^{n} \int_{\mathfrak{Q}_i} f d\mu| < \epsilon$. Therefore, $\int_{\mathfrak{Q}} f d\mu = \sum_{i=1}^{n} \int_{\mathfrak{Q}_i} f d\mu$. $\square$

Definition 4.3. *Let $\mathfrak{Q} \in \mathbb{F}$ and let $f : \mathfrak{Q} \to \mathbb{R}$ be a μ-McShane integrable function on $\mathfrak{Q}$. We say that the map $\mathcal{F} \mapsto A \rightsquigarrow \int_A f d\mu$, defined on each subcell of $\mathfrak{Q}$ is the indefinite μ-McShane integral of f.*

It is very straight to see that $\mathcal{F}$ is an additive function of cells. Next, we shall prove Saks-Henstock type lemma for μ-McShane integral on a cell.

Theorem 4.4. *A function $f : \mathfrak{Q} \to \mathbb{R}$ is μ-McShane integrable on $\mathfrak{Q}$ if and only if there exists an additive cell function π defined on the family of all subcells of $\mathfrak{Q}$ such that for each $\epsilon > 0$ there exists a gauge ν on $\mathfrak{Q}$ with*

$$\sum_{(x_i, \mathfrak{Q}_i) \in \mathfrak{P}} \left| \pi(\mathfrak{Q}_i) - f(x_i)\mu(\mathfrak{Q}_i) \right| < \epsilon,$$

for each ν-fine free tagged partition $\mathfrak{P}$ of $\mathfrak{Q}$.

In this situation π is the indefinite μ-McShane integral of f on $\mathfrak{Q}$.

Proof. Let $f \in \mu$-$M(\mathfrak{Q})$, then for each $\epsilon > 0$ there esists a gauge ν on $\mathfrak{Q}$ such that

$$| \int_{\mathfrak{Q}} f d\mu - \mathfrak{S}(f, \mathfrak{P})| < \frac{\epsilon}{3},$$

whenever $\mathfrak{P}$ is a ν-fine free tagged partition of $\mathfrak{Q}$. Let us fix, a partition $\mathfrak{P}_0$ of $\mathfrak{Q}$ and let $\mathfrak{P} \subset \mathfrak{P}_0$ be a ν-fine free partition on $\mathfrak{Q}$. Then $\mathfrak{P}_0 \setminus \mathfrak{P} = \left\{(x_1, \mathfrak{Q}_1), (x_2, \mathfrak{Q}_2), ..., (x_m, \mathfrak{Q}_m)\right\}$. Moreover by Theorem 3.9, $f \in \mu$-$M(\mathfrak{Q}_j)$ for $j = 1, 2, ..., m$. Therefore, given $\gamma > 0$ and $j \in \{1, 2, ..., m\}$ there exists a gauge ν_j on $\mathfrak{Q}_j$ so that $\nu_j(x) < \nu(x)$ for each $x \in \mathfrak{Q}_j$ and such that

$$\left| \int_{\mathfrak{Q}_j} f d\mu - \mathfrak{S}(f, \mathfrak{P}_j) \right| < \frac{\gamma}{m}.$$

for each ν_j-fine free tagged partition $\mathfrak{P}_j$ of $\mathfrak{Q}_j$. Then $\mathfrak{P}_0 = \mathfrak{P} \cup \bigcup_{j=1}^{m} \mathfrak{P}_j$ is a tagged partition of $\mathfrak{Q}$ that is a ν-fine and

$$\sum_{(x_i,\mathfrak{Q}_i)\in\mathfrak{P}_0} f(x_i)\mu(\mathfrak{Q}_i) = \sum_{(x_i,\mathfrak{Q}_i)\in\mathfrak{P}} f(x_i)\mu(\mathfrak{Q}_i) + \sum_{j=1}^{m} \sum_{(x_i,\mathfrak{Q}_i)\in\mathfrak{P}_j} f(x_i)\mu(\mathfrak{Q}_i).$$

Since π is an indefinite μ-McShane integral of f on $\mathfrak{Q}$ so $\pi(\mathfrak{Q}) = \sum_{(x_i,\mathfrak{Q}_i)\in\mathfrak{P}} \pi(\mathfrak{Q}_i) + \sum_{j=1}^{m} \pi(\mathfrak{Q}_j)$. Consequently,

$$\begin{aligned}
\left|\sum_{(x_i,\mathfrak{Q}_i)\in\mathfrak{P}} \Big(\pi(\mathfrak{Q}_i) - f(x_i)\mu(\mathfrak{Q}_i)\Big)\right| &\le \left|\Big(\pi(\mathfrak{Q}) - \sum_{(x_i,\mathfrak{Q}_i)\in\mathfrak{P}_0} f(x_i)\mu(\mathfrak{Q}_i)\Big)\right| \\
&\quad + \sum_{j=1}^{m} \left|\pi(\mathfrak{Q}_j) - \sum_{(x_i,\mathfrak{Q}_i)\in\mathfrak{P}_j} f(x_i)\mu(\mathfrak{Q}_i)\Big)\right| \\
&< \frac{\epsilon}{3} + m.\frac{\gamma}{m}
\end{aligned}$$

Since γ is arbitrary small number, we have

$$\left|\sum_{(x_i,\mathfrak{Q}_i)\in\mathfrak{P}} \Big(\pi(\mathfrak{Q}_i) - f(x_i)\mu(\mathfrak{Q}_i)\Big)\right| < \epsilon, \tag{1}$$

for each ν-fine free tagged partial partitions $\mathfrak{P}$ on $\mathfrak{Q}$.

Let $\mathfrak{P}^+ = \Big\{(x_i, \mathfrak{Q}_i) \in \mathfrak{P} : \ \pi(\mathfrak{Q}_i) - f(x_i)\mu(\mathfrak{Q}_i) \ge 0\Big\}$ and
$\mathfrak{P}^- = \Big\{(x_i, \mathfrak{Q}_i) \in \mathfrak{P} : \ \pi(\mathfrak{Q}_i) - f(x_i)\mu(\mathfrak{Q}_i) < 0\Big\}$. Clearly $\mathfrak{P}^+$ and $\mathfrak{P}^-$ are ν-fine partial partitions on $\mathfrak{Q}$ and satisfy Equation (1). Hence,

$$\begin{aligned}
\sum_{(x_i,\mathfrak{Q}_i)\in\mathfrak{P}} \left|\pi(\mathfrak{Q}_i) - f(x_i)\mu(\mathfrak{Q}_i)\right| &\le \sum_{(x_i,\mathfrak{Q}_i)\in\mathfrak{P}^+} \Big(\pi(\mathfrak{Q}_i) - f(x_i)\mu(\mathfrak{Q}_i)\Big) - \sum_{(x_i,\mathfrak{Q}_i)\in\mathfrak{P}^-} \Big(\pi(\mathfrak{Q}_i) - f(x_i)\mu(\mathfrak{Q}_i)\Big) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Conversely, suppose there exists an additive cell function π defined on the family of all subcells of $\mathfrak{Q}$ such that $\sum_{(x_i,\mathfrak{Q}_i)\in\mathfrak{P}} \left|\pi(\mathfrak{Q}_i) - f(x_i)\mu(\mathfrak{Q}_i)\right| < \epsilon$ whenever $\mathfrak{P}$ is a ν-fine free tagged partial partitions on $\mathfrak{Q}$. In particular, this inequality holds for a ν-fine partition $\mathfrak{P}_0 = \{(x_i, \mathfrak{Q}_1), ..., (x_m, \mathfrak{Q}_m)\}$ of $\mathfrak{Q}$. Then

$$\begin{aligned}
\left|\pi(\mathfrak{Q}) - \sum_{i=1}^{m} f(x_i)\mu(\mathfrak{Q}_i)\right| &= \left|\sum_{i=1}^{m} \pi(\mathfrak{Q}_i) - \sum_{i=1}^{m} f(x_i)\mu(\mathfrak{Q}_i)\right| \\
&\le \sum_{i=1}^{m} \left|\pi(\mathfrak{Q}_i) - f(x_i)\mu(\mathfrak{Q}_i)\right| \\
&< \epsilon.
\end{aligned}$$

Hence $f \in \mu\text{-}M(\mathfrak{Q})$. $\square$

4.1. Relation with the Lebesgue integral

It is well known that in a real line, Lebesgue and McShane integrals are equivalent. One can see [5, Theorem 10.11]. Next, we shall discuss the relationship of Lebesgue and μ-McShane integrals on $\mathfrak{Q}$. We denote the Lebesgue integral of $f : \mathfrak{Q} \to \mathbb{R}$ with respect to μ by $(L)\int_{\mathfrak{Q}} f d\mu$.

Theorem 4.5. *Let $f : \mathfrak{Q} \to \mathbb{R}$ be a function. If f is Lebesgue integrable on $\mathfrak{Q}$ with respect to μ, then f is μ-McShane integral on $\mathfrak{Q}$ and $(L)\int_{\mathfrak{Q}} f d\mu = \int_{\mathfrak{Q}} f d\mu$.*

Proof. This is an adaptation of proof of Theorem 4.1 in [1]. By Theorem 2.3, for a given $\epsilon > 0$ there exist functions g_1 and g_2 on $\mathfrak{Q}$ that are upper and lower semicontinuos respectively such that $-\infty \leq g_1 \leq f \leq g_2 \leq +\infty$ and $(L)\int_{\mathfrak{Q}} f d\mu = \int_{\mathfrak{Q}} f d\mu$. To claim this, define a gauge ν on $\mathfrak{Q}$ such that $g_1(t) \leq f(x) + \epsilon$ and $g_2(t) \geq f(x) - \epsilon$, for each $t \in \mathfrak{Q}$ with $d(x,t) < \nu(x)$.

Let $\mathfrak{P} = \Big\{(x_1, \mathfrak{Q}_1), ..., (x_m, \mathfrak{Q}_m)\Big\}$ be a ν-fine free partition of $\mathfrak{Q}$. Then, for each $i \in \{1, 2, \ldots, p\}$, we have

$$L\int_{Q_i} g_1 d\mu \leq L\int_{Q_i} f d\mu \leq L\int_{Q_i} g_2 d\mu \tag{2}$$

Moreover, by $g_1(t) \leq f(x_i) + \epsilon$ for each $t \in (\mathfrak{Q}_i)$, it follows $(L)\int_{\mathfrak{Q}_i}(g_1 - \epsilon)d\mu \leq (L)\int_{\mathfrak{Q}_i} f(x_i)d\mu$ and therefore $(L)\int_{\mathfrak{Q}_i} g_1 d\mu - \epsilon\mu(\mathfrak{Q}_i) \leq f(x_i)\mu(\mathfrak{Q}_i)$. Similarly, by $g_2(t) \geq f(x_i) + \epsilon$ for each $t \in (\mathfrak{Q}_i)$, it follows $f(x_i)\mu(\mathfrak{Q}_i) \leq (L)\int_{\mathfrak{Q}_i} g_2 d\mu + \epsilon\mu(\mathfrak{Q}_i)$. So, for $i = 1, 2, ..., p$, we have $(L)\int_{\mathfrak{Q}_i} g_1 d\mu - \epsilon\mu(\mathfrak{Q}_i) \leq f(x_i)\mu(\mathfrak{Q}_i) \leq (L)\int_{\mathfrak{Q}_i} g_2 d\mu + \epsilon\mu(\mathfrak{Q}_i)$. Hence, $(L)\int_{\mathfrak{Q}} g_1 d\mu - \epsilon \leq \mathfrak{S}(f, \mathfrak{P}) \leq (L)\int_{\mathfrak{Q}} g_2 d\mu$. By (2), $(L)\int_{Q} g_1 d\mu \leq (L)\int_{\mathfrak{Q}} f d\mu \leq \int_{\mathfrak{Q}} g_2 d\mu$. Thus, $|\mathfrak{S}(f, \mathfrak{P}) - (L)\int_{\mathfrak{Q}} f d\mu| \leq (L)\int_{\mathfrak{Q}}(g_2 - g_1)d\mu + 2\epsilon < 3\epsilon$.
□

Next, we aim to find if the opposite inclusion of Theorem 4.5 true, as the case is for the real line. Next, we prove that μ-McShane integrability implies Lebesgue integrability in our settings.

Theorem 4.6. *If f is μ-McShane integrable on $\mathfrak{Q}$, then it is Lebesgue integrable on $\mathfrak{Q}$.*

Proof. Let $f : \mathfrak{Q} \to \mathbb{R}$ be μ-McShane integrable function on $\mathfrak{Q}$. By Theorem 3.11, $|f|$ is McShane integrable. It is clear that $|f|$ is μ-Henstock-Kurzweil integrable. Finally, by [7, Theorem 2.64], f is Lebesgue integrable on $\mathfrak{Q}$. □

5. Conclusion

In this article we defined the μ-McShane integral on a complete metric measure space. We described some basic properties of μ-McShane integrals on a complete metric measure space with a non-atomic Radon measure μ, associate with a family of μ-cell functions in respect to the Vitali covering theorem. The relation of Lebesgue and μ-McShane integral is also discussed.

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