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On some paranormed ideal convergent triple sequence spaces via zweier operator

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Abstract. In the present work, we introduce paranormed zweier ideal convergent triple sequence spaces defined by a compact operator ${}_3\mathcal{Z}^I(\tau)$,${}_3\mathcal{Z}^I_0(\tau)$ and ${}_3\mathcal{Z}^I_\infty(\tau)$ where $q = (q_{ijk})$ is a triple sequence of positive numbers and we study some algebraic and topological properties of these spaces.

1. Introduction

A double sequence is a function $x : \mathbb{N}\times\mathbb{N} \to \mathbb{R} or \mathbb{C}$ where, the sets $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ have their usual meaning as the set of natural numbers,real numbers and complex numbers respectively. By the convergence of a double sequence, we mean the convergence in the Pringsheim's sense. A double sequence $x = (x_{ij})$ has a Pringsheim limit L provided that for a given $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that $|x_{ij} - L| < \epsilon$, whenever $i, j > n$. A triple sequence (x_{ijk}) is a generalization of double sequence and defined as a function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R} or \mathbb{C}$. A number of different kinds of studies about triple sequence spaces was initially formalised by well known authors *Sahiner et al.*[31, 32], *Esi et al.*[9, 11], *Datta et al.*[7], *Debnath et al.*[8] and further by many other authors too. Let the class of all triple sequences (x_{ijk}) to be denoted by ${}_3\omega$ and its domain to be $\mathbb{N}^3$ where the subscripts i,j and k belong to $\mathbb{N}$. Throughout the present work, classes ${}_3l_\infty$, ${}_3c$ and ${}_3c_0$ will represent triple sequence spaces which are bounded, convergent, and convergent to zero in Pringsheim's sense respectively, moreover these spaces normed by sup-norm as follows

$$||x||_\infty = \sup_{i,j,k} |x_{ijk}|, \qquad where \quad i, j, k \in \mathbb{N}.$$

In 2007 *Sahiner*,*Gurdal* and *Duden*[31] introduced the notion of convergence of triple sequences.Further,this concept has been studied by many authors([2, 3, 6, 9, 10, 13, 17, 18]).

Fast[12] and Schoenberg[34] worked independently and carried out the concept of statistical convergence. Later on *Kostyrko*, *Sala*t and *Wilczynski*[23] generalized the statistical convergence and as a result the concept of ideal convergence came into existence. Subsequently, *Tripathy*, *Salat* and *Ziman*[33] and many other researchers (like Raj *et. al.* [30], Ayman-Mursaleen and Serra-Capizzano [4], etc) worked on ideal

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convergence as well as statistical convergence. An ideal I is defined to be a family of a non-empty set X i.e $I \subseteq 2^{\mathbf{X}}$ if $\emptyset \in I$ and If $I_1, I_2 \in I$ implies that their union is in I i.e $I_1 \cup I_2 \in I$, and $I_1 \in I, I_2 \subseteq I_1 \implies I_2 \in I$. whereas a filter is a family of sets $F \subseteq 2^{\mathbf{X}} \iff \emptyset \notin F$, $F_1, F_2 \in F$ implies that their intersection is in F i.e $F_1 \cap F_2 \in F$ and $F_1 \subseteq F_2 \implies F_2 \in F$. If $I \neq \emptyset$ and $X \notin I$ then I is said to be non-trivial, admissible if and only if $\{\{x\} : x \in X\} \subseteq I$ and maximal if there is no ideal $J \neq I$ that contains I. For every I to be a non-trivial ideal there must corresponds a filter $F = F(I) = \{Y : X - Y \in I\}$.

Maddox [26] introduced the following subspaces of a linear space ω.

$l(q) := \{x \in \omega : \sum_k |x_k|^{q_k} < \infty\}$,
$l_\infty(q) := \{x \in \omega : \sup_k |x_k|^{q_k} < \infty\}$,
$c(q) := \{x \in \omega : \lim_k |x_k - l|^{q_k} = 0, \ \text{for } l \in \mathbb{C} \ \}$,
$c_0(q) := \{x \in \omega : \lim_k |x_k|^{q_k} = 0, \ \}$,
where $q = (q_k)$ denotes a sequence of strictly positive real numbers.

Subsequently, Lascarides ([24, 25], introduced the following subspaces of a linear space ω.

$l_\infty\{q\} = \{x \in \omega : \ \exists\, 0 < r \text{ s.t } \sup_k |x_k r|^{q_k} t_k < \infty\}$,
$c_0\{q\} = \{x \in \omega : \ \exists\, 0 < r \text{ s.t } \lim_k |x_k r|^{q_k} t_k = 0, \ \}$,
$l\{q\} = \{x \in \omega : \ \exists\, 0 < r \text{ s.t } \sum_{k=1}^{\infty} |x_k r|^{q_k} t_k < \infty\}$,
where $t_k = q_k^{-1}$, $\forall$ for all k.

2. Preliminaries and Definitions

Definition 2.1:[16] A sequence $(x_{ijk}) \in {}_3\omega$ is said to be convergent to a number c in pringsheim's sense if for every $\epsilon > 0$ however small, there exists a natunal number $m \in \mathbb{N}$ such that

$$|x_{ijk} - c| < \epsilon \ \text{ whenever } \ i \geq m, \ \ j \geq m, \ \ k \geq m.$$

Example:[16] Let

$$x_{ijk} = \begin{cases} jk, & i = 3 \\ ik, & j = 5 \\ ij, & k = 7 \\ 8, & \text{otherwise.} \end{cases}$$

Then $(x_{ijk}) \to 8$ in Pringsheim's sense.

Definition 2.2:[16] A sequence $(x_{ijk}) \in {}_3\omega$ is said to be a Cauchy sequence if for every $\epsilon > 0$ however small, there exists a natural number $m \in \mathbb{N}$ such that

$$|x_{ijk} - x_{pqr}| < \epsilon \ \text{ whenever } \ i \geq p \geq m, \ \ j \geq q \geq m, \ \ k \geq r \geq m.$$

Definition 2.3:[16] A sequence $(x_{ijk}) \in {}_3\omega$ is said to be bounded if there exists a number $s > 0$ such that $|x_{ijk}| < s$ for all i,j,k.

Definition 2.4:[15] A sequence $(x_{ijk}) \in {}_3\omega$ is said to be ideal or I-convergent to a number y if for every $\epsilon > 0$ however small, such that

$$\left\{(i, j, k) \in \mathbb{N}^3 : |x_{ijk} - y| \geq \epsilon\right\} \in I.$$

symbolically write as, $I - \lim x_{ijk} = y$.

Definition 2.5:[15] A sequence $(x_{ijk}) \in {}_3\omega$ is said to be I-null if $y = 0$ and symbolically write as $I - \lim x_{ijk} = 0$.

Definition 2.6:[15] A sequence $(x_{ijk}) \in {}_3\omega$ is said to be I-Cauchy if for every $\epsilon > 0$ however small, there exists natural numbers p, q and r depend on ϵ such that

$$\left\{(i, j, k) \in \mathbb{N}^3 : |x_{ijk} - x_{pqr}| \geq \epsilon\right\} \in I.$$

Definition 2.7:[15] A sequence $(x_{ijk}) \in {}_3\omega$ is said to be I-bounded if there exists a number $s > 0$ such that

$$\left\{(i, j, k) \in \mathbb{N}^3 : |x_{ijk}| > s\right\} \in I.$$

Definition 2.8:[15] A triple sequence space S is said to be solid if $(\alpha_{ijk} x_{ijk}) \in E$ whenever $(x_{ijk}) \in S$ and for all sequences (α_{ijk}) of scalars with $|\alpha_{ijk}| \leq 1$, for all $i, j, k \in \mathbb{N}$.

Definition 2.9:[15] A triple sequence space S is said to be monotone if it contains the canonical pre-images of all its step spaces.

Definition 2.10:[15] A triple sequence space S is said to be sequence algebra if $(x_{ijk} \star y_{ijk}) \in S$, whenever $(x_{ijk}) \in S$ and $(y_{ijk}) \in S$.

Here, recalling a Lemma which will be use later for establishing some results:

Lemma: If $I \subset 2^{\mathbb{N}}$ and $K \subseteq \mathbb{N}$. If $K \notin I$, Then $K \cap \mathbb{N} \notin I$.(see[19–22])

Definition 2.11. A function defined on a linear space, $h : X \longrightarrow R$ is called a paranorm, if for all $x, y \in X$,
(i)$h(y) = 0 \; if \; y = \theta$,
(ii) $h(-y) = h(y)$,
(iii) $h(x + y) \leq h(x) + h(y)$,
(iv) If (a_n) is a sequence of scalars with $a_n \to a \; (n \to \infty)$ and $y_n, c \in X$ with $y_n \to c \;\; (n \to \infty)$, in the sense that $h(y_n - c) \to 0 \;\; (n \to \infty)$, in the sense that $h(a_n y_n - ac) \to 0 \;\; (n \to \infty)$.

The notion about the paranorm is closely associated to the linear metric spaces and it is a generalization of that of positive value(See[28]).

Definition 2.12 Let X and Y be two normed linear spaces. An operator T defined by

$$T : X \to Y$$

is said to be a *Compact Linear Operator* (completely continous linear operator) if T is linear and T maps every bounded sequence (x_k) in X onto a sequence $T(x_k)$ in Y which has a convergent subsequence. The set of all bounded linear operators $\mathcal{B}$(X,Y) is normed linear space normed by

$$||T|| = \sup_{x \in X, ||x|| = 1} ||Tx||$$

The set of all compact linear operator C(X,Y) is a closed subspace of $\mathcal{B}$(X,Y) and C(X,Y) is a Banach space if Y is a Banach space.

The K-space is a sequence space S with a linear topology which maps each $q_i \to \mathbf{C}$ such that $q_i(y) = y_i$ and it is continuous for every $i \in \mathbf{N}$. Moreover, if S is complete metric space then a K-space is called Fréchet coordinate space(FK-space) and if FK-space has a normable topology then it becomes a Banach coordinate space(BK-space).

Let, for any two sequence spaces S_1 and S_2 an infinite matrix $A = (c_{ij})$ of real or complex numbers c_{ij} defines a matrix mapping from S_1 to S_2 i.e $A : S_1 \to S_2$, the sequence $Ay = \{(Ay)_n)\}$, is A transform of y in S_2, for every $y = (y_k) \in S_1$

where,

$$(Ay)_n = \sum_j c_{ij} y_j, \quad (i \in \mathbf{N}). \qquad ...(1)$$

We denote the class of matrices by $B = (S_1 : S_2)$, thus if $A \in B$ if and only if the series given in eq. (1) converges for all $i \in \mathbf{N}$ and $y \in S_1$.

Recently, Mursaleen, Altay and Başar[1, 5], Malkowsky[27],Ng and Lee[29] and Wang[36] constructed a new sequence space by means of matrix domain with particular limitation method

Şengönül[35] defined $y_i = px_i + (1-p)x_{i-1}$ as Z^p transform of the sequence $x = (x_i)$ with $1 < p < \infty$ and $x_{-1} = 0$ and $Z^p = (z_{ik})$ denotes matrix as

$$z_{ik} = \begin{cases} p, & if, i = k, \\ 1-p, & if, i-1 = k; (i, k \in \mathbf{N}), \\ 0, & \text{for other cases.} \end{cases}$$

Later on, Şengönül[35] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$

$$\mathcal{Z} = \{x = (x_k) \in \omega : Z^p x \in \lambda\},$$

where $\lambda = c \; and \; c_0$

Here, recalling some theorems by Şengönül[35] which will be use later for establishing some results:

Theorem 2.1. The spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ are the BK-spaces with the norm

$$\|x\|_{\mathcal{Z}} = \|x\|_{\mathcal{Z}_0} = \|Z^p x\|_c.$$

Theorem 2.2. The spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ are linearly isomorphic to c and c_0 respectively, i.e $\mathcal{Z} \cong c$ and $\mathcal{Z}_0 \cong c_0$.

Theorem 2.3. For $p \neq 1$ the inclusions $\mathcal{Z}_0 \subset \mathcal{Z}$ strictly holds.

Let $p = (p_k)$ be the real positive bounded sequence. For any complex number ν, whenever $\nu = \sup_k(p_k) < \infty$, we have $|\nu|^{p_k} \leq \max(1, |\nu|^E)$. where $E = \sup_k(p_k)$ we get, $|x_k + y_k|^{p_k} \leq K(|x_k|^{p_k} + |y_k|^{p_k})$ for $K = \max(1; 2^{E-1})$. [26]

Khan and Ebadullah [14] introduced various zweier sequence spaces.

$$\mathcal{Z}^I = \{k \in \mathbb{N} : \{x = (x_k) \in \omega : I - \lim Z^p x = L \text{ for some L}\}\} \in I,$$
$$\mathcal{Z}_0^I = \{k \in \mathbb{N} : \{x = (x_k) \in \omega : I - \lim Z^p x = 0\}\} \in I,$$

$$\mathcal{Z}^I_\infty = \{k \in \mathbb{N} : \{x = (x_k) \in \omega : \sup_k |Z^p x| < \infty\}\} \in I,$$

Also, denoted by

$$m^I_{\mathcal{Z}} = \mathcal{Z}^I_\infty \cap \mathcal{Z}^I \text{ and } m^I_{\mathcal{Z}_0} = \mathcal{Z}^I_\infty \cap \mathcal{Z}^I_0.$$

3. Main Results

Here we define the main results of this article as classes of sequence spaces via an ideal defined over $\mathbb{N}^3$.

$_3\mathcal{Z}^I(\tau) = \{x = (x_{ijk}) \in {}_3\omega : \{(i,j,k) \in \mathbb{N}^3 : |T(Z^p x) - s|^{q_{ijk}} \geq \epsilon\} \in I\}$
for some s $\in \mathbb{C}$;

$_3\mathcal{Z}^I_0(\tau) = \{x = (x_{ijk}) \in {}_3\omega : \{(i,j,k) \in \mathbb{N}^3 : |T(Z^p x)|^{q_{ijk}} \geq \epsilon\} \in I\}$;

$_3\mathcal{Z}^I_\infty(\tau) = \{x = (x_{ijk}) \in {}_3\omega : \sup_{i,j,,k} |T(Z^p x)|^{q_{ijk}} < \infty\}$.

Also, denoted by

$$_3m^I_{\mathcal{Z}}(\tau) = {}_3\mathcal{Z}_\infty(\tau) \cap {}_3\mathcal{Z}^I(\tau)$$

and

$$_3m^I_{\mathcal{Z}_0}(\tau) = {}_3\mathcal{Z}_\infty(\tau) \cap {}_3\mathcal{Z}^I_0(\tau)$$

where $q = (q_{ijk})$, is a positive real triple sequence.

Throughout the article, we will use some representations to reduce elementary calculations as below $T(Z^p x) = x^{'}, T(Z^p y) = y^{'}$ and $T(Z^p z) = z^{'}$ for all $x, y, z \in {}_3\omega$.

Theorem 3.1. The spaces $_3\mathcal{Z}^I(\tau)$, $_3\mathcal{Z}^I_0(\tau)$, $_3m^I_{\mathcal{Z}}(\tau)$ and $_2m^I_{\mathcal{Z}_0}(\tau)$ are linear spaces.
Proof. We'll only prove for the space $_3\mathcal{Z}^I(\tau)$.
For the other spaces similar way will be follow.
Let, there exits two triple sequences in the space $_3\mathcal{Z}^I(\tau)$ i.e $(x_{ijk}), (y_{ijk}) \in {}_3\mathcal{Z}^I(\tau)$ and α_1, α_2 be scalars and epsilon to be greater than 0 i.e $\epsilon > 0$. however small.
we always get,

$$\{(i,j,k) \in \mathbb{N}^3 : |x^{'}_{ijk} - s_1|^{q_{ijk}} \geq \frac{\epsilon}{2t_1}, \text{ for some } s_1 \in \mathbb{C} \ \} \in I$$

$$\{(i,j,k) \in \mathbb{N}^3 : |y^{'}_{ijk} - s_2|^{q_{ijk}} \geq \frac{\epsilon}{2t_2}, \text{ for some } s_2 \in \mathbb{C} \ \} \in I$$

where

$$t_1 = E.max\{1, \sup_{i,j,k} |\alpha_1|^{q_{ijk}}\}$$

$$t_2 = E.max\{1, \sup_{i,j,k} |\alpha_2|^{q_{ijk}}\}$$

and

$$E = max\{1, 2^{F-1}\} \text{ where } F = \sup_{i,j,k} q_{ijk} \geq 0.$$

Let

$$A_1 = \{(i,j,k) \in \mathbb{N}^3 : |x^{'}_{ijk} - s_1|^{q_{ijk}} < \frac{\epsilon}{2t_1}, \text{ for some } s_1 \in \mathbb{C} \ \} \in I$$

$$A_2 = \{(i,j,k) \in \mathbb{N}^3 : |y^{'}_{ijk} - s_2|^{q_{ijk}} < \frac{\epsilon}{2t_2}, \text{ for some } s_2 \in \mathbb{C} \ \} \in I$$

be such that $A_1^c, A_2^c \in I$.
Then

$$A_3 = \{(i,j,k) \in \mathbb{N}^3 : |(\alpha_1 x'_{ijk} + \alpha_2 y'_{ijk}) - (\alpha_1 s_1 + \alpha_2 s_2)|^{q_{ijk}}) < \epsilon\}$$

$$\supseteq \{(i,j,k) \in \mathbb{N}^3 : |\alpha_1|^{q_{ijk}} |x'_{ijk} - s_1|^{q_{ijk}} < \frac{\epsilon}{2t_1} |\alpha_1|^{q_{ijk}}.E\}$$

$$\cap \{(i,j,k) \in \mathbb{N}^3 : |\alpha_2|^{q_{ijk}} |y'_{ijk} - s_2|^{q_{ijk}} < \frac{\epsilon}{2t_2} |\alpha_2|^{q_{ijk}}.E\}$$

Therefore as a result we get, $A_3^c = A_1^c \cap A_2^c \in I$. subsequently we get the linear combination $(\alpha_1 x'_{ijk} + \alpha_2 y'_{ijk}) \in {}_3\mathcal{Z}^I(\tau)$. by proving this linearity, we can say that the space ${}_3\mathcal{Z}^I(\tau)$ is a linear space.
For the other spaces ${}_3\mathcal{Z}_0^I(\tau)$, ${}_3m^I_{\mathcal{Z}}(\tau)$ and ${}_2m^I_{\mathcal{Z}_0}(\tau)$ by following the similar way one can prove that they are also linear spaces.
Theorem 3.2. Let $(q_{ijk}) \in {}_3l_\infty$. Then ${}_3m^I_{\mathcal{Z}}(\tau)$ and ${}_3m^I_{\mathcal{Z}_0}(\tau)$ are paranormed spaces, paranormed by $h(x') = \sup_{i,j,k} |x'_{ijk}|^{\frac{q_{ijk}}{t}}$ where $t = max\{1, \sup_{i,j,k} q_{ijk}\}$.

Proof. Suppose we take $x' = (x'_{ijk}), y' = (y'_{ijk}) \in {}_3m^I_{\mathcal{Z}}(q)$.
(1) then, $h(x') \neq 0$ if and only if $x' \neq 0$.
(2) $h(x') = h(-x')$ is obvious.
(3) Since $\frac{q_{ijk}}{t} \leq 1$ and $t > 1$,now here we use Minkowski's inequality then we can write,

$$\sup_{i,j,k} |x'_{ijk} + y'_{ijk}|^{\frac{q_{ijk}}{t}} \leq \sup_{i,j,k} |x'_{ijk}|^{\frac{q_{ijk}}{t}} + \sup_{i,j,k} |y'_{ijk}|^{\frac{q_{ijk}}{t}}$$

(4) λ to be complex number then we write (λ_{ijk}) s.t $\lambda_{ijk} \to \lambda$, $(i,j,k \to \infty)$.
Let $x'_{ijk} \in {}_3m^I_{\mathcal{Z}}(\tau)$ such that $|x'_{ijk} - L|^{q_{ijk}} \geq \epsilon$.
Therefore, $h(x' - Le) = \sup_{i,j,k} |x'_{ijk} - L|^{\frac{q_{ijk}}{t}} \leq \sup_{i,j,k} |x'_{ijk}|^{\frac{q_{ijk}}{t}} + \sup_{i,j,k} |L|^{\frac{q_{ijk}}{t}}$, where e=(1,1,1.....).
Hence, $h((\lambda_{ijk} x'_{ijk} - \lambda L)) \leq h((\lambda_{ijk} x'_{ijk})) + h(\lambda L) = \lambda_{ijk} h(x') + \lambda h(L)$ as $(i,j,k \to \infty)$.
Therefore as a result the space, ${}_3m^I_{\mathcal{Z}}(\tau)$ is a paranormed space which was the required proof of the theorem and for the rest of the results one can follow the same steps.

Theorem 3.3. The space ${}_3m^I_{\mathcal{Z}}(\tau)$ is closed in ${}_3l_\infty(\tau)$ as a subspace.

Proof. Suppose, we take $(x'^{(lmn)}_{ijk})$ as a cauchy sequence in the space ${}_3m^I_{\mathcal{Z}}(\tau)$ s.t $x'^{(lmn)} \to x'$.
We claim that $x' \in {}_3m^I_{\mathcal{Z}}(\tau)$.
Since $(x'^{(mn)}_{ijk}) \in {}_3m^I_{\mathcal{Z}}(\tau)$, then there is another triple sequence (a_{lmn}) exists s.t

$$\{(i,j,k) \in \mathbb{N}^3 : |x'^{(lmn)} - a_{lmn}| \geq \epsilon\} \in I$$

To prove our claim we will only prove that
(1)(a_{lmn}) converges to number a.
(2)If $U = \{(i,j,k) \in \mathbb{N}^3 : |x'_{ijk} - a| < \epsilon\}$, implies that $U^c \in I$.

Since $(x'^{(lmn)}_{ijk})$ is a Cauchy sequence in ${}_3m^I_{\mathcal{Z}}(\tau)$ then for a given $\epsilon > 0$ however small, there always exists $(i_0, j_0, k_0) \in\in \mathbb{N}^3$ s.t

$$\sup_{i,j,k} |x'^{(lmn)}_{ijk} - x'^{(pqr)}_{ijk}| < \frac{\epsilon}{3}, \text{ for all (l,m,n),(p,q,r)} \geq (i_0, j_0, k_0)$$

For given $\epsilon > 0$ however small we can have,

$$B_{lmn,pqr} = \{(i,j,k) \in \mathbb{N}^3 : |x'^{(lmn)}_{ijk} - x'^{(pqr)}_{ijk}| < \frac{\epsilon}{3}\}$$

$$B_{pqr} = \{(i,j,k) \in \mathbb{N}^3 : |x'^{(pqr)}_{ijk} - a_{pqr}| < \frac{\epsilon}{3}\}$$

$$B_{lmn} = \{(i,j,k) \in \mathbb{N}^3 : |x'^{(lmn)}_{ijk} - a_{lmn}| < \frac{\epsilon}{3}\}$$

Then $B^c_{lmn,pqr}, B^c_{pqr}, B^c_{lmn} \in I$.
Let $B^c = B^c_{lmn,pqr} \cap B^c_{pqr} \cap B^c_{lmn}$, where

$$B = \{(i,j,k) \in \mathbb{N}^3 : |a_{pqr} - a_{lmn}| < \epsilon\}.$$

Then $B^c \in I$.
We choose $(i_0, j_0, k_0) \in B^c$,then for each $(l,m,n),(p,q,r) \geq (i_0, j_0, k_0)$, we have

$$\{(i,j,k) \in \mathbb{N}^3 : |a_{pqr} - a_{lmn}| < \epsilon\} \supseteq \{(i,j,k) \in \mathbb{N}^3 : |x'^{(pqr)}_{ijk} - a_{pqr}| < \frac{\epsilon}{3}\}$$
$$\cap\{(i,j,k) \in \mathbb{N}^3 : |x'^{(lmn)}_{ijk} - x'^{(pqr)}_{ijk}| < \frac{\epsilon}{3}\}$$
$$\cap\{(i,j,k) \in \mathbb{N}^3 : |x'^{(lmn)}_{ijk} - a_{lmn}| < \frac{\epsilon}{3}\}$$

Then (a_{lmn}) is a Cauchy sequence in $\mathbb{C}$ so it will always converges to a number $a \in \mathbb{C}$ s.t $a_{lmn} \to a$, as $(l,m,n) \to \infty$.

For the other part of our proof let $\delta \in (0,1)$ be given. Then we will only have to show that if $U = \{(i,j,k) \in \mathbb{N}^3 : |x'_{ijk} - a|^{q_{ijk}} < \delta\}$, implies $U^c \in I$.
Since $x'^{(lmn)} \to x'$, then there exists $(p_0, q_0, r_0) \in \mathbb{N}^3$ such that

$$P = \{(i,j,k) \in \mathbb{N}^3 : |x'^{(p_0,q_0,r_0)} - x'| < (\frac{\delta}{3D})^t\} \tag{1}$$

which implies that $P^c \in I$
So we can choose a triplet (p_0, q_0, r_0) as with (1), we can obtain

$$Q = \{(i,j,k) \in \mathbb{N}^3 : |a_{p_0q_0r_0} - a|^{q_{ijk}} < (\frac{\delta}{3D})^t\}$$

s.t $Q^c \in I$
as, $\{(i,j,k) \in \mathbb{N}^3 : |x'^{(p_0q_0r_0)} - a_{p_0q_0r_0}|^{q_{ijk}} \geq \delta\} \in I$. Then we can get a subset S of $\mathbb{N}^3$ s.t $S^c \in I$, where S is defined as

$$S = \{(i,j,k) \in \mathbb{N}^3 : |x'^{(p_0q_0r_0)} - a_{p_0q_0r_0}|^{q_{ijk}} < (\frac{\delta}{3D})^t\}.$$

Let $U^c = P^c \cap Q^c \cap S^c$, where the set U can be defined as $U = \{(i,j,k) \in \mathbb{N}^3 : |x'_{ijk} - a|^{q_{ijk}} < \delta\}$.
Therefore, as a result it is always true for each $(i,j,k) \in U^c$, we get

$$\{(i,j,k) \in \mathbb{N}^3 : |x'_{ijk} - a|^{q_{ijk}} < \delta\} \supseteq \{(i,j,k) \in \mathbb{N}^3 : |x'^{(p_0q_0r_0)} - x|^{q_{ijk}} < (\frac{\delta}{3D})^t\}$$
$$\cap\{(i,j,k) \in \mathbb{N}^3 : |x'^{(p_0q_0r_0)} - a_{p_0q_0r_0}|^{q_{ijk}} < (\frac{\delta}{3D})^t\}$$
$$\cap\{(i,j,k) \in \mathbb{N}^3 : |a_{p_0q_0r_0} - a|^{q_{ijk}} < (\frac{\delta}{3D})^t\}.$$

So, we have proved our claim to be true . Moreover the inclusions $m^I_{\mathcal{Z}}(\tau) \subset l_\infty(\tau)$ and $m^I_{\mathcal{Z}_0}(\tau) \subset l_\infty(\tau)$ are strictly follows and so regarding the Theorem 3.3 another result follows.

Theorem 3.4. The spaces ${}_3m^I_{\mathcal{Z}}(\tau)$ and ${}_3m^I_{\mathcal{Z}_0}(\tau)$ are nowhere dense subsets of ${}_3l_\infty(\tau)$.

Theorem 3.5. The spaces ${}_3m^I_{\mathcal{Z}}(\tau)$ and ${}_3m^I_{\mathcal{Z}_0}(\tau)$ are not separable.

Proof. We'll only prove for the space ${}_3m^I_{\mathcal{Z}}(\tau)$.
For the other spaces similar way will be follow.
Let $S \subset \mathbb{N}^3$ to be infinite and of increasing natural numbers s.t $S \in I$and a sequence (q_{ijk}) to be defined as i.e,

$$q_{ijk} = \begin{cases} 1, & \text{if (i,j,k)} \in S, \\ 2, & \textit{otherwise.} \end{cases}$$

Let $P_0 = \{x' = (x'_{ijk}) : x'_{ijk} = 0 \text{ or } 1, \text{ for } (i,j,k) \in S \text{ and } x'_{ijk} = 0, \text{otherwise}\}$.
which is an uncountable set.
Consider, the set defined as $B_1 = \{B(x', \frac{1}{2}) : x' \in P_0\}$. class of open balls and an open cover C_1 of ${}_3m^I_{\mathcal{Z}}(\tau)$ containing B_1.
Here, B_1 is an uncountable set so the open cover C_1 can't be reduced to be countable subcover for the space ${}_3m^I_{\mathcal{Z}}(\tau)$.
Therefore as a result the space ${}_3m^I_{\mathcal{Z}}(\tau)$ is not separable.

Theorem 3.6. Let, there exits numbers such that $h = \inf_{i,j,k} q_{ijk}$ and $G = \sup_{i,j,k} q_{ijk}$. Then both the conditions below are equivalent.
(i) $G < \infty$ and $h > 0$.
(ii) ${}_3\mathcal{Z}^I_0 = {}_3\mathcal{Z}^I_0(\tau)$

Proof. Suppose there exits two numbers such that $h > 0$ and $G < \infty$, then we have the inequalities $\min\{1, s^h\} \le s^{q_{ijk}} \le \max\{1, s^G\}$which satisfies for any real $s > 0$ and for all triplets $(i,j,k) \in \mathbb{N}^3$. which enables us to prove the equivalence of both of the conditions (i) and (ii) above. Hence the results hold.

Theorem 3.7. For any two positive real triple sequences (q_{ijk}) and (r_{ijk}) the inclusion holds ${}_3m^I_{\mathcal{Z}_0}(\tau) \supseteq {}_3m^I_{\mathcal{Z}_0}(r)$ $\iff \lim_{(i,j,k)\in E} \inf \frac{q_{ijk}}{r_{ijk}} > 0$, where, the subset E is in I and its complement E^c belongs to $\mathbb{N}^3$

Proof. Let $\lim_{(i,j,k)\in E} \inf \frac{q_{ijk}}{r_{ijk}} > 0$, and $(x'_{ijk}) \in {}_3m^I_{\mathcal{Z}_0}(r)$. Then for any real $\beta > 0$ s.t $q_{ijk} > \beta r_{ijk}$, for all triplets (i,j,k) sufficiently large in E. Since $(x_{ijk}) \in {}_3m^I_{\mathcal{Z}_0}(r)$ for any given $\epsilon > 0$ however small, we can always have

$$F_0 = \{(i,j,k) \in \mathbb{N}^3 : |x_{ijk}|^{r_{ijk}} \ge \epsilon\} \in I$$

Let $H_0 = E^c \cup F_0$. Then $H_0 \in I$.
Then for all triplets (i,j,k) sufficiently large in H_0 the inclusion holds

$$\{(i,j,k) \in \mathbb{N}^3 : |x_{ijk}|^{q_{ijk}} \ge \epsilon\} \subseteq \{(i,j,k) \in \mathbb{N}^3 : |x_{ijk}|^{\beta r_{ijk}} \ge \epsilon\} \in I.$$

Therefore $(x'_{ijk}) \in {}_3m^I_{\mathcal{Z}_0}(\tau)$.
The converse part is obvious hence can be folllow in similar way.

Theorem 3.8. For any two positive real triple sequences (q_{ijk}) and (r_{ijk}) the inclusion holds ${}_3m^I_{\mathcal{Z}_0}(r) \supseteq {}_3m^I_{\mathcal{Z}_0}(\tau)$ $\iff \lim_{(i,j,k)\in E} \inf \frac{r_{ijk}}{q_{ijk}} > 0$, where, the subset E is in I and its complement E^c belongs to $\mathbb{N}^3$

Proof. This theorem cab be proved by following the steps of previous resut hence omitted.

Theorem 3.9. For any two positive real triple sequences (q_{ijk}) and (r_{ijk}) the inclusion holds
$_3m_0^I(r) = {}_3m_0^I(\tau) \iff \lim_{(i,j,k)\in E} \inf \frac{q_{ijk}}{r_{ijk}} > 0$, and $\lim_{(i,j,k)\in E} \inf \frac{r_{ijk}}{q_{ijk}} > 0$,
where, the subset E is in $\mathbb{N}^3$ and its complement E^c belongs to I

Proof. This theorem can be proved by combining above results which is obvious hence omitted.

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