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On δ_{ss}-perfect modules

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Abstract. After the definitions of perfect and semiperfect rings, the transportation of them to perfect and semiperfect modules is a significant creation for new characterizations of supplemented modules and the other modified versions of them. Inspired by this idea, we aim to create a route from δ_{ss}-perfect rings to δ_{ss}-perfect modules. A module W is said to be δ_{ss}-perfect if each factor module is of a projective δ_{ss}-cover. Owing to this goal, we obtain new relations for projective (amply) δ_{ss}-supplemented and δ_{ss}-lifting modules. Also, we present various characterization theorems for a (projective) module to be δ_{ss}-perfect.

1. Introduction

First of all let us point that, R and W will be representing an associative ring with identity and a unitary left R-module. We will also use the notations $\leq$, $\leq_{\oplus}$ for a submodule of W and a direct summand of W, respectively.

A submodule S of W is identified *essential* in W, signed with $S \trianglelefteq W$, if the intersection of S with the other submodules of W except for $\{0\}$ is nonzero. The intersection of all essential submodule of W is called *socle* of W and denoted as $Soc(W)$ which is equivalent to the sum of all simple submodules of W. A module W is called *singular* if $W \cong \frac{S}{T}$ where $T \trianglelefteq S$ for a module S. As the dual form of an essential submodule; $S \leq W$ is defined as small submodule provided that $W \neq S + T$ for every proper submodule T of W. The *radical* of a module W, signed by $Rad(W)$, is the intersection of all maximal submodules of W, which is equivalent to the sum of all small submodules of W. A projective module E with a surjective homomorphism $f : E \longrightarrow W$ providing $Ker(f) \ll E$ is called a *projective cover* of W. A module is called *semiperfect* whose each factor module has a projective cover [6]. All concepts given up to now can be found in [1], [16] and [4] for getting detailed informations.

For a module W and for the submodules $S, T \leq W$ if T is the minimal with regard to $W = S + T$, then T is identified a *supplement* submodule of S in W which is equivalent to $S + T = W$ and $S \cap T \ll T$. A *supplemented* module is a module such that each submodule has a supplement. The submodule S of W has *ample supplements* in W if every submodule T of W with $S + T = W$, contains a supplement of S in W. An amply *supplemented* module is a module such that each submodule has ample supplements. A module W is named *lifting* if for each $S \leq W$, there occures a decomposition such that $W = A \oplus B$ where $A \leq S$ and $S \cap B \ll B$. For more detailed information on these concepts we refer to [1], [9], [16] and [18].

In [17], Zhou generalized small submodules of a module to δ-small submodules as follows. A submodule $S \leq W$ is defined *δ-small*, signed by $S \ll_{\delta} W$, if $W \neq S + T$ for every $T \lneq W$ with $\frac{W}{T}$ is singular. This is

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equivalent to the existence of a projective semisimple submodule S' of S with $W = S \oplus T$, whenever $W = N + T$. The notation $\delta(W)$ represents the sum of all δ-small submodules of W. A projective module E with an epimorphism $f : E \longrightarrow W$ is called a *projective δ-cover* of W where $Ker(f) \ll_\delta E$. A module for which each factor module has a projective δ-cover is named *δ-semiperfect* [15]. In [7], Koşan generalized supplemented modules to the δ-supplemented modules using singularity. $T \leq W$ is identified a δ-supplement of a submodule S of W if $S + T = W$ and $S \cap T \ll_\delta T$. The module whose each submodule has a δ-supplement is said to be δ-supplemented. A module W is named amply δ-supplemented if for any submodule S, T of W with $W = S + T$, T involves a δ-supplement of S in W. A *δ-lifting* module W is a module that has a decomposition $W = A \oplus B$ such that $A \leq S$ and $S \cap B \ll_\delta B$ for any submodule S of W. All concepts which are related with δ-supplemented modules and the other modified forms of them can be seen in [7], [13] and [2].

In [5], *ss*-supplemented modules are defined as a proper extension of semisimple ones. A module W is called *ss-supplemented* if every $S \leq W$ has a supplement T in W such that $S \cap T$ is semisimple. Moreover, a local module with a semisimple radical is called a *strongly local module*.

In [14], δ_{ss}-supplemented modules and strongly δ-local modules are defined as a singular type of the concepts given in [5]. A module W is named *δ_{ss}-supplemented* if each $S \leq W$ has a δ-supplement T in W such that $S \cap T$ is semisimple which is equivalent to $S + T = W$, $S \cap T \leq Soc_\delta(T)$ where $Soc_\delta(T)$ is the sum of all simple and δ-small submodules of T. Note that $Soc_\delta(W) = \Sigma\{S \leq W \mid S$ is semisimple and $S \ll_\delta W\} = \delta(W) \cap Soc(W)$. A projective module E with the epimorphism $f : E \longrightarrow W$ such that $Ker(f) \leq Soc_\delta(T)$ is called a *projective δ_{ss}-cover* of W. *δ_{ss}-perfect rings* are the rings whose modules have a projective δ_{ss}-cover. A module W is called *strongly δ-local* if the submodule $\delta(W)$ is maximal, semisimple and δ-small in W. In [12], also *$\oplus$-δ_{ss}-supplemented* modules were defined.

Motivated by δ_{ss}-perfect rings given in [14], in this paper we define δ_{ss}-perfect modules. And owing to this, we obtain a restriction of δ-semiperfect modules. We say a module whose each factor module has a δ_{ss}-cover is δ_{ss}-perfect. Inspired by the terminology, some characterizations of (projective) δ_{ss}-perfect modules are obtained. In particular, we list the results obtained in this study as follows:

1. Let E be a projective δ_{ss}-cover of W. Then the case of being a δ_{ss}-perfect module coincides on the modules E and W.
2. A projective module E is δ_{ss}-perfect if and only if $\frac{E}{Soc_\delta(E)}$ is semisimple and the direct compositions of $\frac{E}{Soc_\delta(E)}$ can be lifted to E.
3. A module W is δ_{ss}-perfect if and only if W is coatomic, W is of a projective δ_{ss}-cover and so does every simple factor module of W.
4. The module $W = \oplus_{i \in I} W_i$ is δ_{ss}-perfect if and only if every projective W_i is δ_{ss}-perfect.
5. A projective module E is δ_{ss}-perfect if and only if E is δ-semiperfect and $\delta(E) \leq Soc(E)$.

2. δ_{ss}-perfect modules

Recall from [14], a projective δ_{ss}-cover of a module W is a projective module E with a surjective homomorphism $f : E \longrightarrow W$ such that whose kernel is semisimple and δ-small in E. In this part of the study, we define δ_{ss}-perfect modules as following and investigate various structural properties of them.

Definition 2.1. *A module whose each factor module is of a projective δ_{ss}-cover is said to be δ_{ss}-perfect.*

Now we present some characterizations of δ_{ss}-perfect modules via δ_{ss}-supplemented modules and several generalizations of them.

Lemma 2.2. *If W is a δ_{ss}-perfect module, then W is an amply δ_{ss}-supplemented module.*

Proof. Let W be a δ_{ss}-perfect module and $W = S + T$ for $S, T \leq W$. By hypothesis, there exists a projective δ_{ss}-cover of $\frac{W}{S}$ with $p : E \longrightarrow \frac{W}{S}$. Assume that $\pi : T \longrightarrow \frac{T}{S \cap T} \cong \frac{W}{S}$ be the canonical epimorphism. So, there exists a homomorphism h from E to T such that $\pi \circ h = p$, as E is projective. Following that we have,

$\frac{W}{S} = p(E) = \pi(h(E)) = \frac{h(E)+S}{S}$. Clearly we get $W = h(E) + S$ and as $h(E) \leq T$, $T = h(E) + (S \cap T)$ by the modular law. Moreover, $S \cap h(E) = h(Ker(p))$ is semisimple and δ-small in $h(E)$ since p is a projective δ_{ss}-cover. Hence $h(E)$ is a δ_{ss}-supplement of S in W contained in T. That means W is an amply δ_{ss}-supplemented module. □

Definition 2.3. *([11]) A module W is δ_{ss}-lifting if for any submodule S of W there exists a decomposition $W = A \oplus B$ such that $A \leq S$ and $S \cap B \leq Soc_\delta(B)$ where $Soc_\delta(B) = \delta(B) \cap Soc(B)$.*

Motivated by the above theorem and Theorem 5.6 given in [14], we give the following useful lemma.

Lemma 2.4. *Let W be a projective module. Then the listed statements given below are equivalent:*

1. W is δ_{ss}-perfect.
2. W is δ_{ss}-lifting.
3. W is $\oplus$-δ_{ss}-supplemented.
4. W is amply δ_{ss}-supplemented.
5. W is δ_{ss}-supplemented.

Proof. (1) $\Longleftrightarrow$ (2) and is evident by [14, Theorem 5.6].
(2) $\Rightarrow$ (3) is clear by definitions.
(1) $\Rightarrow$ (4) is clear by Lemma 2.2.
(3) $\Rightarrow$ (5) and (4) $\Rightarrow$ (5) are clear.
(5) $\Rightarrow$ (2) is clear by [11, Proposition 6]. □

Corollary 2.5. *A ring R is δ_{ss}-perfect if and only if R is δ_{ss}-lifting.*

In [8] Mares proved that a projective cover of a semiperfect module is also semiperfect. Inspired by this, we prove the analogous for δ_{ss}-perfect modules in the next theorem. Owing to this theorem, it is possible to restrict the class of δ_{ss}-perfect modules to the class of projective δ_{ss}-perfect modules.

Theorem 2.6. *Let E be a projective δ_{ss}-cover of W. Then two statements given below are equivalent:*

1. E is a δ_{ss}-perfect module.
2. W is a δ_{ss}-perfect module.

Proof. (1) $\Rightarrow$ (2) : By hypothesis, there exists an epimorphism $f : E \longrightarrow W$ with $Ker(f)$ is δ-small in E and semisimple. Let $\pi : W \longrightarrow \frac{W}{S}$ be the natural homomorphism for any submodule S of W. Since $\frac{E}{Ker(\pi\circ f)} \cong \frac{W}{S}$ and E is δ_{ss}-perfect, then $\frac{W}{S}$ has a projective δ_{ss}-cover. Hence, W is a δ_{ss}-perfect module.

(2) $\Rightarrow$ (1) : Let W be a δ_{ss}-perfect module. It is enough to verify that E is δ_{ss}-supplemented by Lemma 2.4. Let $T \leq E$ and $\pi : W \longrightarrow \frac{W}{f(T)}$ be the natural homomorphism. Then we have the composition map $h := \pi \circ f : E \longrightarrow \frac{W}{f(T)}$ and $\frac{W}{f(T)}$ has a projective δ_{ss}-cover, since W is δ_{ss}-perfect. Following $E = E' \oplus E''$ with $h_1 = h\mid_{E'}: E' \longrightarrow \frac{W}{f(T)}$ is a projective δ_{ss}-cover and $E'' \leq Ker(h)$, by [17, Lemma 2.3]. It follows that $E = E' + Ker(h)$ by $h(E) = h(E')$. Following $E = E' + Ker(h) = E' + Ker(\pi \circ f) = E' + f^{-1}(Ker(\pi)) = E' + f^{-1}(f(T)) = (E' + T) + Ker(f)$ is obtained. As $Ker(f) \ll_\delta E$, there exists a projective semisimple submodule Y of $Ker(f)$ such that $E = (E' + T) \oplus Y = T + (E' \oplus Y)$. Now we claim that $T \cap (E' \oplus Y) \ll_\delta E' \oplus Y$ and $T \cap (E' \oplus Y)$ is semisimple. Here it is easy to see that $T \cap (E' \oplus Y) = T \cap E'$ as $E = (E' + T) \oplus Y$. Since h_1 is a projective δ_{ss}-cover, $E' \cap (T + Ker(f)) \leq Ker(h_1)$ is semisimple and also δ-small in E'. Finally since $T \cap (E' \oplus Y) = T \cap E' \leq E' \cap (T + Ker(f))$, then $T \cap (E' \oplus Y) \ll_\delta E' \oplus Y$ and semisimple by [17, Lemma 1.2] and [4, Cor. 8.1.5]. □

For any prime integer p, although the $\mathbb{Z}$-module $\mathbb{Z}_p$ is δ_{ss}-lifting, it does not have a projective δ_{ss}-cover. As a result, it is possible to say that a δ_{ss}-lifting module need not be δ_{ss}-perfect.

Lemma 2.7. *Let E be a projective δ_{ss}-perfect module. Then $\delta(E) = Soc_\delta(E) = Soc(E)$.*

Proof. *Case* 1 : Let $E = \delta(E)$. Then E is a projective semisimple module by [10, Lemma 2.6] and [14, Prop. 3.1.(4)]. So $\delta(E) = E = Soc_\delta(E) = Soc(E)$ is obtained.

Case 2 : Let $E \neq \delta(E)$. Since E is projective and δ_{ss}-perfect, then E is also δ_{ss}-lifting by Lemma 2.4. So there occures a decomposition $E = A \oplus B$ such that $A \leq \delta(E)$ and $\delta(E) \cap B \leq Soc_\delta(Y)$. For the projection $\pi : E \longrightarrow A$, we have $\pi(\delta(E)) = \pi(\delta(A \oplus B)) = \pi(\delta(A) \oplus \delta(B)) = \delta(A) = A \cap \delta(E) = A$ and so A is projective semisimple by [10, Lemma 2.6]. Therefore $\delta(E) = \delta(A) \oplus \delta(B) = A \oplus (\delta(E) \cap B) \ll_\delta A \oplus B = E$ and $\delta(E)$ is also semisimple since A and $\delta(E) \cap B$ are semisimple. Hence $\delta(E) \leq Soc_\delta(E)$. Thus, $Soc_\delta(E) = \delta(E) \cap Soc(E) = \delta(E)$, we have $\delta(E) \leq Soc(E)$. Otherwise, as E is projective, by [14, Prop. 5.2] $Soc(E) \ll_\delta E$ and so $Soc(E) \leq \delta(E)$ is obtained. □

Proposition 2.8. *If W is a δ_{ss}-perfect module, then $\delta(W) = Soc_\delta(W) \leq Soc(W)$. In particular $\frac{W}{Soc_\delta(W)}$ is semisimple.*

Proof. Let us assume that W is δ_{ss}-perfect. Thus, W has a projective δ_{ss}-cover $f : E \longrightarrow W$ and E is also δ_{ss}-perfect by Lemma 2.7. Moreover, $\delta(E) \ll_\delta E$ and $\delta(E)$ is semisimple. Since $Ker(f) \ll_\delta E$ by [10, Lemma 2.8], then $f(\delta(E)) = \delta(W)$ is semisimple and it is also δ-small in W from [17, Lemma 1.2] and [4, Cor. 8.1.5]. Hence, $\delta(W) = Soc_\delta(W) \leq Soc(W)$ is obtained. In particular, as W is also δ_{ss}-supplemented by Lemma 2.2, then $\frac{W}{Soc_\delta(W)}$ is semisimple by [14, Prop. 4.7]. □

It is evident from given definitions that, δ_{ss}-perfect modules are restricted versions of δ-semiperfect modules. Now we give a relation between these concepts.

Proposition 2.9. *The following statements are equivalent for a projective module W:*

1. *W is δ_{ss}-perfect.*
2. *W is δ-semiperfect and $\delta(W) \subseteq Soc(W)$.*

Proof. $(1 \Longrightarrow 2)$: Assume that W is δ_{ss}-perfect. Then it is clear that W is also δ-semiperfect by definitions and $\delta(W) = Soc(W)$ by Lemma 2.7.

$(2 \Longrightarrow 1)$: Let W be a projective δ-semiperfect module. Then W is amply δ-supplemented by [10, Lemma 2.4]. As $\delta(W) \subseteq Soc(M)$, W is also δ_{ss}-supplemented by [14, Theorem 4.19]. Hence, W is a δ_{ss}-perfect module by Lemma 2.4. □

Example 2.10. *Let F be a field, $I = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ and*

$R = \{(x_1, x_2, ..., x_n, x, x, ...) \mid n \in \mathbb{N},\ x_i \in M_2(F),\ x \in I\}$.
is a ring with the component-wise operations such that

$$\delta(R) = \{(x_1, x_2, ..., x_n, x, x, ...) \mid n \in \mathbb{N},\ x_i \in M_2(F),\ x \in J\}\text{, and}$$

$$Soc(R) = \{(x_1, x_2, ..., x_n, 0, 0, ...) \mid n \in \mathbb{N},\ x_i \in M_2(F)\}\text{ where}$$

$J = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$.

It can be seen in [17, Example 3.3] that, ${}_RR$ is a δ-perfect ring and so it is δ-semiperfect. But by Proposition 2.9, ${}_RR$ is not a δ_{ss}-perfect ring as $\delta(R) \nleq Soc(R)$.

In [10], the authors gave a characterization for projective δ-semiperfect modules as follows. A projective module W is semiperfect if and only if $\delta(W) \ll_\delta W$, $\frac{W}{\delta(W)}$ is semisimple and each direct decomposition of $\frac{W}{\delta(W)}$ can be lifted to W. Motivated by this fact, we give a new characterization for projective δ_{ss}-semiperfect modules. But firstly, let us give a useful lemma that we need.

Lemma 2.11. *The implications given below are equivalent for a projective module W:*

1. *W is δ_{ss}-lifting.*
2. $\frac{W}{Soc_\delta(W)}$ *is semisimple and for any* $\overline{X} = \frac{X+Soc_\delta(W)}{Soc_\delta(W)} \leq_\oplus \frac{W}{Soc_\delta(W)}$, *there exists a direct summand A of W such that* $\overline{X} = \overline{A}$.

Proof. $(1 \Longrightarrow 2)$: Let W be a δ_{ss}-lifting module. Since W is also δ_{ss}-supplemented, then $\frac{W}{Soc_\delta(W)}$ is semisimple by [14, Proposition 4.7]. From assumption, there exists direct summands A, B of X with $X = A \oplus B$, such that $A \leq_\oplus W$ and $B \leq Soc_\delta(W)$. Hence $\frac{X+Soc_\delta(W)}{Soc_\delta(W)} = \frac{A+Soc_\delta(W)}{Soc_\delta(W)}$ is obtained, i.e., $\overline{X} = \overline{A}$.

$(2 \Longrightarrow 1)$: Let S be any submodule of W. Since $\frac{W}{Soc_\delta(W)}$ is semisimple, we have $\frac{S+Soc_\delta(W)}{Soc_\delta(W)} \leq_\oplus \frac{W}{Soc_\delta(W)}$ and from assumption, there exists a submodule $D \leq_\oplus W$ such that $\frac{S+Soc_\delta(W)}{Soc_\delta(W)} = \frac{D+Soc_\delta(W)}{Soc_\delta(W)}$. It follows that $W = D \oplus Y$ for a submodule Y of M and so $\frac{W}{Soc_\delta(W)} = \frac{(D+Y+Soc_\delta(W))}{Soc_\delta(M)} = \frac{(S+Y+Soc_\delta(W))}{Soc_\delta(W)}$. Since $Soc_\delta(W) \ll_\delta W$ from [14, Proposition 3.1(2)], then there exists a projective semisimple submodule of E of $Soc_\delta(W)$ such that $W = (S+Y) \oplus E$. Then, $S + Y$ is projective as a direct summand of W. From [16, 41.14], we have $S + P = S' \oplus Y$ with $S' \leq S$. Thus, $W = S' \oplus (Y \oplus E)$ and even as $\frac{W}{Soc_\delta(W)} = \frac{(S+Soc_\delta(W))}{W)}$, we have $S \cap (Y \oplus Soc_\delta(M)) \leq Soc_\delta(W)$. Hence, we have $S \cap (Y \oplus E) = S \cap Y \leq S \cap (Y \oplus Soc_\delta(W)) \leq Soc_\delta(W) \ll_\delta W$. □

Theorem 2.12. *The listed implications given below are equivalent for a projective module W:*

1. W is δ_{ss}-perfect.
2. $\frac{W}{Soc_\delta(W)}$ is semisimple and each direct summand of $\frac{W}{Soc_\delta(W)}$ is an image of a direct summand of W.
3. $\frac{W}{Soc_\delta(W)}$ is semisimple and each direct composition of $\frac{W}{Soc_\delta(W)}$ is lifted to a direct composition of W.

Proof. $(1) \Longrightarrow (3)$: It is clear by Lemma 2.11.

$(3) \Longrightarrow (2)$: It is clear by statements.

$(2) \Longrightarrow (1)$: Let $\pi : W \longrightarrow \frac{W}{Soc_\delta(W)}$ be the natural homomorphism. It is known that $Ker(\pi) = Soc_\delta(W) \ll_\delta W$ by [14, Prop. 3.1]. It is enough to show that W is δ_{ss}-supplemented by Lemma 2.7. Let $A \leq W$. Then, $\frac{A+Soc_\delta(W)}{Soc_\delta(W)} \oplus \frac{B}{Soc_\delta(B)} = \frac{W}{Soc_\delta(W)}$ as $\frac{W}{Soc_\delta(W)}$ is semisimple. By hypothesis, there exists $D \leq_\oplus W$ with $\frac{D+Soc_\delta(W)}{Soc_\delta(W)} = \frac{B}{Soc_\delta(W)}$ which implies $D + Soc_\delta(W) = B$. It follows that $A + D + Soc_\delta(W) = W$ is obtained. By hypothesis, there exists a projective semisimple submodule E of $Soc_\delta(P)$ with $A + (D \oplus E) = W$. Now, let us show that $A \cap (D \oplus E)$ is semisimple and it is δ-small in $D \oplus E$ to complete the proof. It can be easily verified that $A \cap (D \oplus E) = A \cap D \leq A \cap (D + Soc_\delta(W)) = A \cap B \leq Soc_\delta(W) \ll_\delta W$ as $\frac{A+Soc_\delta(W)}{Soc_\delta(W)} \oplus \frac{B}{Soc_\delta(B)} = \frac{W}{Soc_\delta(W)}$. From here, $A \cap (D \oplus E)$ is semisimple as a submodule of the semisimple module (see in [14, Prop. 3.1]) $Soc_\delta(W)$ and it is also δ-small in W. Moreover, since $D \leq_\oplus W$ and $D \leq X \oplus E$, then we get $A \cap (D \oplus E) \ll_\delta D \oplus E$ by [13, Lemma 1.2] and [17, Lamma 2.1]. □

Since all rings with identity is projective, the above theorem verifies the structure of δ_{ss}-perfect rings also characterized in [14, Theorem 5.3]. So, we can repeat the following corollary owing to this theorem.

Corollary 2.13. *R is a δ_{ss}-perfect ring if and only if* $\frac{R}{Soc(R)}$ *is semisimple and idempotents lift to module* $Soc(R)$.

Proof. As R is projective and $Soc(R) \leq S(R)$ by [14, Prop. 5.2], $Soc_\delta(R) = Soc(R) \cap S(R) = Soc(R)$ is obtained. Thus, the proof is completed by Theorem 2.12. □

In the following theorem the necessary and sufficient conditions are determined for a projective module W to be δ_{ss}-perfect.

Theorem 2.14. *A projective module W is δ_{ss}-perfect if and only if each proper submodule of W is included by a maximal submodule of W and there exists a projective δ_{ss}-cover for each simple factor module of W.*

Proof. ($\Longrightarrow$) : The first statement is evident by [10, Theorem 2.15] as every δ_{ss}-perfect module is δ-semiperfect. Since W is projective δ_{ss}-perfect, then $\delta(W) = Soc_\delta(W)$ by Lemma 2.7. And as W is δ_{ss}-supplemented by Proposition 2.9, then $\frac{W}{\delta(W)} = \frac{W}{Soc_\delta(W)}$ is semisimple by [14, Prop. 4.7]. So the second implication is verified.

($\Longleftarrow$) : For the necessity we show that W provides the conditions of Theorem 2.12. Let us assume that $\frac{W}{Soc_\delta(W)}$ is semisimple.

Case 1 : If W is semisimple, then $\frac{W}{Soc_\delta(W)}$ is semisimple as a factor module of W.

Case 2 : Let W is not semisimple and let $\pi : W \longrightarrow \frac{W}{Soc_\delta(W)}$ be the natural homomorphism. Now, we want to show that $\frac{W}{Soc_\delta(W)}$ is semisimple. Suppose that $\frac{W}{Soc_\delta(W)}$ is not semisimple. Then, there exists a submodule of $\frac{W}{Soc_\delta(W)}$ which is not a direct summand. For this proper essential submodule K of $\frac{W}{Soc_\delta(W)}$, $\pi^{-1}(K)$ is a proper essential submodule of W. From assumption, there exists a maximal submodule A of W containing $\pi^{-1}(K)$ and $\frac{W}{A}$ has a projective δ_{ss}-cover as a simple factor module. Following we have a decomposition $W = X \oplus Y$ such that $\pi|_X : X \longrightarrow \frac{W}{A}$ is a projective δ_{ss}-cover and $Y \leq Ker(\pi|_X)$ by [17, Lemma 2.3]. Therefore, we have $Y \leq Ker(\pi|_X) = X \cap A \leq Soc_\delta(X) \leq Soc_\delta(W)$ and $A \leq W$ is maximal $A + X = W$. So, $\frac{W}{Soc_\delta(W)} = \frac{A+X}{Soc_\delta(W)} = \frac{A}{Soc_\delta(W)} \oplus \frac{X+Soc_\delta(W)}{Soc_\delta(W)}$ as $X \cap A \leq Soc_\delta(W)$. As $\pi^{-1}(K) \leq A$ and $K \trianglelefteq \frac{W}{Soc_\delta(W)}$, then $K \leq \frac{A}{Soc_\delta(W)} \trianglelefteq \frac{W}{Soc_\delta(W)}$. Then, $\frac{X+Soc_\delta(W)}{Soc_\delta(W)} = \frac{Soc_\delta(W)}{Soc_\delta(W)}$ and so we get $X \leq Soc_\delta(W)$. As $Y \leq Soc_\delta(W)$, we have that $W = X + Y \leq X + Soc_\delta(W) = Soc_\delta(W)$. Hence $W = Soc_\delta(W)$ and so W is projective semisimple by [14, Prop. 3.1(4)]. This is a contradiction. Now in the remaining part of the proof it will be shown that each direct decomposition of $\frac{W}{Soc_\delta(W)}$ can be lifted to a direct composition of W. Let $\frac{W}{Soc_\delta(W)} = \oplus_{i\in I} D_i$. Since $\frac{W}{Soc_\delta(W)}$ is semisimple, then each D_i is semisimple as a submodule of $\frac{W}{Soc_\delta(W)}$. So each D_i can be written as $D_i = \oplus_{j\in J} B_j$ where each B_j is simple. Then each B_j has a projective δ_{ss}-cover $f_j : W_j \longrightarrow B_j$ with $Ker(f_i) \leq Soc_\delta(W_j) \leq \delta(W_j)$ as a simple factor module of W. Hence, we have a homomorphism $\oplus_{j\in J} f_j : \oplus_{j\in J} W_j \longrightarrow \oplus_{j\in J} B_j = D_i$ where $\oplus_{j\in J} W_j$ is projective and $Ker(\oplus_{j\in J} f_j) \leq \delta(\oplus_{j\in J} W_j)$. So the composition $\frac{W}{Soc_\delta(W)} = \oplus_{i\in I} D_i$ can be lifted to a direct composition of W by [10, Lemma 2.10]. □

Now we generalize Theorem 2.14 for δ_{ss}-perfect modules.

Theorem 2.15. *Let W be a module. W is a δ_{ss}-perfect module if and only if there exists a projective δ_{ss}-cover for each simple factor module of W and each proper submodule of W is included by a maximal submodule of W.*

Proof. ($\Longrightarrow$) : Let W be a δ_{ss}-perfect module. Then, the first statement is evident by the concept of a δ_{ss}-perfect module. The second one is clear by Proposition 2.8 and [14, Theorem 2.7].

($\Longleftarrow$) : Let $f : E \longrightarrow W$ be a projective δ_{ss}-cover of W. It is enough to show that E is δ_{ss}-perfect. So, it remains to show that E satisfies the sufficiency conditions of Theorem 2.14. Let $\frac{E}{T}$ be any simple factor module of W. Then, $T \leq E$ is maximal.

Case 1 : Let $Ker(f) \leq T$. As $\frac{E}{T}$ is simple, then it is cyclic and so it is free. That means $\frac{E}{T}$ is also projective. So, it has a projective δ_{ss}-cover naturally.

Case 2 : Let $Ker(f) \nleq T$. By the maximality of T, we have $T + Ker(f) = E$. As $Ker(f) \leq Soc_\delta(E) \ll_\delta E$, there exists a semisimple projective $P \leq Ker(f)$ with $P \oplus K = E$. So $\frac{E}{T}$ has a projective δ_{ss}-cover, as it is projective.

Let $A \lneq P$.

Case 1 : Let $\pi(A) = W$. Then $A + Ker(f) = E$. So, there exists a projective semisimple submodule P of $Ker(f)$ with $A \oplus P = E$. Thus, each proper submodule of $P \cong \frac{E}{A}$ is included by a maximal submodule. Hence, A is included by a maximal submodule of E.

Case 2 : Let $\pi(A) \neq W$. Thus, $\pi(A)$ is included by a maximal $T \leq W$. Hence, $f^{-1}(T) \leq P$ is maximal where $A \leq f^{-1}(T)$. □

Theorem 2.16. *Let $\{W_i\}_{i\in I}$ be a community of projective δ_{ss}-perfect modules. Then $W = \oplus_{i\in I} W_i$ is δ_{ss}-perfect if and only if each W_i is δ_{ss}-perfect.*

Proof. ($\Longrightarrow$) : The claim given in the necessity part is clear.

($\Longleftarrow$) : For the claim given in the sufficiency part firstly, point that $\delta(W_i) = Soc(W_i) = Soc_\delta(W_i)$ for each $i \in I$ as each W_i is δ_{ss}-perfect, by Lemma 2.7. And it is a known that, as $W = \oplus_{i\in I} W_i$, $\oplus(\delta(W_i)) = \oplus(Soc(W_i))$

implies that $\delta(W) = Soc(W) \ll_\delta W$ by [14, Prop. 5.2]. Now since each W_i is projective δ_{ss}-perfect, then each W_i is also δ-semiperfect for every $i \in I$. Thus, W be a δ-semiperfect module by [10, Cor. 2.18]. Hence, W is a δ_{ss}-perfect module by Proposition 2.9. □

It is known that a module W is local iff W is a cover of simple module. Moreover, each projective semiperfect module E is of a direct composition of local modules and $Rad(E) \ll E$ [3].

In [10], defining δ-local modules as δ-covers of simple modules, a characterization has been given for projective δ-semiperfect modules as follows.

Lemma 2.17. *(see in [10, Cor. 2.22]) A module W is projective δ-semiperfect if and only if W is a direct sum of δ-local modules and $\delta(W) \ll_\delta W$.*

Motivated by these, first we define δ_{ss}-local modules and give a characterization for projective δ_{ss}-perfect modules via δ_{ss}-local modules.

Definition 2.18. *A module W is δ_{ss}-local, if it is a δ_{ss}-cover of a simple module.*

Remark 2.19. *If W is δ_{ss}-local, then there exists a δ_{ss}-cover from W to a simple module B. Note that B is a projective module as it is cyclic. Since B is δ_{ss}-supplemented, then it is also δ_{ss}-perfect by Lemma 2.4. Hence, W is a δ_{ss}-perfect module by Theorem 2.6.*

Since each projective δ_{ss}-local module is δ_{ss}-perfect then the following corollary is obtained.

Corollary 2.20. *A projective module W is δ_{ss}-perfect if and only if W is a direct sum of projective δ_{ss}-local modules.*

Proof. $(\Longrightarrow)$: Let W is δ_{ss}-perfect and $\pi : W \longrightarrow \frac{W}{Soc_\delta(W)}$ be the natural homomorphism. So W is a δ_{ss}-cover of $\frac{W}{Soc_\delta(W)}$. Then, by Theorem 2.12, $\frac{W}{Soc_\delta(W)}$ is semisimple and it is of a direct composition as a direct sum of simple modules. As W is projective δ_{ss}-perfect, this direct decomposition is lifted to a diret sum of W, that is a direct sum of projective δ_{ss}-local modules.

$(\Longleftarrow)$: Clear by Theorem 2.12 and Remark 2.19. □

Corollary 2.21. *A ring R is δ_{ss}-perfect if and only if R is a direct sum of projective δ-local modules.*

Now we want to characterize the rings whose cyclic modules are δ_{ss}-lifting. But firstly, we give the following useful proposition.

Proposition 2.22. *A module W is δ_{ss}-lifting if and only if W is amply δ_{ss}-supplemented and each δ_{ss}-supplement submodule $T \leq W$ has a decomposition $T = A \oplus B$ with $A \leq_\oplus W$ and B is projective semisimple.*

Proof. $(\Longrightarrow)$: It is clear by the necessity part of Proposition 3.1 given in [10].

$(\Longleftarrow)$: Now, we will show that W is δ_{ss}-lifting. Since W is (amply) δ_{ss}-supplemented then each submodule $S \leq W$ is of a δ_{ss}-supplement T with $S+T = W$ and $S \cap T \leq Soc_\delta(T)$. Therefore, there occures a δ_{ss}-supplement $T^{'} \leq T$ contained in S, that is, $T + T^{'} = W$ and $T \cap T^{'} \leq Soc_\delta(T^{'})$. By hypothesis, $T^{'}$ is of a decomposition $T^{'} = A \oplus B$ where $A \leq_\oplus W$ and B is semisimple projective. Thus, there occures a submodule $A^{'} \leq W$ with $W = A \oplus A^{'}$. By modular law, as $T' \leq T$ and $U \leq T^{'} \leq T$, we have $S = S \cap W = S \cap (T + T^{'}) = T' + (S \cap T)$ and $S = S \cap W = S \cap (A \oplus A^{'}) = A \oplus (A^{'} \cap S)$. Thus, for the projection map $\pi : A \oplus A^{'} \longrightarrow A^{'}$, $A^{'} \cap S = \pi(S) = \pi(T') + \pi(S \cap T) = \pi(B) + \pi(S \cap T)$ and also, $\pi(T') + \pi(S \cap T) \leq Soc_\delta(A^{'})$ since B is projective semisimple and $S \cap T \leq Soc_\delta(T)$ by [[17, Lemma 2.2]; [4, Cor. 8.1.5] and [13, Lemma 1.2]]. Hence, W is δ_{ss}-lifting. □

Theorem 2.23. *The following implications are equivalent for a δ_{ss}-perfect module W:*

1. W is δ_{ss}-lifting.
2. W has a projective δ_{ss}-cover $f : E \longrightarrow W$ such that $f(D) = X \oplus Y$ where $X \leq_\oplus W$ and Y is projective semisimple for any direct summand D of E.

3. W has a projective δ_{ss}-cover $f : E \longrightarrow W$ such that $f(T) = X \oplus Y$ where $X \leq_{\oplus} W$ and Y is projective semisimple for each δ_{ss}-supplement $T \leq E$.

Proof. (1) $\Longrightarrow$ (3) : It is clear by [10, Theorem 3.5].

(3) $\Longrightarrow$ (2) : It is clear.

(2) $\Longrightarrow$ (1) : Since W is a δ_{ss}-perfect module, W is (amply) δ_{ss}-supplemented. For any δ_{ss}-supplement S of W, there exists a decomposition $S = f(X) \oplus L$ where $X \leq_{\oplus} E$ and $L \leq W$ is projective semisimple from [10, Lemma 3.3]. By hypothesis, $f(X) = A \oplus B$ with $A \leq_{\oplus} E$ and $B \leq W$ is projective semisimple. Thus, $S = A \oplus (B \oplus L)$ such that $B \oplus L$ is semisimple and $A \leq_{\oplus} E$. Hence, W is δ_{ss}-lifting by Proposition 2.22. □

Corollary 2.24. *Each cyclic left R-module is δ_{ss}-lifting for a ring R if and only if R is δ_{ss}-perfect and each cyclic left R-module W is of a projective δ_{ss}-cover $f : E \longrightarrow W$ such that $f(T) = X \oplus Y$ where $X \leq_{\oplus} W$ and Y is projective semisimple for each δ_{ss}-supplement $T \leq E$.*

Proof. ($\Longrightarrow$) : By hypothesis the module ${}_RR$ is δ_{ss}-lifting as a cyclic module. So, the ring R is a δ_{ss}-perfect by Corollary 2.5. Also, every (cyclic) R-module W is of a projective δ_{ss}-cover by [14, Corollary 5.7]. Thus, each factor module of W is of a projective δ_{ss}-cover that means W is a δ_{ss}-perfect module. Hence, the proof can be completed according to Theorem 2.23.

($\Longleftarrow$) : Let the ring R be δ_{ss}-perfect and W be a cyclic left R-module. Thus, W is a δ_{ss}-perfect module. So W is δ_{ss}-lifting by Theorem 2.23. □

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