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# On WD and WDMP generalized inverses in rings

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**Abstract.** Motivated by the very recent work of Gao, Y., Chen, J., Wang, J., Zou, H. [Comm. Algebra, 49(8) (2021) 3241-3254; MR4283143], we introduce two new generalized inverses named weak Drazin (WD) and weak Drazin Moore-Penrose (WDMP) inverses for elements in rings. A few of their properties are then provided, and the fact that the proposed generalized inverses coincide with different well-known generalized inverses under certain assumptions is established. Further, we discuss additive properties, reverse-order law and forward-order law for WD and WDMP generalized inverses. Then, we propose a binary relation called the WD order. Some examples are also provided in support of the theoretical results.

## 1. Introduction and Preliminaries

Let $R$ be a proper unitary ring with involution whose unity is 1. A ring $R$ is said to be a *proper ring* if $a^*a = 0 \implies a = 0$ for all $a \in R$. An *involution* $*$ is an anti-isomorphism that satisfies the conditions:

$$(a+b)^* = a^* + b^*, \ \ (ab)^* = b^*a^*, \text{ and } (a^*)^* = a \text{ for all } a, b \in R.$$

Let $a \in R$. Then, the *commutant* and the *double commutant* of $a$ are defined by

$$comm(a) = \{x \in R : ax = xa\},$$

and

$$comm^2(a) = \{x \in R : xy = yx \text{ for all } y \in comm(a)\},$$

respectively. By $N(R)$, we denote the set of all nilpotent elements of $R$. An element $a$ is said to be *Hermitian* if $a^* = a$, and is called *idempotent* if $a^2 = a$. An element $a$ is *quasinilpotent* if $1 + xa \in R^{-1}$ for all $x \in comm(a)$, where $R^{-1}$ denotes the set of all the standard invertible elements of $R$. An element $a \in R$ is *Moore-Penrose invertible* if there exists a unique element $x \in R$ that satisfies the equations:

$$(1.)axa = a, \ (2.)xax = x, \ (3.)(ax)^* = ax, \text{ and } (4.)(xa)^* = xa.$$

Then, $x$ is called as the *Moore-Penrose* [15] inverse of $a$, and is denoted as $x = a^\dagger$. By $R^\dagger$, we denote the set of all Moore-Penrose invertible elements of $R$. An element $a$ is called *Drazin invertible* [7] if there exists a

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unique element $x \in R$ such that $xa^{k+1} = a^k$, $ax = xa$, and $ax^2 = x$, for some positive integer $k$. If the Drazin inverse of $a$ exists, then it is denoted by $a^d$. The smallest positive integer $k$ is called the *Drazin index*, is denoted by $i(a)$. The set of all Drazin invertible elements of $R$ will be denoted by $R^d$. If $i(a) = 1$, then the Drazin inverse of $a$ is called the *group inverse* of $a$, and is denoted by $a^\#$. The set of group invertible elements of $R$ will be denoted by $R^\#$. If $a \in R^\dagger$ and it commutes with its Moore-Penrose inverse (i.e., $aa^\dagger = a^\dagger a$) is called EP element [14]. The set of all EP elements of $R$ will be denoted by $R^{EP}$.

In 2017, Xu *et al*. [26] proved that an element $a$ is *core invertible* if there exists a unique element $x \in R$ satisfying the following conditions:

$$(ax)^* = ax, \ \ ax^2 = x, \ \text{ and } \ xa^2 = a,$$

the core inverse of $a$ is denoted by $a^{\#\!\!\!\bigcirc}$. The set of all core invertible elements of $R$ will be denoted by $R^{\#\!\!\!\bigcirc}$. An element $a$ is *pseudo core invertible* [8] if there exists a unique element $x \in R$ such that

$$(ax)^* = ax, \ \ ax^2 = x, \ \text{ and } \ xa^{k+1} = a^k,$$

for some positive integer $k$. The least positive integer $k$ for which the above equations hold is called the *pseudo core index*, and is denoted by $I(a)$. The pseudo core inverse of $a$ is denoted by $a^{\textcircled{d}}$, and $R^{\textcircled{d}}$ denotes the set of all pseudo core invertible elements of $R$. An element $a \in R$ has generalized *Hirano inverse* [5] if there exists a unique element $x \in R$ such that $x = xax$, $x \in comm^2(a)$, and $(a^2 - ax) \in R^{qnil}$, where $R^{qnil}$ is the set of all quasinilpotent elements of $R$. Chen and Sheibani [5] proved the following results for Hirano invertible elements.

**Theorem 1.1.** *(Theorem 3.1, [5])*
*An element $a \in R$ has Hirano inverse if and only if $a - a^3 \in N(R)$.*

**Theorem 1.2.** *(Theorem 2.1, [5])*
*If $a \in R$ has Hirano inverse, then $a$ has Drazin inverse.*

**Corollary 1.3.** *(Corollary 2.8, [3])*
*Let $a \in A$, where $A$ is a Banach algebra. Then, the followings are equivalent:*

(i) *$a$ has generalized Hirano inverse,*
(ii) *there exists a unique idempotent element $p \in A$ such that $pa = ap$ and $a^2 - p \in A^{qnil}$.*

An element $a$ is said to be *right pseudo core invertible* [25] if there exists a unique element $x \in R$ such that $axa^k = a^k$, $ax^2 = x$, and $(ax)^* = ax$, for some positive integer $k$, is denoted as $a_r^{\textcircled{d}}$. The least positive integer $k$ for which the above equations hold is called the *right pseudo core index*, and is denoted by $I(a)$. The set of all right pseudo core invertible elements of $R$ is denoted by $R_r^{\textcircled{d}}$. In 2019, Zhu [28] introduced the DMP inverse for an element which is recalled next. Let $a \in R^d \cap R^\dagger$. Then any element $x$ satisfying $xax = x$, $xa = a^d a$, and $a^k x = a^k a^\dagger$ for some positive integer $k$, is called *DMP inverse* [28] of $a$. It is unique, and is denoted by $a^{d,\dagger}$. The smallest positive integer $k$ is called the *DMP index* of $a$. The set of all DMP invertible elements of $R$ is denoted by $R^{d,\dagger}$.

In 2016, Wang and Liu [24] proposed a new generalized inverse for matrices called G-Drazin inverse, and is as follows. Let $A \in \mathbb{C}^{n\times n}$. Then, a matrix $X \in \mathbb{C}^{n\times n}$ is called *G-Drazin inverse* of $A$ if

$$AXA = A, \ XA^{k+1} = A^k, \text{ and } A^{k+1}X = A^k,$$

where $k = ind(A)$. It is denoted by $X = A^{GD}$. In general, this inverse is not unique.

Recently, in 2022, Hernández *et al.* [11] introduced another generalized inverse called *GDMP inverse*. The definition of a GDMP inverse of a matrix is stated next. Let $A \in \mathbb{C}^{n\times n}$ and $k = ind(A)$. For each $A^{GD} \in A\{GD\}$, a GDMP inverse of $A$, denoted by $A^{GD\dagger}$, is the $n \times n$ matrix $A^{GD\dagger} = A^{GD}AA^\dagger$. This inverse is also not unique. Similarly, in 2021, Hernández *et al.* [12] introduced two new generalized inverses of rectangular complex matrices, namely 1MP and MP1-inverses, and showed that the binary relations induced for these new generalized inverses are partial orders. The definition of a partial order is recalled next. A binary relation is called *partial order* if it is reflexive, anti-symmetric and transitive. The term 'partial order' was initially proposed in ring theory [2]. Below, we recall some of the significant partial orders that have been introduced in the literature.

(a) $a \leq^{\#} b$, i.e., $aa^{\#} = ba^{\#}$ and $a^{\#}a = a^{\#}b$ is called the sharp partial order [23].

(b) $a \leq \# b$, i.e., $aa^{\#} = ba^{\#}$ and $b^{\circ} \subseteq a^{\circ}$ is called the right sharp partial order [19].

(c) $a\# \leq b$, i.e., $a^{\#}a = a^{\#}b$ and ${}^{\circ}b \subseteq {}^{\circ}a$ is called the left sharp partial order [19].

(d) $a \leq^{*} b$, i.e., $aa^{*} = ba^{*}$ and $a^{*}a = a^{*}b$ is called the star partial order [23].

If $a$ and $b$ are a pair of invertible elements, then $ab$ is also invertible and the inverse of the product $ab$ satisfying

$$(ab)^{-1} = b^{-1}a^{-1},$$

is called as the *reverse-order law*. On the other way,

$$(ab)^{-1} = a^{-1}b^{-1}$$

is known as the *forward-order law*. While the reverse-order law do not hold for different generalized inverses, the forward-order law is not true even for invertible elements. The *additive property* of invertible elements $a$ and $b$ is

$$(a + b)^{-1} = a^{-1} + b^{-1}.$$

Similarly, the *absorption law* for invertible elements $a$ and $b$ is

$$a^{-1}(a + b)b^{-1} = a^{-1} + b^{-1}.$$

In 1966, Greville [10] first obtained necessary and sufficient conditions for which the reverse-order law holds for the Moore-Penrose inverse in matrix form, i.e., $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$. Mosić and Djordjević [21] extended the reverse-order law involving the Moore-Penrose inverse in matrix setting to elements in ring. The same problem was also considered by several authors for other generalized inverses. For example, Deng [6] studied the reverse-order law for the group inverse on Hilbert space. In 2012, Mosić and Djordjević [20] extended the reverse-order law for the group inverse in Hilbert space to ring. In 2017, Zhu *et al.* [30] discussed the reverse-order law for the inverse along an element. In 2019, Xu *et al.* [27] studied the reverse-order law and the absorption law for the $(b, c)$-inverses in rings. Jin and Benitez [13] proved the absorption law for Moore-Penrose inverse, Drazin inverse, group inverse, core inverse and dual core inverse. In 2021, Gao *et al.* [9] provided the reverse-order law, the forward-order law and the absorption law for the generalized core inverse. In 2017, Zhu and Chen [32] provided the forward-order law and additive property for the Drazin inverse in a ring. In 2021, Li *et al.* [18] studied the forward-order law for the core inverse in matrix setting. In 2023, Kumar and Mishra [17] extended the forward-order law for the core inverse in ring setting. In 2022, Kumar *et al.* [16] discussed several results on additive properties, reverse-order law and forward-order law for GD inverse and GDMP inverse of matrices. Zhu *et al.* ([29], [30], [31]) and Zou *et al.* [33] provided several results on additive properties, reverse-order law and forward-order law. Zhu *et al.* [29] obtained the following additive property of the Moore-Penrose inverse.

**Theorem 1.4.** *(Lemma 2.3, [29])*
*Let $a, b \in R^{\dagger}$ such that $a^{*}b = ab^{*} = 0$. Then, $(a + b)^{\dagger} = a^{\dagger} + b^{\dagger}$.*

Very recently, Baksalary *et al.* [1] established necessary and sufficient conditions for two orthogonal projectors to be the Moore-Penrose additive.

This article aims to study reverse-order law, forward-order law, additive property, and absorption law for two new generalized inverses for elements in rings called weak Drazin (WD) inverse and weak Drazin Moore-Penrose (WDMP) inverse.

In this context, the article is organized as follows. First, we define WD and WDMP inverses which are extensions of GD and GDMP inverses, respectively. In Section 2, we illustrate some properties of WD and WDMP inverses. Also, we show that if WD and WDMP inverses exist, then the right pseudo core inverse, Drazin inverse, DMP inverse and Hirano inverse exist. In Section 3, we establish the reverse-order law, the forward-order law and the additive property for WD and WDMP inverse. We also propose a few results assuming the additive property, reverse-order law and forward-order law hold for WD inverse and WDMP inverse. At the end, we propose a binary relation the WD order in Subsection 3.1.

## 2. WD inverse and WDMP inverse

In this section, we propose two new generalized inverses called WD inverse and WDMP inverse for elements in a ring, and discuss some of their properties. We then define a relation between a WDMP inverse and the right pseudo core inverse. In this aspect, we first introduce the definition of a weak Drazin inverse.

**Definition 2.1.** *Let $a \in R$. If there exists an element $x \in R$ such that satisfies the following equations:*

$$axa = a, \quad a^{k+1}x = a^k, \text{ and } xa^{k+1} = a^k,$$

*then $x$ is called a weak Drazin inverse (WD inverse) of $a$. It is denoted by $x = a^{WD}$ and the smallest positive integer $k = ind(a)$ is the weak Drazin index of $a$. The set of all weak Drazin invertible elements of $R$ is denoted by $R^{WD}$.*

The next example demonstrates the existence of a WD inverse.

**Example 2.2.** *Let $R = \mathbb{Z}_{10}$ be a ring with conjugate involution and $2 = a \in R$. Then, $a^{WD} = 3$ and $a^{WD} = 8$. If $3 = a \in R = \mathbb{Z}_{10}$ be a ring with conjugate involution, then $a^{WD} = 7$. And if $R = \mathbb{Z}$ be a ring with conjugate involution and $0, 1 \neq a \in R$. Then, WD inverse of $a$ does not exist.*

The definition of a WDMP inverse is motivated by the definition of a GDMP inverse of a matrix.

**Definition 2.3.** *Let $a \in R^{WD} \cap R^{\dagger}$. Then, a weak Drazin Moore-Penrose inverse (WDMP inverse) of the element $a$ is defined as*

$$a^{WD\dagger} = a^{WD}aa^{\dagger}.$$

*The set of all weak Drazin Moore-Penrose invertible elements of $R$ is denoted by $R^{WD\dagger}$.*

**Remark 2.4.** *For an element $a \in R^{WD} \cap R^{\dagger}$ to be WDMP invertible, we need WD and Moore-Penrose invertibility of $a$. But, WD and Moore-Penrose inverses do not exist for all elements in rings. Hence, WDMP inverse does not exist for all elements in rings. For example, let $R = \mathbb{Z}$ be a ring with conjugate involution and $0, 1 \neq a \in R$. Then, WDMP inverse of $a$ does not exist.*

**Example 2.5.** *Let $R = \mathbb{Z}_{10}$ be a ring with conjugate involution and $2 = a \in R$. Then, $a^{WD\dagger} = 8$.*

A consequence of the above definition of a WD inverse is shown next as a corollary.

**Corollary 2.6.** *Let $a \in R^{WD}$. Then, $a \in R^d$.*

*Proof.* We have $a \in R^{WD}$. So, $aa^{WD}a = a$, $a^{WD}a^{k+1} = a^k$ and $a^{k+1}a^{WD} = a^k$ imply that $a^k \in a^{k+1}R \cap Ra^{k+1}$. It is well-known that if $a^k \in a^{k+1}R \cap Ra^{k+1}$, then $a \in R^d$. Hence, $a$ is Drazin invertible. □

Every idempotent element is weak Drazin invertible which is proved in the next result.

**Theorem 2.7.** *Let $p \in R$ be an idempotent element. Then, $p$ is WD invertible. Moreover, $p^{WD} = p$.*

*Proof.* We know that $p^2 = p$. Now, $ppp = p^2p = p^2 = p$. Similarly, $pp^{k+1} = p^k$ and $p^{k+1}p = p$ for every positive integer $k$. Hence, $p$ is WD invertible and every idempotent element is a self WD inverse. □

The next example is in the direction of Theorem 2.7.

**Example 2.8.** *Let* $R = \dfrac{\mathbb{Z}_2 \langle x, y\rangle}{\langle x^2 - x, y^2 - y\rangle}$ *be the ring generated over* $\mathbb{Z}_2$ *by* $\langle x, y\rangle$ *with relations* $\langle x^2 - x, y^2 - y\rangle$ *and* $a = x + \langle x^2-x, y^2-y\rangle$, $b = y + \langle x^2-x, y^2-y\rangle \in R$ *with conjugate involution. Now,* $a^2 = (x + \langle x^2-x, y^2-y\rangle)(x + \langle x^2-x, y^2-y\rangle) = x^2 + \langle x^2-x, y^2-y\rangle = x^2 + x - x + \langle x^2-x, y^2-y\rangle = x + \langle x^2-x, y^2-y\rangle = a$. *Similarly,* $b^2 = b$. *Both* $a$ *and* $b$ *are idempotent elements. Thus,* $a$ *and* $b$ *are self WD invertible.*

**Remark 2.9.** *Since* $aa^\dagger$ *is an idempotent element, so* $(aa^\dagger)^{WD} = aa^\dagger$.

**Theorem 2.10.** *If* $a \in R^{WD}$, *then there exist two idempotent elements say* $p$ *and* $q$ *such that*

$$axa = a, \ a^k p = 0, \ \text{and } qa^k = 0.$$

*Proof.* From Definition 2.1, we have

$$axa = a, \tag{1}$$

$$a^{k+1}x = a^k, \tag{2}$$

$$xa^{k+1} = a^k. \tag{3}$$

Now, $(ax)^2 = axax = ax$ by (1), so $ax$ is an idempotent element. Similarly, $(xa)^2 = xa$ is idempotent. Hence, $p = 1 - ax$ and $q = 1 - xa$ are also idempotent. From (2), we have $a^{k+1}x = a^k$, which implies $a^k(1 - ax) = 0$. So, $a^k p = 0$. Similarly, $qa^k = 0$. □

We show that $a^{WD}aa^\dagger$ is a solution of the following equations.

**Theorem 2.11.** *Let* $a \in R$ *with involution* $*$. *If* $a \in R^{WD\dagger}$, *then* $y = a^{WD}aa^\dagger$ *is a solution of these equations:*

$$yay = y, \ \ ay = aa^\dagger, \text{ and } ya^k = a^{WD}a^k.$$

*Proof.* Putting $y = a^{WD}aa^\dagger$ in equations $yay = y$, $ay = aa^\dagger$, and $ya^k = a^{WD}a^k$, we get

$$\begin{aligned} a^{WD}aa^\dagger aa^{WD}aa^\dagger &= a^{WD}aa^\dagger(aa^{WD}a)a^\dagger \\ &= a^{WD}aa^\dagger aa^\dagger \\ &= a^{WD}aa^\dagger, \end{aligned} \tag{4}$$

$$aa^{WD}aa^\dagger = (aa^{WD}a)a^\dagger = aa^\dagger, \tag{5}$$

and

$$a^{WD}aa^\dagger a^k = a^{WD}(aa^\dagger a)a^{k-1} = a^{WD}aa^{k-1} = a^{WD}a^k, \tag{6}$$

respectively. From (4), (5) and (6), we can say that $y = a^{WD}aa^\dagger$ is a solution of this system of equations. □

From Corollary 2.6 and Definition 2.3, we can say if $a$ is WDMP invertible, then it is DMP invertible.

**Corollary 2.12.** *Let* $a \in R^{WD\dagger}$. *Then,* $a \in R^{d\dagger}$.

*Proof.* If $a \in R^{WD\dagger}$, then $a \in R^{WD}$ and $a \in R^\dagger$ by Definition 2.3. From Corollary 2.6, $a \in R^d$. Therefore, $a \in R^d \cap R^\dagger$, i.e., there exists an element $x \in R$ such that $x = a^d aa^\dagger$. Hence, $a \in R^{d\dagger}$. □

Now, we prove some properties of a WDMP inverse with the help of Definition 2.3 and Theorem 2.11.

**Lemma 2.13.** *Let* $a \in R^{WD\dagger}$ *and* $y$ *be a WDMP inverse of* $a$. *Then, the following conditions hold:*

(i) $aya^k = a^k$, *for every positive integer* $k$.

(ii) *ay and ya are both idempotent elements.*
(iii) *there exists an idempotent element p such that* $pa^k = 0$.
(iv) $y(ay)^k = y$, *for every positive integer k.*
(v) $a^{k+1}ya = a^{k+1}$, *where* $k = ind(a)$.
(vi) $ya^{k+1}y = a^k a^\dagger$, *where* $k = ind(a)$.
(vii) $a^\dagger ay = a^\dagger$.

*Proof.* (i) From the 3rd property of Theorem 2.11 and the first property of Definition 2.1, we obtain $ya^k = a^{WD}a^k$ and $aa^{WD}a = a$. Now, $aya^k = aa^{WD}a^k = (aa^{WD}a)a^{k-1} = a^k$.
(ii) From Theorem 2.11, $yay = y$ imply that $ay$ and $ya$ are idempotents.
(iii) From the above part (i), we obtain $aya^k = a^k$. Then, $aya^k - a^k = 0$ implies $(1 - ay)a^k = 0$. By the above part (ii), we know that $ay$ is an idempotent element. So, $1 - ay$ is also an idempotent element. So, we can say $pa^k = 0$, where $p = 1 - ay$.
(iv) From Theorem 2.11, we get $yay = y$ and from part (ii) we know that $ay$ is an idempotent element. Hence, $(ay)^k = ay$ for every positive integer $k$. So, $y(ay)^k = yay = y$.
(v) From Definition 2.3, we have $y = a^{WD}aa^\dagger$. So,

$$a^{k+1}ya = a^{k+1}a^{WD}aa^\dagger a = a^{k+1}a^{WD}a = a^{k+1}.$$

(vi) From Definition 2.3, we obtain $y = a^{WD}aa^\dagger$. Hence,

$$a^{WD}aa^\dagger a^{k+1}a^{WD}aa^\dagger = a^{WD}a^{k+1}a^{WD}aa^\dagger = a^k a^{WD}aa^\dagger = a^k a^\dagger.$$

(vii) $a^\dagger ay = a^\dagger aa^{WD}aa^\dagger = a^\dagger aa^\dagger = a^\dagger$.
□

**Theorem 2.14.** *If* $a \in R^{WD\dagger}$, *then a is right pseudo core invertible. Moreover, WDMP inverse of a is the right pseudo core inverse of a.*

*Proof.* We know that $a \in R^{WD\dagger}$. So, by Definition 2.3, we have $yay = y$, $ay = aa^\dagger$. So, $(ay)^* = (aa^\dagger)^* = aa^\dagger = ay$. From Lemma 2.13 (i), we have $aya^k = a^k$. Hence, we have $yay = y$, $(ay)^* = ay$ and $aya^k = a^k$. So, $a$ is right pseudo core invertible. Thus, $y = a^{WD}aa^\dagger$ is the right pseudo core inverse of $a$. □

If $a$ is Hermitian, then the following properties hold.

**Theorem 2.15.** *If* $a \in R$ *is Hermitian and y is a WDMP inverse of a, then the following conditions hold:*

(i) $a^\dagger ay$ *is the group inverse of a.*
(ii) $a^2y = a$.
(iii) $y^2 = a^{WD}a^\dagger = ya^\dagger$.
(iv) $a^\dagger a$ *is a self WDMP inverse.*
(v) $a^{WD}a(aa^\dagger)^{k+1} = (aa^\dagger)^{WD}(aa^\dagger)^k$.
(vi) $aa^{WD}(aa^\dagger)^{k+1} = aa^\dagger$.

*Proof.* (i) Given that $a$ is Hermitian, so $a^* = a$. Now, $a^\dagger ayaa^\dagger ay = a^\dagger ayay = a^\dagger ay$, and from Lemma 2.13 (i), we know that $aya = a$ implies $aa^\dagger aya = aa^\dagger a = a$. Now, $a(a^\dagger ay) = ay = aa^\dagger = (aa^\dagger)^* = (a^\dagger)^* a^* = (a^*)^\dagger a^* = a^\dagger a = a^\dagger aa^{WD}a = a^\dagger aa^{WD}aa^\dagger a = a^\dagger aya$. Hence, $a^\dagger ay$ is the group inverse of $a$.
(ii) $a^2y = a^2a^{WD}aa^\dagger = a^2a^\dagger = a(aa^\dagger) = a^*(aa^\dagger)^* = a^*(a^\dagger)^*a^* = a^* = a$.
(iii) $y^2 = a^{WD}aa^\dagger a^{WD}aa^\dagger = a^{WD}(aa^\dagger)^* a^{WD}aa^\dagger = a^{WD}(a^\dagger)^* a^* a^{WD}aa^\dagger = a^{WD}a^\dagger aa^{WD}aa^\dagger = a^{WD}a^\dagger aa^\dagger = a^{WD}a^\dagger$. And $y^2 = a^{WD}(a^\dagger a)^* a^\dagger = a^{WD}a^*(a^\dagger)^* a^\dagger = a^{WD}aa^\dagger a^\dagger = ya^\dagger$.
(iv) Given that $a$ is Hermitian, so $a^* = a$ implies that $aa^\dagger = a^\dagger a$. We know that $aa^\dagger$ is an idempotent element and $a$ is Hermitian, which imply $(aa^\dagger)^\dagger = aa^\dagger$. From Theorem 2.7, we have $(aa^\dagger)^{WD} = aa^\dagger$. Now, $(aa^\dagger)^{WD\dagger} = (aa^\dagger)^{WD}(aa^\dagger)(aa^\dagger)^\dagger = aa^\dagger aa^\dagger aa^\dagger = aa^\dagger$. So, $a^\dagger a$ is a self WDMP inverse.

(v) From Theorem 2.7, we have $(aa^\dagger)^{WD} = aa^\dagger$. And $aa^\dagger = a^\dagger a$ because $a$ is Hermitian. Therefore, $a^{WD}a(aa^\dagger)^{k+1} = a^{WD}a(aa^\dagger)^k = a^{WD}aa^k(a^\dagger)^k = a^{WD}a^{k+1}(a^\dagger)^k = a^k(a^\dagger)^k = (aa^\dagger)^k = (aa^\dagger)^2 = aa^\dagger aa^\dagger = (aa^\dagger)^{WD}(aa^\dagger)^k$.

(vi) We know that $(aa^\dagger)^{k+1} = aa^\dagger$. So, $aa^{WD}(aa^\dagger)^{k+1} = aa^{WD}aa^\dagger = aa^\dagger$.

□

In 2012, Chen [14] provided the following result for EP elements.

**Theorem 2.16.** *(Theorem 2.4, [4])*
*An element $a \in R$ is EP if and only if $a \in R^\# \cap R^\dagger$ and one of the following equivalent conditions hold:*

(i) $a^n a^\dagger = a^\dagger a^n$, *for some* $n \geq 1$,
(ii) $(a^\#)^n a^\dagger = a^\dagger (a^\#)^n$, *for some* $n \geq 1$,
(iii) $(a^\dagger)^n = (a^\#)^n$, *for some* $n \geq 1$.

Every EP element is WDMP invertible. This is shown next.

**Theorem 2.17.** *If $a \in R^{EP}$, then $a$ is WD and WDMP invertible. Moreover, $a^\#$ is a WD inverse and WDMP inverse of $a$.*

*Proof.* By Theorem 2.16, we have $a \in R^\# \cap R^\dagger$. So, there exists an element $x \in R$ such that

$$axa = a, xax = x, \text{ and } ax = xa.$$

Now, from the above equations, we get $axa = a$, $a^2x = a$, and $xa^2 = a$. So, $a \in R^{WD}$ with WD index $k = 1$. Therefore, $a \in R^{WD} \cap R^\dagger$, and from Definition 2.3, we get $a \in R^{WD\dagger}$. From Theorem 2.11, $y = a^{WD}aa^\dagger$. We have $a^\# = a^\dagger = a^{WD}$. So, $y = a^{WD}aa^\dagger = a^\# aa^\# = a^\#$. □

Now, we recall the notion of *annihilators* of an element in a ring. The left annihilator of $a \in R$ is given by ${}^\circ a = \{x \in R : xa = 0\}$ and the right annihilator of $a$ is given by $a^\circ = \{x \in R : ax = 0\}$. The following lemma combines Lemma 2.5 and Lemma 2.6 of [22].

**Lemma 2.18.** *Let $a, b \in R$.*

(i) *If $aR \subseteq bR$, then ${}^\circ b \subseteq {}^\circ a$.*
(ii) *If $b$ is regular and ${}^\circ b \subseteq {}^\circ a$, then $aR \subset bR$.*
(iii) *If $Ra \subseteq Rb$, then $b^\circ \subseteq a^\circ$.*
(iv) *If $b$ is regular and $b^\circ \subseteq a^\circ$, then $Ra \subseteq Rb$.*

The next result provides a relation between WDMP inverse and annihilators.

**Theorem 2.19.** *Let $a \in R^{WD\dagger}$ and $y$ be a WDMP inverse of $a$. Then,*

(i) $Ry = Ra^*$.
(ii) $y^\circ = (a^*)^\circ$.

*Proof.* (i) From Definition 2.3 and Theorem 2.14, we have $yay = y$ and $(ay)^* = ay$. Now, $Ry = Ryay = Ry(ay)^* \subseteq R(ay)^* = Ry^*a^* \subseteq Ra^*$, i.e., $Ry \subseteq Ra^*$. Conversely, by Lemma 2.13 (i), we have $aya^k = a^k$ for any positive integer $k$. So, $aya = a$, taking involution to both sides, we get $a^*ay = a^*$. So, $Ra^* = Ra^*ay \subseteq Ry$. Hence, $Ry = Ra^*$.

(ii) We have $Ry = Ra^*$ implies $Ry \subseteq Ra^*$. By Lemma 2.18 (iii), $(a^*)^\circ \subseteq y^\circ$. Also, $Ry = Ra^*$ implies $Ra^* \subseteq Ry$. Again, by Lemma 2.18 (iii), $y^\circ \subseteq (a^*)^\circ$. Hence, $y^\circ = (a^*)^\circ$.

□

Next, we present $ay$ and $ya$ are Hirano invertible.

**Remark 2.20.** *Let $a \in R^{WD\dagger}$ and $y$ be a WDMP inverse of $a$. Then, $ay$ and $ya$ are both Hirano invertible.*

The following corollary directly follows from Remark 2.20 and Corollary 1.3.

**Corollary 2.21.** *Let $a \in R^{WD\dagger}$ and $y$ be a WDMP inverse of $a$. Then, there exists an unique idempotent element $p$ such that $ayp = pay$ and $(ay)^2 - p \in R^{qnil}$.*

**Theorem 2.22.** *Let $a \in R^{WD\dagger}$ and $y$ be a WDMP inverse of $a$. Then, $ay$ and $ya$ are both Drazin invertible.*

*Proof.* It is clear from Theorem 1.2 and Remark 2.20. □

We present a theorem for idempotent elements that is used to prove upcoming results.

**Theorem 2.23.** *Let $a, b \in R$, and $x, y \in R$ be two idempotent elements. Then, the following holds:*

*(i)* $(1 - x)a = b$ *if and only if* $xb = 0$ *and* ${}^{\circ}x \subseteq {}^{\circ}(a - b)$,

*(ii)* $a(1 - y) = b$ *if and only if* $by = 0$ *and* $y^{\circ} \subseteq (a - b)^{\circ}$.

*Proof.* (i) Let $(1 - x)a = b$. Pre-multiplying by $x$ on both sides, we get

$$x(1 - x)a = xb, \text{ i.e., } 0 = xb.$$

Now, $m \in {}^{\circ}x$ implies

$$mxa = 0, \text{ i.e., } m(a - b) = 0.$$

So, $m \in {}^{\circ}(a - b)$. Hence, ${}^{\circ}x \subseteq {}^{\circ}(a - b)$ and $xb = 0$.
Conversely: $(1 - x) \in {}^{\circ}x$, yields $(1 - x) \in {}^{\circ}(a - b)$. So, $(1 - x)(a - b) = 0$, which implies $(1 - x)a = (1 - x)b$, i.e., $(1 - x)a = b - xb$. We know that $xb = 0$, so

$$(1 - x)a = b.$$

(ii) Let $a(1 - y) = b$. Post-multiplying by $y$ on both sides, we get

$$a(1 - y)y = by, \text{ i.e., } by = 0.$$

Now, $m \in y^{\circ}$ implies

$$aym = 0, \text{ i.e., } (a - b)m = 0.$$

So, $m \in (a - b)^{\circ}$. Hence, $y^{\circ} \subset (a - b)^{\circ}$ and $by = 0$.
Conversely: $(1 - y) \in y^{\circ}$, yields $(1 - y) \in (a - b)^{\circ}$. So, $(a - b)(1 - y) = 0$, which implies $a(1 - y) = b(1 - y)$, i.e., $a(1 - y) = b - by$. We know that $by = 0$, we get

$$a(1 - y) = b.$$

□

As $a^{WD}a$ and $aa^{WD}$ are both idempotent elements, by Theorem 2.23 we obtain the following corollaries.

**Corollary 2.24.** *Let $a \in R^{WD}$. Then, for any $b, c \in R$,*

(i) $(1 - a^{WD}a)b = c$ *if and only if* $a^{WD}ac = 0$ *and* ${}^{\circ}(a^{WD}a) \subset {}^{\circ}(b - c)$.
(ii) $b(1 - a^{WD}a) = c$ *if and only if* $ca^{WD}a = 0$ *and* $(a^{WD}a)^{\circ} \subset (b - c)^{\circ}$.

Similarly, $ay$ and $ya$, (where $y$ is a WDMP inverse of $a$) are both idempotent elements, hence the following holds.

**Corollary 2.25.** *Let $a \in R^{WD\dagger}$. Then, for any $b, c \in R$,*

(i) $(1 - a^{WD\dagger}a)b = c$ *if and only if* $a^{WD\dagger}ac = 0$ *and* ${}^{\circ}(a^{WD\dagger}a) \subseteq {}^{\circ}(b - c)$.
(ii) $b(1 - a^{WD\dagger}a) = c$ *if and only if* $ca^{WD\dagger}a = 0$ *and* $(a^{WD\dagger}a)^{\circ} \subseteq (b - c)^{\circ}$.

We end this section with the annihilator property of a WDMP inverse.

**Corollary 2.26.** *If $a \in R^{WD\dagger}$ and $y$ be a WDMP inverse of $a$, then*

$$(a^{WD}a)^{\circ} \subseteq (ya)^{\circ} \subseteq (a^{k+1})^{\circ}.$$

*Proof.* We know $a^{k+1}(1 - a^{WD}a) = 0$. By Theorem 2.23, $(a^{WD}a)^{\circ} \subseteq (a^{k+1})^{\circ}$, i.e., $(a^{WD}a)^{\circ} = (a^{WD}aa^{\dagger}a)^{\circ} \subseteq (a^{k+1})^{\circ}$, i.e, $(a^{WD}a)^{\circ} = (ya)^{\circ} \subseteq (a^{k+1})^{\circ}$. □

## 3. Reverse-order law, Forward-order law and Additive property

In this section, we present the additive property, the reverse-order law and the forward-order law for WD inverse and WDMP inverse, respectively. We start this section with an example that shows the additive property does not always hold for WD inverse.

**Example 3.1.** *Let* $2+ <x^2+x>, 4+ <x^2+x> \in \dfrac{\mathbb{Z}_{10}[x]}{<x^2+x>}$ *with conjugate involution. Let* $3+ <x^2+x>$ *be a WD inverse of* $2+ <x^2+x>$ *and* $4+ <x^2+x>$ *be a WD inverse of* $4+ <x^2+x>$*. Then,* $(2+ <x^2+x> +4+ <x^2+x>)^{WD} = (6+ <x^2+x>)^{WD} = 6+ <x^2+x>$*. But*

$$(2+ <x^2+x>)^{WD} + (4+ <x^2+x>)^{WD} = 3+ <x^2+x> +4+ <x^2+x> = 7+ <x^2+x> .$$

Similarly, the fact that the additive property for WDMP inverse does not hold is shown in the next example:

**Example 3.2.** *Let* $2+ <x^2>, 4+ <x^2> \in \dfrac{\mathbb{Z}_{10}[x]}{<x^2>}$ *with conjugate involution. Let* $8+ <x^2>$ *be a WDMP inverse of* $2+ <x^2>$ *and* $4+ <x^2>$ *be a WDMP inverse of* $4+ <x^2>$*. Then,* $(2+ <x^2> +4+ <x^2>)^{WD\dagger} = (6+ <x^2>)^{WD\dagger} = 6+ <x^2>$*. But*

$$(2+ <x^2>)^{WD\dagger} + (4+ <x^2>)^{WD\dagger} = 8+ <x^2> +4+ <x^2> = 2+ <x^2> .$$

Next, we establish a result for the additive property involving WD inverse.

**Theorem 3.3.** *Let* $a, b \in R^{WD}$*. If* $ab = ba = 0$*,* $ab^{WD} = b^{WD}a = 0$*, and* $a^{WD}b = ba^{WD} = 0$*, then* $(a+b)^{WD} = a^{WD} + b^{WD}$*, where* $a^{WD}$ *and* $b^{WD}$ *are WD inverse of* $a$ *and* $b$*, respectively.*

*Proof.* Suppose $k = \max\{ind(a), ind(b)\}$. We have $ab = ba = 0$, so by the binomial expansion $(a+b)^m = a^m + b^m$ for every positive integer $m$. Further, we obtain

$$\begin{aligned}(a+b)(a^{WD}+b^{WD})(a+b) &= (aa^{WD} + ab^{WD} + ba^{WD} + bb^{WD})(a+b)\\ &= aa^{WD}a + ab^{WD}a + ba^{WD}a + bb^{WD}a + aa^{WD}b\\ &+ ab^{WD}b + ba^{WD}b + bb^{WD}b\\ &= aa^{WD}a + bb^{WD}b\\ &= a+b, \end{aligned} \tag{7}$$

$$\begin{aligned}(a^{WD}+b^{WD})(a+b)^{k+1} &= (a^{WD}+b^{WD})(a^{k+1}+b^{k+1})\\ &= a^{WD}a^{k+1} + a^{WD}b^{k+1} + b^{WD}a^{k+1} + b^{WD}b^{k+1}\\ &= a^k + b^k\\ &= (a+b)^k, \end{aligned} \tag{8}$$

and

$$\begin{aligned}(a+b)^{k+1}(a^{WD}+b^{WD}) &= (a^{k+1}+b^{k+1})(a^{WD}+b^{WD})\\ &= a^{k+1}a^{WD} + a^{k+1}b^{WD} + b^{k+1}a^{WD} + b^{k+1}b^{WD}\\ &= a^k + b^k\\ &= (a+b)^k. \end{aligned} \tag{9}$$

From (7), (8) and (9), we get $(a+b)^{WD} = a^{WD} + b^{WD}$. □

The next result discusses the reverse-order law for WD inverse.

**Theorem 3.4.** *Let $a, b \in R^{WD}$. If $ab = ba$ and $bb^{WD}a^{WD} = a^{WD}bb^{WD}$, then $(ab)^{WD} = b^{WD}a^{WD}$, where $a^{WD}$ and $b^{WD}$ are WD inverse of a and b, respectively.*

*Proof.* Suppose $k = max\{ind(a), ind(b)\}$. We have $aa^{WD}a = a$, $a^{WD}a^{k+1} = a^k$, $a^{k+1}a^{WD} = a^k$, and $bb^{WD}b = b$, $b^{WD}b^{k+1} = b^k$, $b^{k+1}b^{WD} = b^k$. Now,

$$\begin{aligned} abb^{WD}a^{WD}ab &= aa^{WD}bb^{WD}ab \\ &= aa^{WD}bb^{WD}ba \\ &= aa^{WD}ba \\ &= aa^{WD}ab \\ &= ab, \end{aligned} \tag{10}$$

$$\begin{aligned} b^{WD}a^{WD}(ab)^{k+1} &= b^{WD}a^{WD}a^{k+1}b^{k+1} \\ &= b^{WD}a^kb^{k+1} \\ &= b^{WD}b^{k+1}a^k \\ &= b^ka^k \\ &= a^kb^k \\ &= (ab)^k, \end{aligned} \tag{11}$$

and

$$\begin{aligned} (ab)^{k+1}b^{WD}a^{WD} &= a^{k+1}b^{k+1}b^{WD}a^{WD} \\ &= a^{k+1}b^ka^{WD} \\ &= b^ka^{k+1}a^{WD} \\ &= b^ka^k \\ &= (ab)^k. \end{aligned} \tag{12}$$

From (10), (11) and (12), we get $(ab)^{WD} = b^{WD}a^{WD}$.
□

The next result can be proved by the steps as in Theorem 3.4 following similarly.

**Theorem 3.5.** *Let $a, b \in R^{WD}$. If $ab = ba$ and $ba^{WD}a = a^{WD}ab$, then $(ab)^{WD} = b^{WD}a^{WD}$, where $a^{WD}$ and $b^{WD}$ are WD inverse of a and b, respectively.*

The forward-order law involving a WD inverse is presented below.

**Theorem 3.6.** *Let $a, b \in R^{WD}$. If $ab = ba$ and $b^{WD}ba = ab^{WD}b$, then $(ab)^{WD} = a^{WD}b^{WD}$, where $a^{WD}$ and $b^{WD}$ are WD inverse of a and b, respectively.*

*Proof.* Suppose $k = max\{ind(a), ind(b)\}$. By Definition 2.1, $aa^{WD}a = a$, $a^{WD}a^{k+1} = a^k$, $a^{k+1}a^{WD} = a^k$, and $bb^{WD}b = b$, $b^{WD}b^{k+1} = b^k$, $b^{k+1}b^{WD} = b^k$. Now,

$$\begin{aligned} aba^{WD}b^{WD}ab &= baa^{WD}b^{WD}ba \\ &= baa^{WD}ab^{WD}b \\ &= bab^{WD}b \\ &= abb^{WD}b \\ &= ab, \end{aligned} \tag{13}$$

$$\begin{aligned} a^{WD}b^{WD}(ab)^{k+1} &= a^{WD}b^{WD}b^{k+1}a^{k+1} \\ &= a^{WD}b^{k}a^{k+1} \\ &= a^{WD}a^{k+1}b^{k} \\ &= a^{k}b^{k} \\ &= (ab)^{k}, \end{aligned} \tag{14}$$

and

$$\begin{aligned} (ab)^{k+1}a^{WD}b^{WD} &= b^{k+1}a^{k+1}a^{WD}b^{WD} \\ &= b^{k+1}a^{k}b^{WD} \\ &= a^{k}b^{k+1}b^{WD} \\ &= a^{k}b^{k} \\ &= (ab)^{k}. \end{aligned} \tag{15}$$

From (13), (14) and (15), we get $(ab)^{WD} = a^{WD}b^{WD}$. □

The following example demonstrates Theorem 3.4, Theorem 3.5 and Theorem 3.6.

**Example 3.7.** *Let* $R = \dfrac{\mathbb{R}^{3\times3}[x]}{<x^2-1>}$ *with conjugate involution. Suppose* $A = P\begin{bmatrix}2&0&0\\0&1&0\\0&0&0\end{bmatrix}P^{-1} + <x^2-1>$ *and* $B = P\begin{bmatrix}3&0&0\\0&2&0\\0&0&0\end{bmatrix}P^{-1} + <x^2-1> \in R$. *Then,* $AB = P\begin{bmatrix}6&0&0\\0&2&0\\0&0&0\end{bmatrix}P^{-1} + <x^2-1>$ *and we have* $A^{WD} = P\begin{bmatrix}1/2&0&0\\0&1&0\\0&0&y_1\end{bmatrix}P^{-1} + <x^2-1>$ *and* $B^{WD} = P\begin{bmatrix}1/3&0&0\\0&1/2&0\\0&0&y_2\end{bmatrix}P^{-1} + <x^2-1>$, *where* $y_1, y_2 \in \mathbb{R}$ *are arbitrary. Now,* $(AB)^{WD} = P\begin{bmatrix}1/6&0&0\\0&1/2&0\\0&0&y_3\end{bmatrix}P^{-1} + <x^2-1>$, *where* $y_3 \in \mathbb{R}$ *is arbitrary and we can always choose* $y_3 = y_1y_2$. *Hence,* $(AB)^{WD} = A^{WD}B^{WD} = B^{WD}A^{WD}$.

Next example shows that the given conditions in Theorem 3.4 and Theorem 3.5 are sufficient but not necessary.

**Example 3.8.** *Let* $R = \{a_0 + a_1i + a_2j + a_3k : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$, *where* $i^2 = j^2 = k^2 = -1$ *and* $ij = -ji = k, jk = -kj = i, ki = -ik = j$, *be a ring (quaternion polynomial ring) with conjugate involution. Let* $\dfrac{-1}{a_1}i$ *be a WD inverse of* $a_1i$ *and* $\dfrac{-1}{a_2}j$ *be a WD inverse of* $a_2j$, *where* $a_1, a_2 \neq 0$. *Then,* $(a_1ia_2j)^{WD} = (a_1a_2ij)^{WD} = (a_1a_2k)^{WD} = \dfrac{-1}{a_1a_2}k$, *and* $(a_2j)^{WD}(a_1i)^{WD} = \dfrac{(-1)}{a_2}j\dfrac{(-1)}{a_1}i = \dfrac{1}{a_1a_2}ji = \dfrac{-1}{a_1a_2}k$. *Hence,* $(a_1ia_2j)^{WD} = (a_2j)^{WD}(a_1i)^{WD}$. *But* $a_1ia_2j \neq a_2ja_1i$.

The triple reverse-order law involving a WD inverse is presented below.

**Theorem 3.9.** *Let* $a, b, c \in R^{WD}$ *be commute with each other. If* $cc^{WD}b = bcc^{WD}$ *and* $c^{WD}b^{WD}a = ac^{WD}b^{WD}$, *then* $(abc)^{WD} = c^{WD}b^{WD}a^{WD}$, *where* $a^{WD}$, $b^{WD}$ *and* $c^{WD}$ *are WD inverse of* $a$, $b$ *and* $c$, *respectively.*

*Proof.* Suppose $k = max\{ind(a), ind(b), ind(c)\}$. We will show that $(abc)^{WD} = c^{WD}b^{WD}a^{WD}$ by using the definition of WD inverse. Now,

$$\begin{aligned} abcc^{WD}b^{WD}a^{WD}abc &= bcac^{WD}b^{WD}a^{WD}abc \\ &= bcc^{WD}b^{WD}aa^{WD}abc \\ &= bcc^{WD}b^{WD}abc \\ &= cc^{WD}bb^{WD}bca \\ &= cc^{WD}bca \\ &= cc^{WD}cab \\ &= abc, \end{aligned} \tag{16}$$

$$\begin{aligned} c^{WD}b^{WD}a^{WD}(abc)^{k+1} &= c^{WD}b^{WD}a^{WD}a^{k+1}b^{k+1}c^{k+1} \\ &= c^{WD}b^{WD}a^{k}b^{k+1}c^{k+1} \\ &= c^{WD}b^{WD}b^{k+1}c^{k+1}a^{k} \\ &= c^{WD}c^{k+1}b^{k}a^{k} \\ &= c^{k}a^{k}b^{k} \\ &= (abc)^{k}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} (abc)^{k+1}b^{WD}a^{WD} &= a^{k+1}b^{k+1}c^{k+1}c^{WD}b^{WD}a^{WD} \\ &= a^{k+1}b^{k+1}c^{k}b^{WD}a^{WD} \\ &= c^{k}a^{k+1}b^{k+1}b^{WD}a^{WD} \\ &= c^{k}a^{k+1}b^{k}a^{WD} \\ &= c^{k}b^{k}a^{k+1}a^{WD} \\ &= c^{k}b^{k}a^{k} \\ &= (abc)^{k}. \end{aligned} \tag{18}$$

From (16), (17) and (18), we get $(abc)^{WD} = c^{WD}b^{WD}a^{WD}$. □

Similarly, the triple forward law can be proved, and is stated below.

**Theorem 3.10.** *Let $a, b, c \in R^{WD}$ be commute with each others. If $aa^{WD}b = baa^{WD}$ and $a^{WD}b^{WD}c = ca^{WD}b^{WD}$, then $(abc)^{WD} = a^{WD}b^{WD}c^{WD}$, where $a^{WD}$, $b^{WD}$ and $c^{WD}$ are WD inverse of $a$, $b$ and $c$, respectively.*

If $a^{WD}ab^{WD} = a^{WD}$ and $a^{WD}bb^{WD} = b^{WD}$, then $a^{WD}(a+b)b^{WD} = a^{WD}ab^{WD} + a^{WD}bb^{WD} = a^{WD} + b^{WD}$, i.e., $a^{WD}(a+b)b^{WD} = a^{WD} + b^{WD}$. If the absorption law holds for WD inverse, i.e., $a^{WD}(a+b)b^{WD} = a^{WD} + b^{WD}$, then a WD inverse satisfies a few conditions mentioned in the next result.

**Theorem 3.11.** *Let $a, b \in R^{WD}$. If $a^{WD}(a+b)b^{WD} = a^{WD} + b^{WD}$, then $aa^{WD}bb^{WD} = aa^{WD}$, $a^{WD}ab^{WD}b = b^{WD}b$, $a^{k}bb^{WD} = a^{k}$ and $a^{WD}ab^{k} = b^{k}$, where $a^{WD}$ and $b^{WD}$ are WD inverse of $a$ and $b$, respectively.*

*Proof.* Suppose $k = max\{ind(a), ind(b)\}$. We have

$$a^{WD}(a+b)b^{WD} = a^{WD} + b^{WD}. \tag{19}$$

Pre-multiplying equation (19) by $a^{k+1}$, we get

$$a^{k+1}a^{WD}(a+b)b^{WD} = a^{k+1}a^{WD} + a^{k+1}b^{WD}$$

which implies

$$a^k(a+b)b^{WD} = a^k + a^{k+1}b^{WD}, \text{ i.e., } a^{k+1}b^{WD} + a^kbb^{WD} = a^k + a^{k+1}b^{WD}.$$

Hence, $a^kbb^{WD} = a^k$. Again, pre-multiplying equation (19) by $a$, we have $aa^{WD}(a+b)b^{WD} = aa^{WD} + ab^{WD}$ which yields $ab^{WD} + aa^{WD}bb^{WD} = aa^{WD} + ab^{WD}$. So, $aa^{WD}bb^{WD} = aa^{WD}$. Post-multiplying by $b^{k+1}$ in equation (19), we obtain

$$a^{WD}(a+b)b^{WD}b^{k+1} = a^{WD}b^{k+1} + b^{WD}b^{k+1}$$

which implies

$$a^{WD}(a+b)b^k = a^{WD}b^{k+1} + b^k, \text{ i.e., } a^{WD}ab^k + a^{WD}b^{k+1} = a^{WD}b^{k+1} + b^k.$$

We therefore have $a^{WD}ab^k = b^k$. Again, post-multiplying by $b$ in equation (19), we obtain $a^{WD}(a+b)b^{WD}b = a^{WD}b + b^{WD}b$ which yields

$$a^{WD}ab^{WD}b + a^{WD}bb^{WD}b = a^{WD}b + b^{WD}b, \text{ i.e., } a^{WD}ab^{WD}b + a^{WD}b = a^{WD}b + b^{WD}b.$$

Thus, we have $a^{WD}ab^{WD}b = b^{WD}b$.
□

**Corollary 3.12.** *Let $a, b \in R^{WD}$, and $a^{WD}$, $b^{WD}$ be WD inverse of $a$, $b$, respectively. If $a^{WD}(a+b)b^{WD} = a^{WD} + b^{WD}$, then*

(i) *$b^kR \subseteq a^{WD}R$ and $a^kR = a^kbR$.*
(ii) *$^\circ(a^{WD}) \subseteq {}^\circ(b^k)$ and $^\circ(a^k) = {}^\circ(a^kb)$.*
(iii) *$Rab^k = Rb^k$ and $Ra^k \subseteq Rb^{WD}$.*
(iv) *$(b^{WD})^\circ \subseteq (a^k)^\circ$ and $(ab^k)^\circ = (b^k)^\circ$.*

*Proof.* First, we assume that $k = max\{ind(a), ind(b)\}$.

(i) We know that $b^k = a^{WD}ab^k$. So, $b^kR = a^{WD}ab^kR \subseteq a^{WD}aR \subseteq a^{WD}R$, which implies $b^kR \subseteq a^{WD}R$. We have $a^kbb^{WD} = a^k$. Hence, $a^kR = a^kbb^{WD}R \subseteq a^kbR \subseteq a^kR$. So, $a^kR = a^kbR$.
(ii) We have $b^kR \subseteq a^{WD}R$, from Lemma 2.18 (i), we get $^\circ(a^{WD}) \subseteq {}^\circ(b^k)$. And $a^kR = a^kbR$ implies $a^kR \subseteq a^kbR$. Again, from Lemma 2.18 (i), we get $^\circ(a^kb) \subseteq {}^\circ(a^k)$. Similarly, from $a^kbR \subseteq a^kR$, we get $^\circ(a^k) \subseteq {}^\circ(a^kb)$. Hence, $^\circ(a^k) = {}^\circ(a^kb)$.
(iii) Similar to part (i).
(iv) Similar to part (ii) by Lemma 2.18 (iii).
□

**Theorem 3.13.** *Let $a, b \in R^{WD}$. If $b^{WD}aa^{WD} = a^{WD}$ and $b^{WD}ba^{WD} = b^{WD}$, then $b^{WD}(a+b)a^{WD} = a^{WD} + b^{WD}$. Furthermore, $b^kaa^{WD} = b^k$, $b^{WD}ba^k = a^k$, $bb^{WD}aa^{WD} = bb^{WD}$ and $b^{WD}ba^{WD}a = a^{WD}a$, where $a^{WD}$ and $b^{WD}$ are WD inverse of $a$ and $b$, respectively.*

*Proof. Suppose $k = max\{ind(a), ind(b)\}$. We have $b^{WD}aa^{WD} = a^{WD}$ and $b^{WD}ba^{WD} = b^{WD}$. Now, $b^{WD}(a+b)a^{WD} = b^{WD}aa^{WD} + b^{WD}ba^{WD} = a^{WD} + b^{WD}$. Thus, we have*

$$b^{WD}(a+b)a^{WD} = a^{WD} + b^{WD}. \tag{20}$$

*Pre-multiplying by $b^{k+1}$ in equation (20), we get $b^{k+1}b^{WD}(a+b)a^{WD} = b^{k+1}a^{WD} + b^{k+1}b^{WD}$, i.e., $b^k(a+b)a^{WD} = b^{k+1}a^{WD} + b^k$, i.e., $b^kaa^{WD} = b^k$. Similarly, if post-multiplying by $a^{k+1}$ in (20), we get $b^{WD}ba^k = a^k$. Again, pre and post-multipying by $b$ and $a$ in (20), we get $bb^{WD}aa^{WD} = bb^{WD}$ and $b^{WD}ba^{WD}a = a^{WD}a$, respectively.* □

The next result is in the direction of Theorem 3.3.

**Theorem 3.14.** *Let $a, b \in R^{WD\dagger}$, and $a^{WD\dagger}$, $b^{WD\dagger}$ be WDMP inverse of $a$, $b$, respectively. If $ab = ba = 0$, $a^*b = ab^* = 0$, $ab^{WD} = b^{WD}a = 0$, and $a^{WD}b = ba^{WD} = 0$, then $(a+b)^{WD\dagger} = a^{WD\dagger} + b^{WD\dagger}$, where $a^{WD}$ and $b^{WD}$ are WD inverse of $a$ and $b$, respectively.*

*Proof.* From Theorem 3.3 and Theorem 1.4, we have $(a+b)^{WD} = a^{WD} + b^{WD}$ and $(a+b)^{\dagger} = a^{\dagger} + b^{\dagger}$.

$$\begin{aligned}
(a+b)^{WD\dagger} &= (a+b)^{WD}(a+b)(a+b)^{\dagger} \\
&= (a^{WD} + b^{WD})(a+b)(a^{\dagger} + b^{\dagger}) \\
&= (a^{WD}a + a^{WD}b + b^{WD}a + b^{WD}b)(a^{\dagger} + b^{\dagger}) \\
&= (a^{WD}a + b^{WD}b)(a^{\dagger} + b^{\dagger}) \\
&= a^{WD}aa^{\dagger} + a^{WD}ab^{\dagger} + b^{WD}ba^{\dagger} + b^{WD}bb^{\dagger} \\
&= a^{WD}aa^{\dagger} + a^{WD}ab^{*}(b^{\dagger})^{*}b^{\dagger} + b^{WD}ba^{*}(a^{\dagger})^{*}a^{\dagger} + b^{WD}bb^{\dagger} \\
&= a^{WD}aa^{\dagger} + b^{WD}bb^{\dagger} \\
&= a^{WD\dagger} + b^{WD\dagger}.
\end{aligned}$$

□

An immediate consequence of the above result is shown next as a corollary.

**Corollary 3.15.** *Let $a, b \in R^{WD\dagger}$ be Hermitian, and $a^{WD\dagger}$, $b^{WD\dagger}$ be WDMP inverse of $a$, $b$, respectively. If $ab = ba = 0$, $ab^{WD} = b^{WD}a = 0$, and $a^{WD}b = ba^{WD} = 0$, then $(a+b)^{WD\dagger} = a^{WD\dagger} + b^{WD\dagger}$, where $a^{WD}$ and $b^{WD}$ are WD inverse of a and b, respectively.*

*Proof.* We have $a^* = a$ and $b^* = b$, so $ab = ba = 0$ implies $ba^* = 0$ and $ab^* = 0$. Now, from Theorem 3.14, we get $(a+b)^{WD\dagger} = a^{WD\dagger} + b^{WD\dagger}$. □

Zhu *et al.* [29] provided a result for the reverse-order law of the Moore-Penrose inverse in a ring.

**Lemma 3.16.** *(Lemma 2.2, [29]) Let $a, b \in R^{\dagger}$ with $ab = ba$ and $a^*b = ba^*$. Then, $ab \in R^{\dagger}$ and $(ab)^{\dagger} = b^{\dagger}a^{\dagger} = a^{\dagger}b^{\dagger}$.*

**Theorem 3.17.** *Let $a, b \in R^{WD\dagger}$, and $a^{WD\dagger}$, $b^{WD\dagger}$ be WDMP inverse of $a$, $b$, respectively. If $ab = ba$, $a^*b = b^*a$, $a^{WD}abb^{\dagger} = bb^{\dagger}a^{WD}a$, and $bb^{WD}a^{WD} = a^{WD}bb^{WD}$, then $(ab)^{WD\dagger} = b^{WD\dagger}a^{WD\dagger}$, where $a^{WD}$ and $b^{WD}$ are WD inverse of a and b, respectively.*

*Proof.* From Definition 2.3, we get $a, b \in R^{\dagger}$, and from hypothesis $ab = ba$, $a^*b = b^*a$. Using Lemma 3.16, we obtain $(ab)^{\dagger} = b^{\dagger}a^{\dagger} = a^{\dagger}b^{\dagger}$. And from the above Theorem 3.4, $(ab)^{WD} = b^{WD}a^{WD}$. Further, we get

$$\begin{aligned}
(ab)^{WD\dagger} &= (ab)^{WD}ab(ab)^{\dagger} \\
&= b^{WD}a^{WD}abb^{\dagger}a^{\dagger} \\
&= b^{WD}bb^{\dagger}a^{WD}aa^{\dagger} \\
&= b^{WD\dagger}a^{WD\dagger}.
\end{aligned}$$

□

When $a$ and $b$ are both idempotent and Hermitian, then the forward-order law for WDMP inverse holds under a few conditions. This is shown next.

**Theorem 3.18.** *Let $a, b \in R^{WD\dagger}$ be both idempotent and Hermitian. If $ab = ba$ and $b^{WD}ba = ab^{WD}b$, then $(ab)^{WD\dagger} = a^{WD\dagger}b^{WD\dagger}$, where $a^{WD\dagger}$ and $b^{WD\dagger}$ are WDMP inverse of a and b, respectively.*

*Proof.* Since $a$ and $b$ are both idempotent and Hermitian, so $a^\dagger = a$ and $b^\dagger = b$, and $ab = ba$, we get $(ab)^\dagger = b^\dagger a^\dagger = a^\dagger b^\dagger$. Now, from Theorem 3.6, we get

$$\begin{aligned}(ab)^{WD\dagger} &= (ab)^{WD}ab(ab)^\dagger \\ &= a^{WD}b^{WD}abb^\dagger a^\dagger \\ &= a^{WD}b^{WD}abba \\ &= a^{WD}b^{WD}ba \\ &= a^{WD}ab^{WD}b \\ &= a^{WD}aa^\dagger b^{WD}bb^\dagger \\ &= a^{WD\dagger}b^{WD\dagger}.\end{aligned}$$

Hence, $(ab)^{WD\dagger} = a^{WD\dagger}b^{WD\dagger}$. □

We end this section with a straightforward derivation.

**Theorem 3.19.** *Let $a, b \in R^{WD\dagger}$, and $a^{WD\dagger}$, $b^{WD\dagger}$ be WDMP inverse of $a$, $b$, respectively.*

1. *If $(ab)^{WD\dagger} = b^{WD\dagger}a^{WD\dagger}$, then the following conditions hold:*
   (i) $b(ab)^{WD\dagger}a = bb^\dagger a^{WD}a$,
   (ii) $b(ab)^{WD\dagger}a^{k+1} = bb^\dagger a^k$,
   (iii) $b^{k+1}(ab)^{WD\dagger}a = b^{k+1}b^\dagger a^{WD}a$.
2. *If $(ab)^{WD\dagger} = a^{WD\dagger}b^{WD\dagger}$, then the following conditions hold:*
   (i) $a(ab)^{WD\dagger}b = aa^\dagger b^{WD}b$,
   (ii) $a^{k+1}(ab)^{WD\dagger}b = a^{k+1}a^\dagger b^{WD}b$,
   (iii) $a(ab)^{WD\dagger}b^{k+1} = aa^\dagger b^k$.

### 3.1. WD order

This section presents a binary relation called WD order and its properties. We will start with the definition of the WD order.

**Definition 3.20.** *Let $a, b \in R^{WD}$ . Then, $a$ is said to be below $b$ under the WD order if*

$$a^{WD}a = a^{WD}b, \text{ and } aa^{WD} = ba^{WD}.$$

*It is denoted by $a \leq_{WD} b$.*

**Remark 3.21.** *If $a \leq_{WD} b$, then pre and post-multiplying $a^{WD}a = a^{WD}b$ and $aa^{WD} = ba^{WD}$ by $a^{k+1}$, respectively, we obtain $a^kb = ba^k = a^{k+1}$.*

**Remark 3.22.** *If $a$ is invertible element, then there is only element itself, i.e., $a$, for which $a \leq_{WD} b$ holds.*

**Theorem 3.23.** *The WD order is a reflexive and anti-symmetry order.*

*Proof.* The reflexivity is trivial. Now, we are proving anti-symmetric. Let $a \leq_{WD} b$ and $b \leq_{WD} a$, i.e., $a^{WD}a = a^{WD}b$, $aa^{WD} = ba^{WD}$, and $b^{WD}b = b^{WD}a, bb^{WD} = ab^{WD}$, respectively. Now,

$$\begin{aligned}a &= aa^{WD}a \\ &= ba^{WD}a \quad (\text{since } aa^{WD} = ba^{WD}) \\ &= bb^{WD}ba^{WD}a \quad (\text{since } b = bb^{WD}b) \\ &= bb^{WD}aa^{WD}a \quad (\text{since } b^{WD}b = b^{WD}a) \\ &= bb^{WD}a \\ &= bb^{WD}b \\ &= b.\end{aligned}$$

Therefore, the WD order is anti-symmetric. □

**Lemma 3.24.** *Let $a, b \in R^{WD}$ . If $b^{WD}aa^{WD} = a^{WD} = aa^{WD}b^{WD}$, then $b^{WD} \in a\{WD\}$, where $a^{WD}$ and $b^{WD}$ are WD inverse of a and b, respectively.*

*Proof.* Pre and post-multiplying $b^{WD}aa^{WD} = a^{WD}$ by $a$, we get $ab^{WD}a = a$. Post-multiplying $b^{WD}aa^{WD} = a^{WD}$ by $a^{k+1}$, we obtain $b^{WD}a^{k+1} = a^k$. Again, pre-multiplying $aa^{WD}b^{WD} = a^{WD}$ by $a^{k+1}$, we obtain $a^{k+1}b^{WD} = a^k$. So, $b^{WD} \in a\{WD\}$. □

Next result show that the WD order perform a partial order under some conditions.

**Theorem 3.25.** *Let $a, b \in R^{WD}$ . If $b^{WD}aa^{WD} = a^{WD} = aa^{WD}b^{WD}$, then the relation $a \leq_{WD} b$ is a partial order, where $a^{WD}$ and $b^{WD}$ are WD inverse of a and b, respectively.*

*Proof.* To show $a \leq_{WD} b$ is a partial order. It is sufficient to prove that the relation $a \leq_{WD} b$ reflexive, anti-symmetric and transitive. From Theorem 3.23, the WD order is a reflexive and anti-symmetry order. Now, we have to prove WD order is transitive. Suppose that $a \leq_{WD} b$ and $b \leq_{WD} c$, i.e., $a^{WD}a = a^{WD}b$, $aa^{WD} = ba^{WD}$, and $b^{WD}b = b^{WD}c, bb^{WD} = cb^{WD}$, respectively. Then,

$$\begin{aligned} aa^{WD} &= ba^{WD} \\ &= bb^{WD}ba^{WD} \\ &= cb^{WD}ba^{WD} \text{ (since } bb^{WD} = cb^{WD}) \\ &= cb^{WD}aa^{WD} \text{ (since } aa^{WD} = ba^{WD}) \\ &= ca^{WD} \text{ (since } b^{WD}aa^{WD} = a^{WD}). \end{aligned}$$

Similarly, $a^{WD}a = a^{WD}c$. So, $a \leq_{WD} c$. Therefore, the relation $a \leq_{WD} b$ is a partial order. □

## 4. Conclusion

The important findings are summarized as follows:

- The notion of WD inverse and WDMP inverse have been introduced in rings.
- Some relations among WD inverse, Drazin inverse, DMP inverse, WDMP inverse, right pseudo core inverse and Hirano inverse have been established.
- Finally, we have presented a few sufficient conditions such that the reverse-order law and forward-order law for WD and WDMP generalized inverses hold.

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