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Characterization of (m,n)-regular ordered semihypergroups

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Abstract. We, first, characterize (m,n)-hyperideals of ordered semihypergroups in terms of its $(m,0)$-hyperideals and $(0,n)$-hyperideals as well as minimality of (m,n)-hyperideals of ordered semihypergroups in terms of its minimal $(m,0)$-hyperideals and minimal $(0,n)$-hyperideals. Further characterization of (m,n)-regular ordered semihypergroups in terms of its $(m,0)$-hyperideals, $(0,n)$-hyperideals, (m,n)-hyperideals and (m,n)-quasi-hyperideals is studied. After introducing relations $\mathcal{B}_m^n$ and $\mathcal{Q}_m^n$ on ordered semihypergroups, we prove that $\mathcal{B}_m^n \subseteq \mathcal{Q}_m^n$ and provide conditions for the equality to hold in this inclusion. Finally we show that, in any ordered semihypergroups, $\mathcal{Q}_m^n = \mathcal{H}_m^n$ and, in any (m,n)-hypersimple ordered semihypergroup S, $\mathcal{B}_m^n = \mathcal{Q}_m^n = \mathcal{H}_m^n = S \times S$.

1. Introduction and Preliminaries

The notion of a hyperstructure by defining hypergroup was introduced by Marty [10] in 1934. The beauty of hyperstructure is that after operating hyperoperation on two elements a set is obtained, while in classical structures only element is obtained, which is the main reason for the researchers to attract towards such type of structures. Thus the notion of algebraic hyperstructures is a generalization of classical notion of algebraic structures. The concept of ordered semihypergroup was introduced by Heidari and Davvaz in [4]. Thereafter it was studied by several authors. Davvaz et. al. [1, 2, 4, 11] studied some properties of hyperideals, bi-hyperideals and quasi-hyperideals in ordered semihypergroups. As a generalization of bi-ideals, Lajos [7] investigated semigroups by (m,n)-ideals. Further in [3], the notion of an (m,n)-quasi-hyperideal was introduced by Hila et al. and they investigated characterizations and minimality of (m,n)-quasi-hyperideals in semihypergroups.

Most of the results on bi-ideals of semigroups had been proved by S. Lajos. In [8], S. Lajos proved that a non-empty subset B of a regular semigroup S is a bi-ideal of S if and only if there exist a right ideal R and a left ideal L of S such that $B = RL$. In [5], Kehayopulu proved that a nonempty subset B of a regular semihypergroup S is a bi-hyperideal of S if and only if it is represented in the form $B = A \circ C$ for some right hyperideal A and a left hyperideal C of S.

In this paper, we extend this result in the setting of (m,n)-hyperideals of regular order semihypergroups. We also characterize (m,n)-regular ordered semihypergroups by (m,n)-quasi-hyperideals. In the last section of the paper, we introduce some relations $\mathcal{B}_m^n$ and $\mathcal{Q}_m^n$ on ordered semihypergroups and prove that $\mathcal{B}_m^n \subseteq \mathcal{Q}_m^n$.

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We also find conditions under which equality holds in the above inclusion. Thereafter charaterizations of (m,n)-hypersimple ordered semihypergroups in terms of its $(m,0)$-hyperideals, $(0,n)$-hyperideals and (m,n)-quasi-hyperideals are studied. Finally we show that $\mathcal{Q}_m^n = \mathcal{H}_m^n$ in any ordered semihypergroup and $\mathcal{B}_m^n = \mathcal{Q}_m^n = \mathcal{H}_m^n = S \times S$ in any (m,n)-hypersimple ordered semihypergroup S respectively.

A hyperoperation on a set $S(\neq \emptyset)$ is a map $\circ : S \times S \to \mathcal{P}^\star(S)$, where $\mathcal{P}^\star(S)$ denotes the power set of S except $\{\emptyset\}$. Then $(S,\circ)$ is a hypergroupoid. The image of the pair (a,b) in $S \times S$ is denoted by $a \circ b$.

A hypergroupoid $(S,\circ)$ is called a semihypergroup if for all $x_1, x_2, x_3 \in S$

$$(x_1 \circ x_2) \circ x_3 = x_1 \circ (x_2 \circ x_3).$$

It means that $\bigcup\limits_{t \in x_1 \circ x_2} t \circ x_3 = \bigcup\limits_{r \in x_2 \circ x_3} x_1 \circ r$.

For any $T_1, T_2 \in \mathcal{P}^\star(S)$, we denote

$$T_1 \circ T_2 = \bigcup_{t \in T_1, t' \in T_2} t \circ t'.$$

Instead of $\{x_1\} \circ T_1$ and $T_2 \circ \{x_1\}$ we shall write, in whatever follows, $x_1 \circ T_1$ and $T_2 \circ x_1$, respectively. We shall write A^n for $A \circ A \circ A \circ ... \circ A$ (n-copies of A) in the sequel without further mention.

Definition 1.1. *[4] Let $\leq$ be an ordered relation on a set $S(\neq \emptyset)$. The triplet $(S,\circ,\leq)$ is called an ordered semihypergroup if $(S,\circ)$ is a semihypergroup and $(S,\leq)$ is a partially ordered set such that:*

For every $t_1, t_2, t \in S$, $t_1 \leq t_2$ implies $t_1 \circ t \leq t_2 \circ t$ and $t \circ t_1 \leq t \circ t_2$. Here $t_1 \circ t \leq t_2 \circ t$ means that for any $w \in t_1 \circ t$ there exists $w' \in t_2 \circ t$ such that $w \leq w'$.

Let S be an ordered semihypergroup. For a non-empty subset L of S, we denote $(L] = \{x \in S \mid x \leq l \text{ for some } l \in L\}$. A subset $K \neq \emptyset$ of S is called

(i) a subsemihypergroup of S if $K \circ K \subseteq K$;
(ii) an idempotent of S if $K = (K \circ K]$;
(iii) a left (right)-hyperideal [2] of S if $S \circ K \subseteq K(K \circ S \subseteq K)$ and $(K] \subseteq K$;
(iv) an hyperideal of S if K is both a left-hyperideal and a right-hyperideal of S;
(v) a bi-ideal of S if K is a subsemihypergroup of S, $K \circ H \circ K \subseteq K$ and $(K] \subseteq K$;
(vi) quasi-hyperideal of S if $(K \circ H] \cap (H \circ K] \subseteq K$ and $(K] \subseteq K$.

Definition 1.2. *[9] A subsemihypergroup K of S is said to be an (m,n)-hyperideal of S if*

(1) $K^m \circ S \circ K^n \subseteq K$; and
(2) $(K] \subseteq K$.

Dually an $(m,0)$-hyperideal and an $(0,n)$-hyperideal of S are defined. The set of all (m,n) (resp.$(m,0)$, $(0,n)$) -hyperideals of S shall be denoted, in whatever follows, by $\mathcal{K}_{(m,n)}$(resp. $\mathcal{K}_{(m,0)}$, $\mathcal{K}_{(0,n)}$).

Definition 1.3. *[6] A subsemihypergroup C of S is said to be an (m,n)-quasi-hyperideal of S if*

(1) $(C^m \circ S] \cap (S \circ C^n] \subseteq C$; and
(2) $(C] \subseteq C$.

The set of all (m,n)-quasi-hyperideals of S shall be denoted, in whatever follows, by $C_{(m,n)}$. Clearly each (m,n)-quasi-hyperideal of S is an (m,n)-hyperideal but the converse is not true in general.

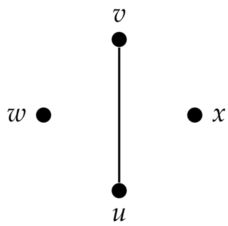
Example 1.4. *Let $S = \{u,v,w,x\}$. Define hyperoperation $\circ$ and order $\leq$ on S as follows:*

$\circ$	u	v	w	x
u	$\{u,v\}$	$\{u,v\}$	$\{u,v\}$	$\{u,v\}$
v	$\{u,v\}$	$\{u,v\}$	$\{u,v\}$	$\{u,v\}$
w	$\{u,v\}$	$\{u,v\}$	$\{u,v\}$	$\{v\}$
x	$\{u,v\}$	$\{u,v\}$	$\{v\}$	$\{w\}$

$$\leq = \{(u,u),(v,v),(w,w),(x,x),(u,v)\}.$$

The covering relation $\prec$ and the figure of S are as follows;

$$\prec = \{(u, v)\}$$



It is easy to check that $K = \{u, v, x\}$ is (1,1)-hyperideal of S but not a (1,1)-quasi-hyperideal of S.

An ordered semihypergroup S is called regular (left-regular, right-regular) [2] if for each $t \in S$, $t \in (t \circ S \circ t]$($t \in (S \circ t \circ t]$, $t \in (t \circ t \circ S]$).

Let S be an ordered semihypergroup and A_1, A_2 are subsets of $\mathcal{P}^*(S)$. Then (1) $(A_1] \circ (A_2] \subseteq (A_1 \circ A_2]$; (2) $((A_1] \circ (A_2]] = (A_1 \circ A_2] = (A_1 \circ (A_2]] = ((A_1] \circ A_2]$; (3) $(A_1] \cup (A_1] \subseteq (A_1 \cup A_2]$; and (4) $(A_1 \cap A_2] \subseteq (A_1] \cap (A_2]$.

Definition 1.5. *[9] An element t of S is said to be an (m, n)-regular ($(m, 0)$-regular, $(0, n)$-regular) element if $t \in (t^m \circ S \circ t^n]$ ($t \in (t^m \circ S]$, $t \in (S \circ t^n]$). An ordered semihypergroup S is said to be (m, n)-regular ($(m, 0)$-regular, $(0, n)$-regular) if each element of S is (m, n)-regular ($(m, 0)$-regular, $(0, n)$-regular) or equivalently for each subset L of S, $L \subseteq (L^m \circ S \circ L^n]$ ($L \subseteq (L^m \circ S]$, $L \subseteq (S \circ L^n]$).*

Definition 1.6. *[3] An ordered semihypergroup S is said to be hypersimple(resp. left hypersimple, right hypersimple) if S does not contain any proper hyperideal(resp. left hyperideal, right hyperideal).*

Definition 1.7. *[9] An ordered semihypergroup S is said to be (m, n)-hypersimple ($(m, 0)$-hypersimple, $(0, n)$-hypersimple) if S dose not contain any proper (m, n)-hyperideal ($(m, 0)$-hyperideal, $(0, n)$-hyperideal).*

Let S be an ordered semihypergroup and $m,\ n$ be non-negative integers. Relations $\mathcal{B}_m^n$ and $\mathcal{Q}_m^n$ were defined in [6, 9] as follows:

$$\mathcal{B}_m^n = \{(z, w) \in S \times S \mid [z]_{m,n} = [w]_{m,n}\}$$
$$\mathcal{Q}_m^n = \{(z, w) \in S \times S \mid [z]_{q(m,n)} = [w]_{q(m,n)}\}.$$

Where $[z]_{m,n}, [w]_{m,n} \in \mathcal{K}_{(m,n)}$.

2. (m, n)-regularity

Throughout this paper S is an ordered semihypergroup until and unless otherwise specified and $m, n \in \mathbb{Z}^+$.

Lemma 2.1. *The following statements hold in (m, n)-regular ordered semihypergroup S:*

(1) *Every (m,n)-hyperideal is an (m, n)-quasi-hyperideal of S;*
(2) *For every (m, n)-quasi-hyperideal C of S, $C = (C^m \circ S \circ C^n]$.*

Proof. (1). Let K be an (m, n)-hyperideal of S. Take any $\emptyset \neq T \subseteq (K^m \circ S] \cap (S \circ K^n]$. Then $T \subseteq (K^m \circ S]$ and $T \subseteq (S \circ K^n]$. (m, n)-regularity of S gives $T \subseteq (T^m \circ S \circ T^n] \subseteq (((K^m \circ S])^m \circ S \circ ((S \circ K^n])^n] \subseteq (K^m \circ S \circ K^n] \subseteq K$. Therefore $(K^m \circ S] \cap (S \circ K^n] \subseteq K$. Hence $K \in \mathcal{C}_{(m,n)}$.

(2). Obvious □

Theorem 2.2. *[9] An ordered semihypergroup of S is (m, n)-regular if and only if $D \cap K = (K^m \circ D^n]$ for each $(m, 0)$-hyperideal K and for each $(0, n)$-hyperideal D of S.*

Lemma 2.3. *Let the ordered semihypergroup S is (m,n)-regular, then the $(m,0)$-hyperideals and the $(0,n)$-hyperideals of S are idempotent and for each $(m,0)$-hyperideal K and each $(0,n)$-hyperideal D of S, $(K \circ D] \in \mathcal{C}_{(m,n)}$.*

Proof. Let K and D be $(m,0)$-hyperideal and $(0,n)$-hyperideal of S. Therefore $K^m \circ S \subseteq K$, $S \circ D^n \subseteq D$. As S is (m,n)-regular, we have

$$\begin{aligned} K &\subseteq (K^m \circ S \circ K^n] = (K^m \circ S \circ K^{n-1} \circ K] \\ &\subseteq (K^m \circ S \circ K^{n-1} \circ (K^m \circ S \circ K^n]] \\ &= (K^m \circ (S \circ K^{n-1}) \circ K^m \circ (S \circ K^n)] \\ &\subseteq (K^m \circ S \circ K^m \circ S] \\ &\subseteq (K \circ K] \end{aligned}$$

and

$$\begin{aligned} (K \circ K] &\subseteq ((K^m \circ S \circ S^n] \circ (K^m \circ S \circ K^n]] \\ &= (K^m \circ (S \circ K^n \circ K^m \circ S \circ K^n)] \\ &\subseteq (K^m \circ S] \\ &\subseteq (K] = K. \end{aligned}$$

Therefore $(K \circ K] = K$. Similarly $(D \circ D] = D$. Since $((K \cap D)^m \circ S] \cap (S \circ (K \cap D)^n] \subseteq (K^m \circ S] \cap (S \circ D^n] \subseteq (K] \cap (D] = K \cap D$, we have $K \cap D \in \mathcal{C}_{m,n}$. Therefore, by Theorem 2.1, $(K^m \circ D^n] \in \mathcal{C}_{(m,n)}$. As $(K \circ K] = K$ and $(D \circ D] = D$, $(K \circ D] \in \mathcal{C}_{(m,n)}$. □

By Lemma 2.3, we may rewrite Theorem 2.2 as follows:

Theorem 2.4. *An ordered semihypergroup S is (m,n)-regular if and only if $D \cap K = (K \circ D]$ for each $(m,0)$-hyperideal K and for each $(0,n)$-hyperideal D of S.*

Let $K \neq \emptyset$ be any subset of S. Then by $[K]_{m,n}$ and $[K]_{q(m,n)}$, the (m,n)-hyperideal and the (m,n)-quasi-hyperideal of S generated by a subset K of S are given by [6, 9] as follows:

$$[K]_{m,n} = \Big(\bigcup_{i=1}^{m+n} K^i \cup K^m \circ S \circ K^n \Big];$$

$$[K]_{q(m,n)} = \Big(\bigcup_{i=1}^{\max\{m,n\}} K^i \Big] \cup \Big((K^m \circ H] \cap (H \circ K^n] \Big).$$

Proposition 2.5. *For each (m,n)-hyperideal A of (m,n)-regular ordered semihypergrop S, there exists $(m,0)$-hyperideal K and $(0,n)$-hyperideal D of S such that $A = (K \circ D]$.*

Proof. Let A be an (m,n)-hyperideal of S. Then $(A^m \circ S \circ A^n] \subseteq A$. As S is (m,n)-regular, $A \subseteq (A^m \circ S \circ A^n]$. Therefore $A = (A^m \circ S \circ A^n]$ and also $[A]_{m,0} = (A^m \circ S] = K$ and $[A]_{0,n} = (S \circ A^n] = D$. Thus

$$([A]_{m,0} \circ [A]_{0,n}] = ((A^m \circ S] \circ (S \circ A^n]] \subseteq (A^m \circ S \circ A^n] = A$$

and

$$\begin{aligned} A &= (A^m \circ S \circ A^n] \\ &= (((A^m \circ S \circ A^n])^m \circ S \circ A^n] \\ &= (\underbrace{(A^m \circ S \circ A^n] \circ (A^m \circ S \circ A^n] \circ \ldots \circ (A^m \circ S \circ A^n]}_{m\text{-times}} \circ S \circ A^n] \\ &\subseteq (A^m \circ S \circ A^n \circ A^m \circ S \circ A^n \circ \ldots \circ A^m \circ S \circ A^n \circ S \circ A^n] \\ &\subseteq (A^m \circ S \circ S \circ A^n] \\ &= ((A^m \circ S] \circ (S \circ A^n]] \\ &= K \circ D. \end{aligned}$$

□

Proposition 2.6. *Let K be any $(m,0)$-hyperideal of a (m,n)-regular ordered semihypergroup S and $D \in \mathcal{P}^*(S)$. Then $(K \circ D] \in \mathcal{K}_{(m,n)}$.*

Proof. Let K be any $(m,0)$-hyperideal of S and $\emptyset \neq D \subseteq S$. Now

$$\begin{aligned}
&((K \circ D])^m \circ S \circ ((K \circ D])^n \\
&= \underbrace{(K \circ D] \circ (K \circ D] \circ \ldots \circ (K \circ D]}_{m\text{-times}} \circ S \circ \underbrace{(K \circ D] \circ (K \circ D] \circ \ldots \circ (K \circ D]}_{n\text{-times}} \\
&\subseteq (K \circ D \circ K \circ D \circ .. \circ K \circ D] \circ S \circ (K \circ D \circ K \circ D \circ .. \circ K \circ D] \\
&\subseteq (K \circ D] \circ S \circ (S \circ D] \\
&\subseteq (K \circ S \circ S \circ S \circ D] \\
&\subseteq (K^m \circ S \circ D] \quad \text{(by Lemma 2.3)} \\
&\subseteq (K \circ D].
\end{aligned}$$

Therefore $(K \circ D] \in \mathcal{K}_{(m,n)}$. □

By Propositions 2.5 and 2.6, we have:

Theorem 2.7. *Let $A \neq \emptyset$ be a subset of an (m,n)-regular ordered semihypergroup S. Then $A \in \mathcal{K}_{(m,n)}$ if and only there exist $(m,0)$-hyperideal K and $(0,n)$-hyperideal D of S such that $A = (D \circ K]$.*

Theorem 2.8. *[9] Let $A \neq \emptyset$ be any subset of S. Then*

(1) $\left(([A]_{m,n})^m \circ S \circ ([A]_{m,n})^n\right] = \left(A^m \circ S \circ A^n\right]$.
(2) $\left(([A]_{m,0})^m \circ S\right] = (A^m \circ S]$.
(3) $\left(S \circ ([A]_{0,n})^n\right] = (S \circ A^n]$.

Lemma 2.9. *If S is an (m,n)-regular ordered semihypergroup, then*

(1) *For each $z \in S$, $[z]_{m,n} = ([z]_{m,0} \circ [z]_{0,n}]$;*
(2) $(A^m \circ S] \cap (S \circ A^n] = (A^m \circ S \cap S \circ A^n]$.

Proof. (1). Let $t \in [z]_{m,n}$. As S is (m,n)-regular, $t \in (t^m \circ S \circ t^n]$. By Theorem 2.7, $t \in (t^m \circ S \circ t^n] \subseteq (([z]_{m,n})^m \circ S \circ ([z]_{m,n})^n] = (z^m \circ S \circ z^n] = (z^m \circ (S \circ z^{n-1}) \circ z] \subseteq (z^m \circ S \circ z]$. Since $z \in [z]_{m,0}$, $z^m \circ S \subseteq ([z]_{m,0})^m \circ S \subseteq [z]_{m,0}$. As $z \in [z]_{0,n}$, $z^m \circ S \circ z \in [z]_{m,0} \circ [z]_{0,n}$. Therefore $t \in (t^m \circ S \circ t^n] \subseteq (z^m \circ S \circ z] \subseteq ([z]_{m,0} \circ [z]_{0,n}]$. Hence $[z]_{m,n} \subseteq ([z]_{m,0} \circ [z]_{0,n}]$.

For the reverse inclusion, take any $y \in ([z]_{m,0} \circ [z]_{0,n}]$. Then $y \leq w$ such that $w \in r \circ s$ for some $r \in [z]_{m,0}$ and $s \in [z]_{0,n}$. Now the following cases may arise:

Case 1. $r \in (z^k]$, where $k \in \{1, 2, ..., m\}$ and $s \in (S \circ z^n]$. Then

$$\begin{aligned}
& r \circ s \\
& \subseteq (z^k] \circ (S \circ z^n] \\
& \subseteq \left(((z^m \circ S \circ z^n])^k\right] \circ (S \circ z^n] \\
& = \Big(\underbrace{(z^m \circ S \circ z^n] \circ (z^m \circ S \circ z^n] \circ ... \circ (z^m \circ S \circ z^n]}_{k\text{-times}}\Big] \circ (S \circ z^n] \\
& \subseteq \Big((z^m \circ S \circ z^n] \circ \underbrace{(z^m \circ S \circ z^n] \circ (z^m \circ S \circ z^n] \circ ... \circ (z^m \circ S \circ z^n]}_{k-2\text{-times}}\Big] \circ (S \circ z^n] \\
& \vdots \\
& \subseteq (z^m \circ S \circ z^n] \\
& \subseteq [z]_{m,n},
\end{aligned}$$

as required.
Case 2. $r \in (z^m \circ S]$ and $s \in (z^k]$, where $k \in \{1, 2, ..., m\}$. Similar to case 1.
Case 3. $r \in (z^m \circ S]$ and $s \in (S \circ z^n]$. Then $r \circ s \subseteq (z^m \circ S] \circ (S \circ z^n] \subseteq (z^m \circ S \circ S \circ z^n] \subseteq (z^m \circ S \circ z^n] \subseteq [z]_{m,n}$, as required.
Case 4. $r \in (z^p]$ and $s \in (z^q]$, where $p \in \{1, 2, ..., m\}$ and $q \in \{1, 2, ..., n\}$. Then $r \circ s \subseteq (z^p] \circ (z^q] \subseteq (z^p \circ z^q] = (z^{p+q}] \subseteq \left(([z]_{m,n})^{p+q}\right] \subseteq \left([z]_{m,n}\right] = [z]_{m,n}$, as required.

(2). We have $(A^m \circ S \cap S \circ A^n] \subseteq (A^m \circ S] \cap (S \circ A^n]$. For reverse inclusion, take any $t \in (A^m \circ S] \cap (S \circ A^n]$. Then $t \in (A^m \circ S]$ and $t \in (S \circ A^n]$. As S is (m, n)-regular, $t \in (t^m \circ S \circ t^m] \subseteq (((A^m \circ S])^m \circ S \circ ((S \circ A])^n] \subseteq (A^m \circ S \circ A^n]$. Since $(A^m \circ S \circ A^n] \subseteq (A^m \circ S]$ and $(A^m \circ S \circ A^n] \subseteq (S \circ A^n]$, $(A^m \circ S \circ A^n] \subseteq (A^m \circ S] \cap (S \circ A^n]$. Thus $t \in (A^m \circ S] \cap (S \circ A^n]$. Hence $(A^m \circ S \cap S \circ A^n] = (A^m \circ S] \cap (S \circ A^n]$. □

Definition 2.10. *An (m, n)-quasi-hyperideal C of S is said to be minimal if for each $C' \in C_{(m,n)}$ such that $C' \subseteq C$ implies $C' = C$.*

Similarly, we may define a minimal $(m, 0)$-hyperideal and a minimal $(0, n)$-hyperideal of S. The set all minimal (m, n)-hyperideals (minimal $(m, 0)$ and $(0, n)$-hyperideals) of S shall be denoted, in whatever follows, by $\mathcal{M}_{(m,n)}$ ($\mathcal{M}_{(m,0)}$ and $\mathcal{M}_{(0,n)}$).

Theorem 2.11. *Let $A \neq \emptyset$ be any subset of an (m, n)-regular ordered semihypergroup S. Then $A \in \mathcal{M}_{(m,n)}$ if and only if $A = (K \circ D]$ for some $K \in \mathcal{M}_{(m,0)}$ and $D \in \mathcal{M}_{(0,n)}$.*

Proof. Let $A \in \mathcal{M}_{(m,n)}$. Then for each $z \in A$, $[z]_{m,n} = A$. Thus, by Lemma 2.9, $A = ([z]_{m,0} \circ [z]_{0,n}]$. To show that $[z]_{m,0} \in \mathcal{M}_{(m,0)}$ take any $K \in \mathcal{M}_{(m,0)}$ such that $K \subseteq [z]_{m,0}$. As S is (m, n)-regular, by Theorem 2.4, $[z]_{m,0} \cap [z]_{0,n} = ([z]_{m,0} \circ [z]_{0,n}]$. Again, by Theorem 2.4, $(K \circ [z]_{0,n}] = K \cap [z]_{0,n} \subseteq [z]_{m,0} \cap [z]_{0,n} = ([z]_{m,0} \circ [z]_{0,n}] = A$. By Proposition 2.6, $(K \circ [z]_{0,n}] \in \mathcal{K}_{(m,n)}$. Since $(K \circ [z]_{0,n}] \subseteq A$, by minimality of A, we have $(K \circ [z]_{0,n}] = A$ and $[z]_{m,0} \cap [z]_{0,n} = K \cap [z]_{0,n}$. Now since $z \in [z]_{m,0} \cap [z]_{0,n}$, $z \in K \cap [z]_{0,n}$ implying that $z \in K$. So $[z]_{m,0} \subseteq K$, and, hence $K = [z]_{m,0}$. Thus $[a]_{m,0} \in \mathcal{M}_{(m,0)}$. Similarly one may show that $[z]_{0,n} \in \mathcal{M}_{(0,n)}$.

Conversely assume that $A = (K \circ D]$ for some $K \in \mathcal{M}_{(m,0)}$ and $C \in \mathcal{M}_{(0,n)}$ of S. By Theorem 2.11, we have $A \in \mathcal{K}_{(m,n)}$. To show that $A \in \mathcal{M}_{(m,n)}$, take any $A' \in \mathcal{K}_{(m,n)}$ such that $A' \subseteq A$. Then $(A'^m \circ S] \subseteq (A^m \circ S] \subseteq (((K \circ D])^m \circ S] = ((K \circ D] \circ (K \circ D] \circ ... \circ (K \circ D] \circ S] \subseteq (K \circ D \circ K \circ D \circ ... \circ K \circ D \circ D] \subseteq (K \circ D] \subseteq ((K^m \circ S \circ K^n] \circ S] = (K^m \circ S \circ K^n \circ S] \subseteq (K^m \circ S] \subseteq (K] = K$. As $(A'^m \circ S]$ is $(m, 0)$-hyperideal of S and $K \in \mathcal{M}_{(m,n)}$, we have $(A'^m \circ S] = K$. Similarly $(S \circ A'^n] = C$. Now $A = (K \circ D] = ((A'^m \circ S] \circ (S \circ A'^n]] = (A'^m \circ S \circ S \circ A'^n] \subseteq (A'^m \circ S \circ A'^n] \subseteq (A'] = A'$. Hence $A \in \mathcal{K}_{(m,n)}$. □

Combining Theorems 2.4, 2.8 and 2.11 we have:

Theorem 2.12. *Let $A \neq \emptyset$ be any subset of an (m, n)-regular ordered semihypergroup S. Then $A \in \mathcal{M}_{(m,n)}$ if and only if $A = K \cap D$ for some $K \in \mathcal{M}_{(m,0)}$ and $D \in \mathcal{M}_{(0,n)}$.*

Remark 2.13. *In any ordered semihypergroup left[right]-hyperideals, bi-hyperideals and quasi-hyperideals are $(m, 0)[(0, n)]$-hyperideals, (m, n)-hyperideals and (m, n)-quasi-hyperideals respectively.*

Lemma 2.14. *Let S be any (m, n)-regular ordered semihypergroup. Then*

(1) *$(m, 0)$-hyperideals [(0,n)-hyperideals] and right-hyperideals [left-hyperideals] coincide;*
(2) *(m, n)-hyperideals and bi-hyperideals coincide;*
(3) *(m, n)-quasi-hyperideals and quasi-hyperideals coincide.*

Proof. (1). Let K be an $(m, 0)$-hyperideal of S. As S is (m, n)-regular, $K \circ S \subseteq ((K^m \circ S \circ K^n] \circ S] = (K^m \circ S \circ K^n \circ S] \subseteq (K^m \circ S] \subseteq (K] = K$. Thus K is a right hyperideal of S.

(2). Let $D \in \mathcal{K}_{(m,n)}$. As S is (m, n)-regular, $D \circ S \circ D \subseteq (D^m \circ S \circ D^n] \circ S \circ (D^m \circ S \circ D^n] = (D^m \circ S \circ D^n] \circ (S] \circ (D^m \circ S \circ D^n] \subseteq (D^m \circ S \circ D^n \circ S \circ D^m \circ S \circ D^n] \subseteq (D^m \circ S \circ D^n] \subseteq (D] = D$. Thus D is a bi-hyperideal of S.

(3). Let $C \in \mathcal{C}_{(m,n)}$. As S is (m, n)-regular, $(C \circ S] \cap (S \circ C \subseteq ((C^m \circ S \circ C^n] \circ S] \cap (S \circ (C^m \circ S \circ C^n]] = (C^m \circ S \circ C^n \circ S] \cap (S \circ C^m \circ S \circ C^n] \subseteq (C^m \circ S] \cap (S \circ C^n] \subseteq (C] = C$. Thus C is quasi-hyperideal of S. □

Theorem 2.15. *[6] Let $K \neq \emptyset$ be any subset of S. Then*

(1) $\left(([K]_{q(m,n)})^m \circ S\right] = (K^m \circ S]$;
(2) $\left(S \circ ([K]_{q(m,n)})^n\right] = (S \circ K^n]$.

Lemma 2.16. *For $m \geq 2$ or $n \geq 2$, the ordered semihypergroup S is (m, n)-regular if and only if $C = (C^2]$ for each $C \in \mathcal{C}_{(m,n)}$.*

Proof. Let S be (m, n)-regular ordered semihypergroup and Q be any (m, n)-quasi-hyperideal of S. Then, by Lemma 2.1, $Q = (Q^m \circ H \circ Q^n]$. Now

$$\begin{aligned} C &= (C^m \circ S \circ C^n] \\ &= (C^m \circ S \circ ((C^m \circ S \circ C^n])^n] \\ &= (C^m \circ S \circ (C^m \circ S \circ C^n] \circ (C^m \circ S \circ C^n] \circ ... \circ (C^m \circ S \circ C^n]] \\ &\subseteq (C^m \circ S \circ (C^m \circ S \circ C^n \circ C^m \circ S \circ C^n \circ ... \circ C^m \circ S \circ C^n]] \\ &= ((C^m \circ S \circ C^n] \circ (C^m \circ S \circ C^n]] \\ &= (C \circ C]. \end{aligned}$$

Reverse inclusion is obvious because $C \in \mathcal{C}_{(m,n)}$. Hence $C = (C^2]$.

Conversely assume that $C = (C^2]$ for each $C \in \mathcal{C}_{(m,n)}$. Take any element $z \in S$. Then, as $[z]_{q(m,n)} \in \mathcal{C}_{(m,n)}$, we have

$$\begin{aligned} [z]_{q(m,n)} &= ([z]_{q(m,n)} \circ [z]_{q(m,n)}] \\ &= ([z]_{q(m,n)} \circ ([z]_{q(m,n)} \circ [z]_{q(m,n)}]] \\ &= ([z]_{q(m,n)} \circ [z]_{q(m,n)} \circ [z]_{q(m,n)}] \\ &\vdots \\ &= (([z]_{q(m,n)})^{m+n+1}] \\ &\subseteq (([z]_{q(m,n)})^m \circ S \circ ([z]_{q(m,n)})^n] \\ &\subseteq (z^m \circ S \circ z^n]. \end{aligned}$$

Since $z \in [z]_{q(m,n)}$, $z \in (z^m \circ S \circ z^n]$. Hence S is (m, n)-regular. □

Corollary 2.17. *For $m \geq 2$ or $n \geq 2$, the ordered semihypergroup S is (m,n)-regular if and only if $B = (B^2]$ for each $B \in \mathcal{K}_{(m,n)}$.*

The following example shows that the condition $m \geq 2$ or $n \geq 2$ in Lemma 2.16 and Corollary 2.17 is necessary.

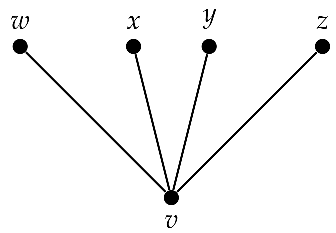
Example 2.18. *Let $H = \{v, w, x, y, z\}$. Define hyperoperation $\circ$ and order $\leq$ on S as follows:*

$\circ$	v	w	x	y	z
v	v	a	v	v	v
w	v	$\{v,w\}$	v	$\{v,y\}$	v
x	v	$\{v,z\}$	$\{v,x\}$	$\{v,w\}$	$\{v,z\}$
y	v	$\{v,w\}$	$\{v,y\}$	$\{v,y\}$	$\{v,w\}$
z	v	$\{v,z\}$	v	$\{v,x\}$	v

$$\leq = \{(v,v),(w,w),(x,x),(z,z),(z,z),(v,w),(v,x),(v,y),(v,z)\}.$$

The covering relation $\prec$ and the figure of S are as follows;

$$\prec = \{(v,w),(v,x),(v,y),(v,z)\}$$



Now $(S, \circ, \leq)$ is a regular ordered semihypergroup. It is easy to check that $A = \{v,z\}$ is a bi-hyperideal as well as a quasi-hyperideal of S but $A \neq (A^2]$.

Proposition 2.19. *For $m \geq 2$ or $n \geq 2$, the ordered semihypergroup S is (m,n)-regular if and only if $K \cap D = (K \circ D] \cap (D \circ K]$ for each $K, D \in \mathcal{K}_{(m,n)}$.*

Proof. Let $K, D \in \mathcal{K}_{(m,n)}$. Then $K \cap D \in \mathcal{K}_{(m,n)}$. So, by Corollary 2.17, $K \cap D = ((K \cap D)^2] = ((K \cap D) \circ (K \cap D)] \subseteq (K \circ D]$. Similarly $K \cap D \subseteq (D \circ K]$. Therefore $K \cap D \subseteq (K \circ D] \cap (D \circ K]$. Now

$$\begin{aligned}
&((K \circ D])^m \circ S \circ ((K \circ D])^n \\
&= \underbrace{(K \circ D] \circ (K \circ D] \circ \ldots \circ (K \circ D]}_{m\text{-times}} \circ S \circ \underbrace{(K \circ D] \circ (K \circ D] \circ \ldots \circ (K \circ D]}_{n\text{-times}} \\
&= \Big((K \circ D) \circ (K \circ D \circ K \circ D \circ \ldots \circ K \circ D \circ S \circ K \circ D \circ K \circ D \circ \ldots \circ K) \circ D\Big] \\
&\subseteq (K \circ D \circ S \circ D] \\
&= (K \circ D^m \circ S \circ D^n] \\
&\subseteq (K \circ D].
\end{aligned}$$

Therefore $(K \circ D] \in \mathcal{K}_{(m,n)}$. Similarly $(D \circ K] \in \mathcal{K}_{(m,n)}$. Therefore $(K \circ D] \cap (D \circ K] \in \mathcal{K}_{(m,n)}$. Again, by Corollary 2.17, $(K \circ D] \cap (D \circ K] = (((K \circ D] \cap (D \circ K])^2] \subseteq ((K \circ D] \circ (D \circ K] \subseteq ((K \circ S \circ D] \subseteq K$. Similarly $((K \circ D] \circ (D \circ K] \subseteq D$. Therefore $(K \circ D] \cap (D \circ K] \subseteq K \cap D$. Hence $K \cap D = (K \circ D] \cap (D \circ K]$.

Converse is obvious. □

Theorem 2.20. *Let $m \geq 2$ or $n \geq 2$). Then for any $K, D, C \in \mathcal{K}_{(m,n)}$ such that $(K \circ D] \cap (D \circ K] \subseteq C$ implies $K \subseteq C$ or $D \subseteq C$ if and only if S is (m,n)-regular and $\mathcal{K}_{(m,n)}$ forms a chain.*

Proof. To prove the direct part let $z \in S$. Since $[z]_{m,n} \in \mathcal{K}_{(m,n)}$ and $\left(([z]_{m,n})^2\right] \subseteq \left(([z]_{m,n})^2\right]$, by hypothesis, $[z]_{m,n} \subseteq \left(([z]_{m,n})^2\right]$. Now

$$\begin{aligned}
[z]_{m,n} \subseteq & \left(([z]_{m,n})^2\right] \\
\subseteq & \left([z]_{m,n} \circ (([z]_{m,n})^2]\right] \\
= & \left([z]_{m,n} \circ ([z]_{m,n})^2\right] \\
= & \left(([z]_{m,n})^3\right] \\
& \vdots \\
\subseteq & \left(([z]_{m,n})^{m+n+1}\right] \\
\subseteq & \left(([z]_{m,n})^m \circ S \circ ([z]_{m,n})^n\right] \\
\subseteq & \left((([z]_{m,n})^m \circ S] \circ (([z]_{m,n})^n]\right] \\
= & \left(z^m \circ S \circ z^n\right].
\end{aligned}$$

Since $z \in [z]_{m,n} \Rightarrow z \in (z^m \circ S \circ z^n]$. Hence S is (m,n)-regular. Next we show that $\mathcal{K}_{(m,n)}$ forms a chain. For this take any $K_1, K_2 \in \mathcal{K}_{(m,n)}$. By Proposition 2.19, $K_1 \cap K_2 = (K_1 \circ K_2] \cap (K_2 \circ K_1]$. Since $K_1 \cap K_2 \in \mathcal{K}_{(m,n)}$, by hypothesis, $K_1 \subseteq K_1 \cap K_2$ or $K_2 \subseteq K_1 \cap K_2$. If $K_1 \subseteq K_1 \cap K_2$, then $K_1 \subseteq K_2$. For the other case if $K_2 \subseteq K_1 \cap K_2$, then $K_2 \subseteq K_1$, as required.

Conversely assume that for any $K, D, C \in \mathcal{K}_{(m,n)}$ such that $(K \circ D] \cap (D \circ K] \subseteq C$. As S is (m,n)-regular, by Proposition 2.19, $K \cap D = (K \circ D] \cap (D \circ K] \subseteq C$. Now, by hypothesis, either $K \subseteq D$ or $D \subseteq K$. Therefore either $K \cap D = K$ or $K \cap D = D$. Hence either $K \subseteq C$ or $D \subseteq C$. □

Theorem 2.21. *[9] Let S be an ordered semihypergroup. Then*

(1) *S is $(m,0)$-regular if and only if $K = (K^m \circ S]\ \forall\ K \in \mathcal{K}_{(m,0)}$;*
(2) *S is $(0,n)$-regular if and only if $D = (S \circ D^n]\ \forall\ D \in \mathcal{K}_{(0,n)}$;*
(3) *S is (m,n)-regular if and only if $L = (L^m \circ S \circ L^n]\ \forall\ L \in \mathcal{K}_{(m,n)}$.*

Theorem 2.22. *An ordered semihypergroup S is (m,n)-regular if and only if $B \cap L \subseteq (B^m \circ L^n]$ for each $B \in \mathcal{K}_{(m,n)}$ and for each $L \in \mathcal{K}_{(0,n)}$.*

Proof. The statement is trivially true for $m = 0 = n$. If $m = 0$ and $n \neq 0$ or $m \neq 0$ and $n = 0$, then the result follows by Theorem 2.21. So, let $m \neq 0$, $n \neq 0$, $B \in \mathcal{K}_{(m,n)}$ and $L \in \mathcal{K}_{(0,n)}$. As S is (m,n)-regular, we have

$$\begin{aligned}
& (B \cap L) \\
& \subseteq ((B \cap L)^m \circ S \circ (B \cap L)^n] \\
& \subseteq (B^m \circ S \circ L^n] \\
& \subseteq (B^m \circ L] \\
& = (B^m \circ (L \circ L]] \quad \text{(by Lemma 2.3)} \\
& = (B^m \circ L \circ L] \\
& \vdots \\
& \subseteq (B^m \circ L^n].
\end{aligned}$$

Therefore $B \cap L \subseteq (B^m \circ L^n]$.

Conversely assume that $B \cap L \subseteq (B^m \circ L^n]$ for each $B \in \mathcal{K}_{(m,n)}$ and for each $L \in \mathcal{K}_{(m,n)}$. Take any $z \in S$. As $[z]_{m,n} \in \mathcal{K}_{(m,n)}$ and $S \in \mathcal{K}_{(m,n)}$, we have

$$\begin{aligned} [z]_{m,n} &= [z]_{m,n} \cap S = (([z]_{m,n})^m \circ S^n] \quad \text{(by hypothesis)} \\ &\subseteq (([z]_{m,n})^m \circ S] = (z^m \circ S] \qquad \text{(by Theorem 2.8)}. \end{aligned}$$

Similarly $[z]_{0,n} \subseteq (S \circ z^n]$. As $(z^m \circ S] \in \mathcal{K}_{(m,n)}$ and $(S \circ z^n] \in \mathcal{K}_{(m,n)}$, by hypothesis

$$\begin{aligned} \{z\} &\subseteq [z]_{m,n} \cap [z]_{0,n} \subseteq (z^m \circ S] \cap (S \circ z^n] \\ &= (((z^m \circ S])^m \circ ((S \circ z^n])^n] \\ &\subseteq (z^m \circ S \circ z^n]. \end{aligned}$$

Hence S is (m,n)-regular. □

Similarly one may prove the following:

Theorem 2.23. *An ordered semihypergroup S is (m,n)-regular if and only if $B \cap K \subseteq (B^m \circ K^n]$ for each $B \in \mathcal{K}_{(m,n)}$ and for each $K \in \mathcal{K}_{(m,0)}$.*

Theorem 2.24. *An ordered semihypergroup S is (m,n)-regular if and only if $C \cap L \subseteq (C^m \circ L^n]$ for each $C \in \mathcal{C}_{(m,n)}$ and for each $L \in \mathcal{K}_{(0,n)}$.*

Proof. Statement follows by Theorem 2.22 because each (m,n)-quasi-hyperideal of S is an (m,n)-hyperideal of S.

Conversely assume that $C \cap L \subseteq (C^m \circ L^n]$ for each $C \in \mathcal{C}_{(m,n)}$ and $L \in \mathcal{K}_{(0,n)}$. Let $z \in S$. As $[z]_{q(m,n)} \in \mathcal{C}_{(m,n)}$ and S is a $(0,n)$-hyperideal of S, we have

$$\begin{aligned} [z]_{q(m,n)} &= [z]_{m,n} \cap S = (([z]_{q(m,n)})^m \circ S^n] \qquad \text{(by hypothesis)} \\ &\subseteq (([z]_{q(m,n)})^m \circ S] = (a^m \circ S] \qquad \text{(by Theorem 2.15)}. \end{aligned}$$

Similarly $[z]_{0,n} \subseteq (S \circ z^n]$. As $(z^m \circ S] \in \mathcal{C}_{(m,n)}$ and $(S \circ z^n] \in \mathcal{K}_{(0,n)}$, by hypothesis, we have

$$\{z\} \subseteq [z]_{q(m,n)} \cap [z]_{0,n} \subseteq (z^m \circ S] \cap (S \circ z^n] = (((z^m \circ S])^m \circ ((S \circ z^n])^n] \subseteq (z^m \circ S \circ z^n].$$

Hence, S is (m,n)-regular. □

Similarly we may prove the following:

Theorem 2.25. *An ordered semihypergroup S is (m,n)-regular if and only if $C \cap K \subseteq (C^m \circ K^n]$ for each $C \in \mathcal{C}_{(m,n)}$ and for each $K \in \mathcal{K}_{(m,0)}$.*

Theorem 2.26. *An ordered semihypergroup S is (m,n)-regular if and only if $L \cap K = (K^m \circ L] \cap (K \circ L^n]$ for each $K \in \mathcal{K}_{(m,0)}$ and $L \in \mathcal{K}_{(0,n)}$.*

Proof. The statement is trivially true for $m = 0 = n$. If $m \neq 0$ and $n = 0$, then we have to show that S is $(m,0)$-regular if and only if $K = (K^m \circ S]$ that follows directly by Theorem 2.17. Similarly when $m = 0$ and $n \neq 0$, then the result follows by Theorem 2.21. So, let $m \neq 0$, $n \neq 0$, and take any $K \in \mathcal{K}_{(m,0)}$ and $L \in \mathcal{K}_{(0,n)}$. As S is (m,n)-regular, By Theorem 2.2 and Lemma 2.3, $K \cap L = (K^m \circ L^n] = (K^m \circ L]$ and $K \cap L = (K^m \circ L^n] = (K \circ L^n]$. So $L \cap K = (K^m \circ L] \cap (K \circ L^n]$.

Conversely assume that $L \cap K = (K^m \circ L] \cap (K \circ L^n]$ for each $K \in \mathcal{K}_{(m,0)}$ and $L \in \mathcal{K}_{(0,n)}$. Let $z \in S$. As $[z]_{m,0} \in \mathcal{K}_{(m,0)}$ and S is a $(0,n)$-hyperideal of S, we have

$$\begin{aligned} [z]_{m,0} &= [z]_{m,0} \cap S = (([z]_{m,0})^m \circ S] \cap ([z]_{m,0} \circ S^n] \qquad \text{(by hypothesis)} \\ &\subseteq (([z]_{m,0})^m \circ S] = (z^m \circ S] \qquad \text{(by Theorem 2.8)}. \end{aligned}$$

Similarly $[z]_{0,n} \subseteq (S \circ z^n]$. As $(z^m \circ S] \in \mathcal{K}_{(m,0)}$ and $(S \circ z^n] \in \mathcal{K}_{(0,n)}$, by hypothesis, we have

$$\begin{aligned}
&[z]_{m,0} \cap [z]_{0,n}\\
&\subseteq (z^m \circ S] \cap (S \circ z^n]\\
&= (((z^m \circ S])^m \circ (S \circ z^n]] \cap ((z^m \circ S] \circ ((S \circ z^n])^n]\\
&= (\underbrace{(z^m \circ S] \circ (z^m \circ S] \ldots (z^m \circ S]}_{m\text{-times}} \circ (S \circ z^n]]\\
&\cap ((z^m \circ S] \circ \underbrace{(S \circ z^n] \circ (S \circ z^n] \ldots (S \circ z^n]}_{n\text{-times}}]\\
&\subseteq (\underbrace{(z^m \circ S] \circ (z^m \circ S] \ldots (z^m \circ S]}_{m-1\text{-times}} \circ (z^m \circ S \circ z^n]]\\
&\cap ((z^m \circ S \circ z^n] \circ \underbrace{(S \circ z^n] \circ (S \circ z^n] \ldots (S \circ z^n]}_{n-1\text{-times}}]\\
&\vdots\\
&\subseteq (z^m \circ S \circ z^n].
\end{aligned}$$

Hence S is (m,n)-regular. □

Corollary 2.27. *The following are equival1ent in an ordered semihypergroup S:*

1. *(1) S is (m,n)-regular;*
2. *(2) $B \cap L \subseteq (B^m \circ L^n]$ for each $B \in \mathcal{K}_{(m,n)}$ and for each $L \in \mathcal{K}_{(0,n)}$;*
3. *(3) $C \cap L \subseteq (C^m \circ L^n]$ for each $C \in \mathcal{C}_{(m,n)}$ and for each $L \in \mathcal{K}_{(0,n)}$;*
4. *(4) $K \cap B \subseteq (K^m \circ B^n]$ for each $B \in \mathcal{K}_{(m,n)}$ and for each $K \in \mathcal{K}_{(m,0)}$;*
5. *(5) $K \cap C \subseteq (K^m \circ C^n]$ for each $C \in \mathcal{C}_{(m,n)}$ and for each $K \in \mathcal{K}_{(m,0)}$;*
6. *(6) $L \cap K = (K^m \circ L] \cap (K \circ L^n]$ for each $K \in \mathcal{K}_{(m,0)}$ and for each $L \in \mathcal{K}_{(0,n)}$.*

3. Relations $\mathcal{B}_m^n$ and $\mathcal{Q}_m^n$ in Ordered Semihypergroups

Definition 3.1. *Let S be an ordered semihypergroup and m, n be non-negative integers. For any $z, v \in S$, define a relation η on S as follows:*

- *(1) either $z = v$ or*
- *(2) $z \in (v^m \circ S \circ v^n]$ and $v \in (z^m \circ S \circ z^n]$.*

Lemma 3.2. *The relation η is an equivalence relation on S.*

Proof. η is clearly reflexive and symmetric. To show that η is transitive, suppose that $(z,v) \in \eta$ and $(v,w) \in \eta$. If $z = v$ or $v = w$, then we are done. If $z \neq v$ and $v \neq w$, then $z \in (v^m \circ S \circ v^n]$, $v \in (z^m \circ S \circ z^n]$, $v \in (w^m \circ S \circ w^n]$ and $w \in (v^m \circ S \circ v^n]$. Now

$$z \in (v^m \circ S \circ v^n] \subseteq (((w^m \circ S \circ w^n])^m \circ S \circ ((w^m \circ S \circ w^n])^n] \subseteq (w^m \circ S \circ w^n]$$

and

$$w \in (v^m \circ S \circ v^n] \subseteq (((z^m \circ S \circ z^n])^m \circ S \circ ((z^m \circ S \circ z^n])^n] \subseteq (z^m \circ S \circ z^n].$$

It gives $z \in (w^m \circ S \circ w^n]$ and $w \in (z^m \circ S \circ z^n]$. So we have $(z,w) \in \eta$ and η is transitive. Hence proved. □

Theorem 3.3. *Let $z, v \in S$. Then $z\ \eta\ v$ if and only if $[z]_{m,n} = [v]_{m,n}$.*

Proof. Assume that $z = v$, then we are done. So, we assume that $z \neq v$. Then $z \in (v^m \circ S \circ v^n]$ and $v \in (z^m \circ S \circ z^n]$. Now for any $i \in \{1, 2, ..., m + n\}$, we have

$$\begin{aligned} z^i &\subseteq ((v^m \circ S \circ v^n])^i \\ &= \underbrace{(v^m \circ S \circ v^n] \circ (v^m \circ S \circ v^n] \ldots (v^m \circ S \circ v^n]}_{i\text{-times}} \\ &\subseteq (v^m \circ S \circ v^n \circ v^m \circ S \circ v^n] \underbrace{(v^m \circ S \circ v^n] \circ (v^m \circ S \circ v^n] \ldots (v^m \circ S \circ v^n]}_{i-2\text{-times}} \\ &\subseteq (v^m \circ S \circ v^n] \circ \underbrace{(v^m \circ S \circ v^n] \circ (v^m \circ S \circ v^n] \ldots (v^m \circ S \circ v^n]}_{i-2\text{-times}} \\ &\vdots \\ &= (v^m \circ S \circ v^n]. \end{aligned}$$

Therefore $z^i \subseteq (v^m \circ S \circ v^n]$ for each $i \in \{1, 2, ..., m + n\}$. So $\bigcup_{i=1}^{m+n} z^i \subseteq (v^m \circ S \circ v^n]$. Thus $(\bigcup_{i=1}^{m+n} z^i] \subseteq (v^m \circ S \circ v^n]$. Also $z^m \circ S \circ z^n \subseteq ((v^m \circ S \circ v^n])^m \circ S \circ ((v^m \circ S \circ v^n])^n \subseteq (v^m \circ S \circ v^n]$. Therefore

$$[z]_{m,n} = (\bigcup_{i=1}^{m+n} z^i \cup z^m \circ S \circ z^n] \subseteq ((v^m \circ S \circ v^n] \cup (v^m \circ S \circ v^n]] \subseteq [v]_{m,n}.$$

Similarly $[v]_{m,n} \subseteq [z]_{m,n}$. Hence $[z]_{m,n} = [v]_{m,n}$.

Conversely assume that $[z]_{m,n} = [v]_{m,n}$. If $z = v$, then we are done. So assume that $z \neq v$. Then the following cases may arise.

Case 1. If $z \in (v^m \circ S \circ v^n]$ and $v \in (z^m \circ S \circ z^n]$, then by definition, $(z, v) \in \eta$.

Case 2. $z \in (v^k]$, where $k \in \{1, 2, ..., m + n\}$ and $v \in (z^m \circ S \circ z^n]$. Now

$$\begin{aligned} z \in (v^k] &\subseteq \Big(((z^m \circ S \circ z^n])^k\Big] \\ &= \Big(\underbrace{(z^m \circ h_1 \circ z^n] \circ (z^m \circ h_1 \circ z^n] \circ \ldots \circ (z^m \circ h_1 \circ z^n]}_{k\text{-times}}\Big] \\ &\subseteq \Big(\underbrace{(v^{mk} \circ S \circ v^{nk}] \circ (v^{mk} \circ S \circ v^{nk}] \circ \ldots \circ (v^{mk} \circ S \circ v^{nk}]}_{k\text{-times}}\Big] \\ &\subseteq (v^m \circ S \circ v^n]. \end{aligned}$$

Thus $z \, \eta \, v$.

Case 3. $z \in (v^p]$ and $v \in (v^q]$ where $p, q \in \{1, 2, ..., m + n\}$. Now $z \in (v^p] \subseteq (z^{pq}] \subseteq (v^{p^2q}] \subseteq (z^{p^2q^2}] \subseteq (v^{p^3q^2}]$.... Choose an integer $l > 0$ such that $p^{l+1}q^l > m + n + 1$. Thus $z \in (v^m \circ S \circ v^n]$. Similarly $v \in (z^m \circ S \circ z^n]$. Hence $z \, \eta \, v$.

Case 4. If $z \in (v^m \circ S \circ v^n]$ and $v \in (z^k]$, where $k \in \{1, 2, ..., m + n\}$.

$$\begin{aligned} v \in (z^k] &\subseteq ((v^m \circ S \circ v^n])^k \\ &= \Big(\underbrace{(v^m \circ S \circ v^n] \circ (v^m \circ S \circ v^n] \ldots (v^m \circ S \circ v^n]}_{k\text{-times}}\Big] \\ &\subseteq \Big(\underbrace{(z^{mk} \circ S \circ z^{nk}] \circ (z^{mk} \circ S \circ z^{nk}] \ldots (z^{mk} \circ S \circ z^{nk}]}_{k\text{-times}}\Big] \\ &\subseteq (z^m \circ S \circ z^n]. \end{aligned}$$

Thus $v \, \eta \, z$. □

Definition 3.4. *Let S be an ordered semihypergroup. Then for any $z, v \in S$, define a relation ζ on S as follows:*

(1) either $z = v$ or

(2) $z \in (v^m \circ S] \cap (S \circ v^n]$ and $v \in (z^m \circ S] \cap (S \circ z^n]$.

Lemma 3.5. *The relation ζ is an equivalence relation on S.*

Proof. Clearly ζ is reflexive and symmetric. To show transitivity, assume that $(z, v) \in \zeta$ and $(v, c) \in \zeta$. If $z = v$ or $v = c$, then we are done. So, let $z \neq v$ and $v \neq c$, then $z \in (v^m \circ S] \cap (S \circ v^n], v \in (z^m \circ S] \cap (S \circ z^n]$ and $v \in (c^m \circ S] \cap (S \circ c^n], c \in (v^m \circ S] \cap (S \circ c^n]$. Now we have

$$z \in (v^m \circ S] \subseteq (((c^m \circ S])^m \circ S] \subseteq (c^m \circ S]$$

and

$$z \in (S \circ v^n] \subseteq (S \circ ((S \circ c^n])^n \subseteq (S \circ c^n].$$

Thus $z \in (c^m \circ S] \cap (S \circ c^n]$. Similarly $c \in (z^m \circ S] \cap (S \circ z^n]$. It implies that $(z, c) \in \zeta$ and ζ is transitive. Hence proved. □

Theorem 3.6. *Let $z, v \in S$. Then $z \, \zeta \, v$ if and only if $[z]_{q(m,n)} = [v]_{q(m,n)}$.*

Proof. Assume that $z = v$, then we are done. So, we assume that $z \neq v$. Then $z \in (v^m \circ S] \cap (S \circ v^n]$ and $v \in (z^m \circ S] \cap (S \circ z^n]$. It implies that $z \in (v^m \circ S] \cap (S \circ v^n] \subseteq [v]_{q(m,n)}$ and $v \in (z^m \circ S] \cap (S \circ z^n] \subseteq [z]_{q(m,n)}$. So $[z]_{q(m,n)} \subseteq [v]_{q(m,n)}$ and $[v]_{q(m,n)} \subseteq [z]_{q(m,n)}$. Hence $[z]_{q(m,n)} = [v]_{q(m,n)}$.

Conversely assume that $[z]_{q(m,n)} = [v]_{q(m,n)}$. If $z = v$, then we are done. So assume that $z \neq v$. Then the following cases may arise:

Case 1. If $z \in (v^m \circ S] \cap (S \circ v^n]$ and $w \in (z^m \circ S] \cap (S \circ z^n]$, then by definition $(z, v) \in \zeta$.

Case 2. $z \in (v^k]$, where $k \in \{1, 2, ..., \max\{m, n\}\}$ and $v \in (z^m \circ S] \cap (S \circ z^n]$.

$$\begin{aligned}
z &\in (v^k] \\
&\subseteq \Big(((z^m \circ S])^k\Big] \\
&= \Big(\underbrace{(z^m \circ S] \circ (z^m \circ S] \circ ... \circ (z^m \circ S]}_{k\text{-times}}\Big] \\
&\subseteq \Big(\underbrace{(v^{mk} \circ S] \circ (v^{mk} \circ S] \circ ... \circ (v^{mk} \circ S]}_{k\text{-times}}\Big] \\
&= \Big((v^m \circ v^{mk-m} \circ S] \circ \underbrace{(v^{mk} \circ S] \circ ... \circ (v^{mk} \circ H]}_{k-1\text{-times}}\Big] \\
&\subseteq \Big((v^m \circ v^{mk-m} \circ S \circ v^{mk} \circ S] \circ \underbrace{(v^{mk} \circ S] \circ ... \circ (v^{mk} \circ S]}_{k-2\text{-times}}\Big] \\
&\subseteq \Big((v^m \circ S] \circ \underbrace{(v^{mk} \circ S] \circ ... \circ (v^{mk} \circ S]}_{k-2\text{-times}}\Big] \\
&\vdots \\
&\subseteq (v^m \circ S].
\end{aligned}$$

Therefore $z \in (v^m \circ S]$ and similarly $z \in (S \circ v^n]$. So, $z \in (v^m \circ S] \cap (S \circ v^n]$. Thus $z \, \zeta \, v$.

Case 3. $z \in (v^p]$ and $v \in (z^q]$, where $p, q \in \{1, 2, ..., \max\{m, n\}\}$. Now $z \in (v^p] \subseteq (z^{pq}] \subseteq (v^{p^2 q}] \subseteq (z^{p^2 q^2}] \subseteq (v^{p^3 q^2}]....$ Choose an integer $l > 0$ such that $p^{l+1} q^l > max\{m, n\}+1$. Thus $z \in (v^m \circ S] \cap (S \circ v^n]$. Similarly $v \in (z^m \circ S] \cap (S \circ z^n]$.

Hence $z\ \zeta\ v$.
Case 4. If $z \in (v^m \circ S] \cap (S \circ v^n]$ and $v \in (z^k]$, where $k \in \{1, 2, ..., max\{m+n\}\}$. Now

$$
\begin{aligned}
& v \in (z^k] \\
& \subseteq \Big(((v^m \circ S])^k \Big] \\
& = \Big(\underbrace{(v^m \circ S] \circ (v^m \circ S] \circ ... \circ (v^m \circ S]}_{k\text{-times}} \Big] \\
& \subseteq \Big(\underbrace{(z^{mk} \circ S] \circ (z^{mk} \circ S] \circ ... \circ (z^{mk} \circ S]}_{k\text{-times}} \Big] \\
& = \Big((z^m \circ z^{mk-m} \circ S] \circ \underbrace{(z^{mk} \circ S] \circ ... \circ (z^{mk} \circ S]}_{k-1\text{-times}} \Big] \\
& = \Big((z^m \circ z^{mk-m} \circ S \circ z^{mk} \circ S] \circ \underbrace{(z^{mk} \circ S] \circ ... \circ (z^{mk} \circ S]}_{k-2\text{-times}} \Big] \\
& \subseteq \Big((z^m \circ S] \circ \underbrace{(z^{mk} \circ S] \circ ... \circ (z^{mk} \circ S]}_{k-2\text{-times}} \Big] \\
& \vdots \\
& \subseteq (z^m \circ S].
\end{aligned}
$$

Therefore $v \in (z^m \circ S]$. Similarly $v \in (S \circ z^n]$. Thus $z\ \zeta\ v$. □

Remark 3.7. *By Theorem 3.3 and Theorem 3.6, it is clear that,* $\eta = \mathcal{B}^n_m$ *and* $\zeta = \mathcal{Q}^n_m$.

Lemma 3.8. *Let S be an ordered semihypergroup. Then* $\mathcal{B}^n_m \subseteq \mathcal{Q}^n_m$.

Proof. Let $(z, w) \in \mathcal{B}^n_m$. Then $[z]_{m,n} = [w]_{m,n}$. So $\{z\} \subseteq [w]_{m,n}$ and $\{w\} \subseteq [z]_{m,n}$. Therefore $(z^m \circ S] \subseteq (([w]_{m,n})^m \circ S] = (w^m \circ S]$ and $(S \circ z^n] \subseteq (S \circ ([w]_{m,n})^n] = (S \circ w^n]$. Thus $(z^m \circ S] \cap (S \circ z^n] \subseteq (w^m \circ S] \cap (S \circ w^n]$. Now

$$
\begin{aligned}
& [z]_{q(m,n)} \\
& = (\bigcup_{i=1}^{max\{m,n\}} z^i] \cup ((z^m \circ S] \cap (S \circ z^n]) \\
& \subseteq (\bigcup_{i=1}^{max\{m,n\}} ([w]_{m,n})^i] \cup ((z^m \circ S] \cap (S \circ z^n]) \quad \text{(because } \{z\} \subseteq [w]_{m,n}) \\
& \subseteq [w]_{m,n} \cup ((w^m \circ S] \cap (S \circ w^n]) \quad \text{(as } (z^m \circ S] \cap (S \circ z^n] \subseteq (w^m \circ S] \cap (S \circ w^n]) \\
& = (\bigcup_{i=1}^{m+n} w^i \cup w^m \circ S \circ w^n] \cup \Big((w^m \circ S] \cap (S \circ w^n] \Big) \\
& \subseteq (\bigcup_{i=1}^{m+n} ([w]_{q(m,n)})^i \cup ([w]_{q(m,n)})^m \circ S \circ ([w]_{q(m,n)})^n] \cup \Big((w^m \circ S] \cap (S \circ w^n] \Big) \\
& = [w]_{q(m,n)} \cup \Big(\Big(([w]_{q(m,n)})^m \circ S \Big] \circ ([w]_{q(m,n)})^n \Big] \cup \Big((w^m \circ S] \cap (S \circ w^n] \Big) \\
& = [w]_{q(m,n)} \cup \Big(\Big(w^m \circ S \Big] \circ ([w]_{q(m,n)})^n \Big] \cup \Big((w^m \circ S] \cap (S \circ w^n] \Big) \quad \text{(by Theorem 2.15)} \\
& = [w]_{q(m,n)} \cup \Big(w^m \circ S \circ ([w]_{q(m,n)})^n \Big] \cup \Big((w^m \circ S] \cap (S \circ w^n] \Big)
\end{aligned}
$$

$= [w]_{q(m,n)} \cup \Big(w^m \circ \Big(S \circ ([w]_{q(m,n)})^n\Big]\Big] \cup \Big((w^m \circ S] \cap (S \circ w^n]\Big)$
$= [w]_{q(m,n)} \cup \Big(w^m \circ \Big(S \circ w^n\Big]\Big] \cup \Big((w^m \circ S] \cap (S \circ w^n]\Big)$ (by Theorem 2.15)
$= [w]_{q(m,n)} \cup \Big(w^m \circ S \circ w^n\Big] \cup \Big((w^m \circ S] \cap (S \circ w^n]\Big)$
$= [w]_{q(m,n)}.$

Similarly, as $\{w\} \subseteq [z]_{m,n}$, one may show that $[w]_{q(m,n)} \subseteq [z]_{q(m,n)}$. Thus $[z]_{q(m,n)} = [w]_{q(m,n)}$ i.e. $(z, w) \in \mathcal{Q}^n_m$. Hence $\mathcal{B}^n_m \subseteq \mathcal{Q}^n_m$, as required. □

Lemma 3.9. *[6] Let S be an order semihypergroup and $a, b \in S$ are $\mathcal{Q}^n_m$-related. Then, $(a^m \circ S] = (b^m \circ S]$, $(S \circ a^n] = (S \circ b^n]$ and $(a^m \circ S \circ a^n] = (b^m \circ S \circ b^n]$.*

Theorem 3.10. *In an (m, n)-regular ordered semihypergroup S, $\mathcal{Q}^n_m = \mathcal{B}^n_m$.*

Proof. Let $(a, x) \in \mathcal{Q}^n_m$. Then $[a]_{q(m,n)} = [x]_{q(m,n)}$ it follows that $\{a\} \subseteq [x]_{q(m,n)}$ and $\{x\} \subseteq [a]_{q(m,n)}$. As $(a, x) \in \mathcal{Q}^n_m$. So, by Lemma 2.4, $(a^m \circ S \circ a^n] = (x^m \circ S \circ x^n]$. Now

$$\begin{aligned}
[a]_{m,n} &= (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ S \circ a^n] \\
&\subseteq (\bigcup_{i=1}^{m+n}([x]_{q(m,n)})^i \cup a^m \circ S \circ a^n] && \text{(because } \{a\} \subseteq [x]_{m,n}) \\
&\subseteq ([x]_{q(m,n)} \cup a^m \circ S \circ a^n] \\
&= ([x]_{q(m,n)}] \cup (a^m \circ S \circ a^n] \\
&= [x]_{q(m,n)} \cup (x^m \circ S \circ x^n] && \text{(as } a^m \circ S \circ a^n = x^m \circ S \circ x^n) \\
&= (\bigcup_{i=1}^{max\{m,n\}} x^i] \cup ((x^m \circ S] \cap (S \circ x^n]) \\
&\subseteq (\bigcup_{i=1}^{m+n} x^i] \cup \Big((x^m \circ S] \circ (S \circ x^n]\Big) && \text{(by Theorem 2.4)} \\
&= (\bigcup_{i=1}^{m+n} x^i \cup x^m \circ S \circ x^n] \\
&= [x]_{m,n}.
\end{aligned}$$

Similarly $[x]_{m,n} \subseteq [a]_{m,n}$. So $[a]_{m,n} = [x]_{m,n} \Rightarrow (a, x) \in \mathcal{B}^n_m$. Thus $\mathcal{Q}^n_m \subseteq \mathcal{B}^n_m$. Hence, by Lemma 3.8, $\mathcal{Q}^n_m = \mathcal{B}^n_m$. □

Proposition 3.11. *If B_x and B_y are two (m, n)-regular $\mathcal{B}^n_m$-classes contained in the same $\mathcal{Q}^n_m$-class of ordered semihypergroup S, then $B_x = B_y$.*

Proof. As x and y are (m, n)-regular elements of S, $x \in (x^m \circ S \circ x^n]$ and $y \in (y^m \circ S \circ y^n]$. So $\{x\}^i \subseteq (x^m \circ S \circ x^n]$ and $\{y\}^i \subseteq (y^m \circ S \circ y^n]$ for each $i \in \{1, 2, \cdots, m + n\}$. Thus $\bigcup_{i=1}^{m+n} x^i \subseteq (x^m \circ S \circ x^m]$ and $\bigcup_{i=1}^{m+n} y^i \subseteq (y^m \circ S \circ y^m]$. Therefore, $[x]_{m,n} = (x^m \circ S \circ x^n]$ and $[y]_{m,n} = (y^m \circ S \circ y^n]$. Since x and y are contained in the same $\mathcal{Q}^n_m$-class, by Lemma 3.9, $(x^m \circ S \circ x^n] = (y^m \circ S \circ y^n]$. So $[x]_{m,n} = [y]_{m,n}$. Therefore $x\mathcal{B}^n_m y$. Hence, $B_x = B_y$. □

Proposition 3.12. *Let S be an ordered semihypergroup. If*

(1) *S is $(m, 0)$-hypersimple, then $\mathcal{Q}^n_m = \mathcal{B}^n_m$.*
(2) *S is $(0, n)$-hypersimple, then $\mathcal{Q}^n_m = \mathcal{B}^n_m$.*
(3) *S is (m, n)-hypersimple, then $\mathcal{Q}^n_m = \mathcal{B}^n_m$.*

Proof. (1). Let $A \in \mathcal{K}_{(m,n)}$ and S be an $(m, 0)$-hypersimple. Since $(A^m \circ H] \in \mathcal{K}_{(m,0)}$, by hypothesis, $(A^m \circ S] = S$. Therefore $(A^m \circ S] \cap (S \circ A^n] = (S \circ A^n] = ((A^m \circ S] \circ A^n] = (A^m \circ S \circ A^n] \subseteq (A] = A$. Thus $A \in \mathcal{K}_{(m,n)}$ and so each (m, n)-hyperideal of S is an (m, n)-quasi-hyperideal. Hence $\mathcal{Q}^n_m = \mathcal{B}^n_m$.

(2). On the lines similar to the proof of (1), we may prove (2).

(3). Let S be an (m, n)-hypersimple ordered semihypergroup. Then there does not exists any proper (m, n)-hyperideal and, thus, the only (m, n)-hyperideals are the (m, n)-quasi-hyperideals. Hence each (m, n)-hyperideal of S is an (m, n)-quasi-hyperideal. □

Theorem 3.13. *[6] Let S be an ordered semihypergroup and C be an (m, n)-quasi-hyperideal of S. Then*

(1) *For each $v \in S$, $[v]_{q(m,n)} = [v]_{m,0} \cap [v]_{0,n}$; and*

(2) *$C = [C]_{m,0} \cap [C]_{0,n}$.*

Theorem 3.14. *Let S be an ordered semihypergroup. Then, $\mathcal{Q}^n_m = \mathcal{H}^n_m$.*

Proof. Let $(v, v') \in \mathcal{Q}^n_m$. Then, $[v]_{q(m,n)} = [v']_{q(m,n)}$. By Theorem 3.13 $[v]_{m,0} \cap [v]_{0,n} = [v']_{m,0} \cap [v']_{0,n}$. As $v \in [v]_{m,0} \cap [v]_{0,n}$, we have $v \in [v']_{m,0} \cap [v']_{0,n} \subseteq [v']_{m,0}$. Thus $[v]_{m,0} \subseteq [v']_{m,0}$. Similarly $[v']_{m,0} \subseteq [v]_{m,0}$. So $[v]_{m,0} = [v']_{m,0}$. By the similar argument we also have $[v]_{0,n} = [v']_{0,n}$. Hence $(v, v') \in \mathcal{H}^n_m$ i.e. $\mathcal{Q}^n_m \subseteq \mathcal{H}^n_m$.

For the reverse inclusion, let $(v, v') \in \mathcal{H}^n_m$. Therefore $[v]_{m,0} = [v']_{m,0}$ and $[v]_{0,n} = [v']_{0,n}$. Thus, $[v]_{m,0} \cap [v]_{0,n} = [v']_{m,0} \cap [v']_{0,n}$. By Theorem 3.13 $[v]_{q(m,n)} = [v']_{q(m,n)}$. Thus $(v, v') \in \mathcal{Q}^n_m$. Hence $\mathcal{H}^n_m \subseteq \mathcal{Q}^n_m$, as required. □

Theorem 3.15. *An ordered semihypergroup S is (m, n)-hypersimple if and only if S is both $(m, 0)$-hpersimple and $(0, n)$-hypersimple.*

Proof. Assume that S is (m, n)-hypersimple. Take any $(m, 0)$-hyperideal A of S. Clearly $(A^m \circ S \circ A^n] \subseteq (A^m \circ S] \subseteq A$ implies A is also an (m, n)-hyperideal of S. Hence $A = S$ i.e. S is $(m, 0)$-hypersimple. Similarly S is $(0, n)$-hypersimple.

Conversely assume that S is both $(m, 0)$ and $(0, n)$-hypersimple. Let A be any (m, n)-hyperideal of S. Since $(A^m \circ S]$ and $(S \circ A^n]$ are $(m, 0)$ and $(0, n)$-hyperideals of S respectively, by assumption $(A^m \circ S] = S$ and $(S \circ A^n] = S$. Therefore $(A^m \circ S \circ A^n] = (S \circ A^n] = S$. As $(A^m \circ S \circ A^n] \subseteq A$, we have $S \subseteq A$. Hence S is (m, n)-hypersimple. □

Proposition 3.16. *An ordered semihypergroup S is (m, n)-hypersimple if and only if S is (m,n)-quasi-hypersimple.*

Proof. Let S be (m, n)-hypersimple and C be any (m, n)-quasi-hyperideal of S. Clearly $(C^m \circ S \circ C^n] \subseteq (C^m \circ S] \cap (S \circ C^n] \subseteq C$ implies C is an (m, n)-hyperideal of S. Since S is (m, n)-hypersimple, $C = S$ i.e. S is (m, n)-quasi-hypersimple.

Conversely assume that S is (m, n)-quasi-hypersimple. As each $(m, 0)$-hyperideal (resp. each $(0, n)$-hyperideal) of S is an (m, n)-quasi-hyperideal of S, S is both $(m, 0)$-hypersimple and $(0, n)$-hypersimple. Thus, by Theorem 3.15, S is (m, n)-hypersimple. □

Theorem 3.17. *Let S be an ordered semihypergroup. If S is (m, n)-hypersimple, then*

$$\mathcal{B}^n_m = \mathcal{Q}^n_m = \mathcal{H}^n_m = S \times S.$$

Proof. Firstly to show that $\mathcal{B}^n_m = S \times S$, take any $v, v' \in S$. Then $[v]_{m,n}$ and $[v']_{m,n}$ are (m, n)-hyperideals of S. (m, n)-hypersimplicity of S gives $[v]_{m,n} = S$ and $[v']_{m,n} = S$. So we have $[v]_{m,n} = [v']_{m,n}$ i.e. $(v, v') \in \mathcal{B}^n_m$. Hence $\mathcal{B}^n_m = S \times S$.

Next to show that $\mathcal{Q}^n_m = S \times S$, take any $v, v' \in S$. Since S is (m, n)-hypersimple, by Proposition 3.16, S is also (m, n)-quasi-hypersimple and $[v]_{q(m,n)}$, $[v']_{q(m,n)}$ are (m, n)-quasi-hyperideals of S. It gives $[v]_{q(m,n)} = S$ and $[v']_{q(m,n)} = S$. Thus $[v]_{q(m,n)} = [v']_{q(m,n)}$ i.e. $(v, v') \in \mathcal{Q}^n_m$. Hence $\mathcal{Q}^n_m = S \times S$.

Finally to show that $\mathcal{H}^n_m = S \times S$, take any $v, v' \in S$. Then $[v]_{m,0}$ and $[v']_{m,0}$ are $(m, 0)$-hyperideals of S. Since S is (m, n)-hypersimple, by Theorem 3.15, S is $(m, 0)$-hypersimple. Therefore $[v]_{m,0} = S$ and $[v']_{m,0} = S$. Thus $[v]_{m,0} = [v']_{m,0}$ i.e. $(v, v') \in {}_m\mathcal{I}$. Similarly $(v, v') \in \mathcal{I}_n$. Hence $\mathcal{H}^n_m = S \times S$. □

4. Conclusion

The main objective of the current paper is to characterize (m, n)-hyperideals in terms of $(m, 0)$-hyperideals and $(0, n)$-hyperideals of ordered semihypergroups. We have also investigated the minimality of (m, n)-hyperideals in terms of the minimality of $(m, 0)$-hyperideals and $(0, n)$-hyperideals of ordered semihypergroups. In this article we generalize the results for ordered semihypergroups proved earlier by S. Lajos [8] for semigroups and by N. Kehayopulu [5] for semihypergroups. The characterization of (m, n)-regular ordered semihypergroups in terms of (m, n)-quasi-hyperideals has also been studied. In the last section of this paper we introduce some relations $\mathcal{B}_m^n$ and $\mathcal{Q}_m^n$ in ordered semihypergroups and look into their relation in different class of ordered semihypergroups. To ennoble the understanding of the above relations we prove that $\mathcal{Q}_m^n = \mathcal{H}_m^n$ in any ordered semihypergroups and $\mathcal{B}_m^n = \mathcal{Q}_m^n = \mathcal{H}_m^n = S \times S$ in any (m, n)-hypersimple ordered semihypergroup S.

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