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Nonlinear mixed triple derivable mapping on prime ∗-algebras

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Abstract. Let $\mathcal{A}$ be a unite prime $*$-algebra containing a non-trivial projection. In this paper, we prove that a map $\phi : \mathcal{A} \to \mathcal{A}$ satisfies $\phi([A \diamond B, C]) = [\phi(A) \diamond B, C] + [A \diamond \phi(B), C] + [A \diamond B, \phi(C)]$ for all $A, B, C \in \mathcal{A}$ if and only if there exists an element $\lambda \in \mathcal{Z}_S(\mathcal{A})$ such that $\phi(A) = d(A) + i\lambda A$, where $d : \mathcal{A} \to \mathcal{A}$ is an additive $*$-derivation and $A \diamond B = AB^* + BA^*$. Also, we give the structure of this map on factor von Neumann algebras.

1. Introduction

Let $\mathcal{A}$ be a $*$-algebra over the complex field $\mathbb{C}$, and for $A, B \in \mathcal{A}$, denote by $[A, B]_* = AB - BA^*$ and $A \bullet B = AB + BA^*$ the skew Lie product and skew Jordan product of A and B, respectively. In some sense, the skew Lie product and skew Jordan product are used to characterize the algebraic structure. There is a vast of literature related to these products in many topics, (see [1–9]). Recall that an additive map ϕ from $\mathcal{A}$ into itself is called an additive derivation if $\phi(AB) = \phi(A)B + A\phi(B)$ for $A, B \in \mathcal{A}$. Besides, if $\phi(A^*) = \phi(A)^*$ for all $A \in \mathcal{A}$, then ϕ is an additive $*$-derivation. Let $\phi : \mathcal{A} \to \mathcal{A}$ be a map (without the additivity assumption). If $\phi([A, B]_*) = [\phi(A), B]_* + [A, \phi(B)]_*$ for all $A, B \in \mathcal{A}$ for $A, B \in \mathcal{A}$, then ϕ is called a nonlinear skew Lie derivation. If $\phi(A \bullet B) = \phi(A) \bullet B + A \bullet \phi(B)$ for all $A, B \in \mathcal{A}$, then ϕ is called a nonlinear skew Jordan derivation. Yu and Zhang [12] proved that every nonlinear skew Lie derivation on factor von Neumann algebras is an additive $*$-derivation. A. Taghavi et al. [10] showed that each nonlinear skew Jordan derivation on factor von Neumann algebras is an additive $*$-derivation. In addition, these results are extended to the cases of nonlinear $*$-Lie triple derivations and nonlinear $*$-Jordan triple derivations by Li et al [14] and V. Darvish et al [13], respectively. Recently, many researchers have shown great interest in the study of mixed products associated with skew Lie product or skew Jordan product, such as $[[A, B], C]_*$, $A \bullet B \circ C$, $[A \bullet B, C]_*$ and so on, where $A \circ B = AB + BA$ and $[A, B] = AB - BA$, (see[18–21]). Let $\phi : \mathcal{A} \to \mathcal{A}$ be a map (without the additivity assumption), then ϕ is called a second nonlinear mixed Jordan triple derivable mapping on $\mathcal{A}$ if

$$\phi(A \circ B \bullet C) = \phi(A) \circ B \bullet C + A \circ \phi(B) \bullet C + A \circ B \bullet \phi(C)$$

for all $A, B, C \in \mathcal{A}$. Pang et al [15] proved that the second nonlinear mixed Jordan triple derivable mapping on factor von Neuamnn algebras is an additive $*$-derivation. In addition, N. Rehman et al. [16] extended

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the results of [15] on prime $*$-algebras. Let $\phi : \mathcal{A} \to \mathcal{A}$ be a map (without the additivity assumption), then ϕ is called a nonlinear mixed Jordan triple derivable mapping on $\mathcal{A}$ if

$$\phi(A \bullet B \circ C) = \phi(A) \bullet B \circ C + A \bullet \phi(B) \circ C + A \bullet B \circ \phi(C)$$

for all $A, B, C \in \mathcal{A}$. Ning and Zhang [17] proved that each nonlinear mixed Jordan triple derivable mapping on factor von Neuamnn algebras is an additive $*$-derivation.

The objective of this paper is to investigate the form of a nonlinear mixed triple derivable mapping on prime $*$-algebras. Let $\phi : \mathcal{A} \to \mathcal{A}$ be a map (without the additivity assumption). If ϕ satisfying

$$\phi([A \diamond B, C]) = [\phi(A) \diamond B, C] + [A \diamond \phi(B), C] + [A \diamond B, \phi(C)]$$

for all $A, B, C \in \mathcal{A}$, then ϕ is called a nonlinear mixed triple derivable mapping on $\mathcal{A}$, where $A \diamond B = AB^* + BA^*$ is the bi-skew Jordan product of $A, B \in \mathcal{A}$. Obviously, this new product is different from the skew Lie product and the skew Jordan product, which has received a fair amount of attentions in some research topics, (see [22–28]). Let $\phi : \mathcal{A} \to \mathcal{A}$ be a map (without the additivity assumption), ϕ is called a nonlinear bi-skew Jordan derivation if $\phi(A \diamond B) = \phi(A) \diamond B + A \diamond \phi(B)$ for all $A, B \in \mathcal{A}$. V. Darvish et al [22] proved that each nonlinear bi-skew Jordan derivations on prime $*$-algebra is an additive $*$-derivation. In [23], they further study nonlinear bi-skew Jordan triple derivations on prime $*$-algebra, obtained the same result. Define a map $\phi : \mathcal{A} \to \mathcal{A}$ such that $\phi(A) = [A, T] - \mathrm{i}A$, where $T^* = -T$. Obviously, ϕ is a nonlinear mixed triple derivable mapping, but it does not an additive $*$-derivation. Let $\mathcal{A}$ be a prime $*$-algebra, i.e. $A = 0$ or $B = 0$ if $A\mathcal{A}B = 0$, and $\mathcal{A}_{sa} = \{A \in \mathcal{A} : A^* = A\}$. Denote by $\mathcal{Z}(\mathcal{A})$ the central of $\mathcal{A}$ and $\mathcal{Z}_S(\mathcal{A}) = \mathcal{Z}(\mathcal{A}) \cap \mathcal{A}_{sa}$. In this paper, we will give the structure of the nonlinear mixed triple derivable mapping on prime $*$-algebras.

2. Main Result

Theorem 2.1. *Let $\mathcal{A}$ be a unite prime $*$-algebra containing a non-trivial projection, and let $\phi : \mathcal{A} \to \mathcal{A}$ be a nonlinear mixed triple derivable mapping, that is, ϕ satisfies*

$$\phi([A \diamond B, C]) = [\phi(A) \diamond B, C] + [A \diamond \phi(B), C] + [A \diamond B, \phi(C)]$$

for any $A, B, C \in \mathcal{A}$ if and only if there exists an element $\lambda \in \mathcal{Z}_S(\mathcal{A})$ such that $\phi(A) = d(A) + \mathrm{i}\lambda A$, where $d : \mathcal{A} \to \mathcal{A}$ is an additive $*$-derivation.

In all that follows, we assume that $\mathcal{A}$ is a prime $*$-algebra containing a non-trivial projection with the unit I, and ϕ is a nonlinear mixed triple derivable mapping on $\mathcal{A}$. Write $P_1 \in \mathcal{A}$ to be the non-trivial projection and $P_2 = I - P_1$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ for $i, j = 1, 2$. Clearly, we only need to prove the necessity. Now we will prove Theorem 2.1 by several lemmas.

Lemma 2.2. $\phi(0) = 0$, $\phi(\mathcal{Z}(\mathcal{A})) \subseteq \mathcal{Z}(\mathcal{A})$.

Proof. Clearly, $\phi(0) = 0$. For any $A \in \mathcal{A}_{sa}$, $a \in \mathcal{Z}(\mathcal{A})$, we have

$$0 = \phi(0) = \phi([A \diamond I, a]) = [A \diamond I, \phi(a)].$$

Then, $[A, \phi(a)] = 0$ for any $A \in \mathcal{A}_{sa}$. This implies that $\phi(a) \in \mathcal{Z}(\mathcal{A})$, and so $\phi(\mathcal{Z}(\mathcal{A})) \subseteq \mathcal{Z}(\mathcal{A})$. □

Lemma 2.3. $\phi(\sum_{i,j=1}^{2} A_{ij}) = \sum_{i,j=1}^{2} \phi(A_{ij})$ *for all $A_{ij} \in \mathcal{A}_{ij}$ with $1 \le i, j \le 2$.*

Proof. Write $T = \phi(\sum_{i,j=1}^{2} A_{ij}) - \sum_{i,j=1}^{2}\phi(A_{ij})$. For $1 \le k \ne l \le 2$, it follow from $[P_k \diamond A_{kk}, P_l] = 0$, $[P_k \diamond A_{ll}, P_l] = 0$ and $[P_k \diamond A_{kl}, P_l] = 0$ that

$$\begin{aligned}\phi([P_k \diamond \sum_{i,j=1}^{2} A_{ij}, P_l]) &= \phi([P_k \diamond A_{lk}, P_l]) \\ &= [\phi(P_k) \diamond \sum_{i,j=1}^{2} A_{ij}, P_l] + [P_k \diamond \sum_{i,j=1}^{2} \phi(A_{ij}), P_l] \\ &+ [P_k \diamond \sum_{i,j=1}^{2} A_{ij}, \phi(P_l)]\end{aligned}$$

and

$$\begin{aligned}\phi([P_k \diamond \sum_{i,j=1}^{2} A_{ij}, P_l]) &= [\phi(P_k) \diamond \sum_{i,j=1}^{2} A_{ij}, P_l] + [P_k \diamond \phi(\sum_{i,j=1}^{2} A_{ij}), P_l] \\ &+ [P_k \diamond \sum_{i,j=1}^{2} A_{ij}, \phi(P_l)].\end{aligned}$$

Thus $[P_k \diamond T, P_l] = P_k T^* P_l - P_l T P_k = 0$, which implies that $P_l T P_k = 0$. For any $X_{kl} \in \mathcal{A}_{kl}$ with $1 \le k \ne l \le 2$, by $[X_{kl} \diamond A_{kk}, P_l] = 0$, $[X_{kl} \diamond A_{kl}, P_l] = 0$ and $[X_{kl} \diamond A_{lk}, P_l] = 0$, we have

$$\begin{aligned}\phi([X_{kl} \diamond \sum_{i,j=1}^{2} A_{ij}, P_l]) &= \phi([X_{kl} \diamond A_{ll}, P_l]) \\ &= [\phi(X_{kl}) \diamond \sum_{i,j=1}^{2} A_{ij}, P_l] + [X_{kl} \diamond \sum_{i,j=1}^{2} \phi(A_{ij}), P_l] \\ &+ [X_{kl} \diamond \sum_{i,j=1}^{2} A_{ij}, \phi(P_l)].\end{aligned}$$

On the other hand,

$$\begin{aligned}\phi([X_{kl} \diamond \sum_{i,j=1}^{2} A_{ij}, P_l]) &= [\phi(X_{kl}) \diamond \sum_{i,j=1}^{2} A_{ij}, P_l] + [X_{kl} \diamond \phi(\sum_{i,j=1}^{2} A_{ij}), P_l] \\ &+ [X_{kl} \diamond \sum_{i,j=1}^{2} A_{ij}, \phi(P_l)].\end{aligned}$$

Then $[X_{kl} \diamond T, P_l] = 0$. Thus $X_{kl} T^* P_l - P_l T X_{kl}^* = 0$. This implies that $P_l T P_l = 0$, and so $T = 0$. □

Lemma 2.4. *For all $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ with $(i \ne j)$, we have*
(1) $\phi(A_{12} + B_{12}) = \phi(A_{12}) + \phi(B_{12})$;
(2) $\phi(A_{21} + B_{21}) = \phi(A_{21}) + \phi(B_{21})$.

Proof. (1) Write $T = \phi(A_{12} + B_{12}) - (\phi(A_{12}) + \phi(B_{12}))$. For any $X_{kl} \in \mathcal{A}_{kl}$ with $1 \le k \ne l \le 2$, it follows from $[X_{kl} \diamond A_{12}, P_l] = 0$ that

$$\begin{aligned}\phi([X_{kl} \diamond (A_{12} + B_{12}), P_l]) &= \phi([X_{kl} \diamond A_{12}, P_l]) + \phi([X_{kl} \diamond B_{12}, P_l]) \\ &= [\phi(X_{kl}) \diamond (A_{12} + B_{12}), P_l] + [X_{kl} \diamond (\phi(A_{12}) + \phi(B_{12})), P_l] \\ &+ [X_{kl} \diamond (A_{12} + B_{12}), \phi(P_l)]\end{aligned}$$

and

$$\phi([X_{kl} \diamond (A_{12} + B_{12}), P_l]) = [\phi(X_{kl}) \diamond (A_{12} + B_{12}), P_l] + [X_{kl} \diamond \phi(A_{12} + B_{12}), P_l] + [X_{kl} \diamond (A_{12} + B_{12}), \phi(P_l)].$$

Then $[X_{kl} \diamond T, P_l] = 0$. This implies that $P_l T P_l = 0$. For any $A_{12} \in \mathcal{A}_{12}$, by $[P_1 \diamond A_{12}, P_2] = 0$, we have

$$\begin{aligned}\phi([P_1 \diamond (A_{12} + B_{12}), P_2]) &= \phi([P_1 \diamond A_{12}, P_2]) + \phi([P_1 \diamond B_{12}, P_2])\\ &= [\phi(P_1) \diamond (A_{12} + B_{12}), P_2] + [P_1 \diamond (\phi(A_{12}) + \phi(B_{12})), P_2]\\ &+ [P_1 \diamond (A_{12} + B_{12}), \phi(P_2)].\end{aligned}$$

On the other hand,

$$\phi([P_1 \diamond (A_{12} + B_{12}), P_2]) = [\phi(P_1) \diamond (A_{12} + B_{12}), P_2] + [P_1 \diamond \phi(A_{12} + B_{12}), P_2] + [P_1 \diamond (A_{12} + B_{12}), \phi(P_2)].$$

Then $[P_1 \diamond T, P_2] = 0$. Hence $P_2 T P_1 = 0$.

It follows from the above expression that $\phi(A_{12} + B_{12}) - (\phi(A_{12}) + \phi(B_{12})) \in \mathcal{A}_{12}$. Let $T_{12} = \phi(A_{12} + B_{12}) - (\phi(A_{12}) + \phi(B_{12}))$. Then, there exists $S_{21} \in \mathcal{A}_{21}$ such that $S_{21} = \phi(-A_{12}^* - B_{12}^*) - (\phi(-A_{12}^*) + \phi(-B_{12}^*))$. By $[(P_2 + A_{12}^*) \diamond (P_1 + B_{12}), P_2] = A_{12} + B_{12} - A_{12}^* - B_{12}^*$ and Lemma 2.3, we can obtain that

$$\begin{aligned}\phi(A_{12} + B_{12} - A_{12}^* - B_{12}^*) &= \phi([(P_2 + A_{12}^*) \diamond (P_1 + B_{12}), P_2])\\ &= [\phi(P_2 + A_{12}^*) \diamond (P_1 + B_{12}), P_2] + [(P_2 + A_{12}^*) \diamond \phi(P_1 + B_{12}), P_2]\\ &+ [(P_2 + A_{12}^*) \diamond (P_1 + B_{12}), \phi(P_2)]\\ &= [(\phi(P_2) + \phi(A_{12}^*)) \diamond (P_1 + B_{12}), P_2]\\ &+ [(P_2 + A_{12}^*) \diamond (\phi(P_1) + \phi(B_{12})), P_2] + [(P_2 + A_{12}^*) \diamond (P_1 + B_{12}), \phi(P_2)]\\ &= \phi([A_{12}^* \diamond P_1, P_2]) + \phi([P_2 \diamond B_{12}, P_2]) = \phi(A_{12} - A_{12}^*) + \phi(B_{12} - B_{12}^*)\\ &= \phi(A_{12}) + \phi(B_{12}) + \phi(-A_{12}^*) + \phi(-B_{12}^*).\end{aligned}$$

Thus $T_{12} + S_{21} = 0$. It follows that $T_{12} = 0$, and so $T = 0$. Similarly, we can show that (2) holds. □

Lemma 2.5. *For all $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ with $(i = 1, 2)$, we have*
(1) $\phi(A_{11} + B_{11}) = \phi(A_{11}) + \phi(B_{11})$;
(2) $\phi(A_{22} + B_{22}) = \phi(A_{22}) + \phi(B_{22})$.

Proof. (1) Write $T = \phi(A_{11} + B_{11}) - (\phi(A_{11}) + \phi(B_{11}))$. For any $X_{12} \in \mathcal{A}_{12}$, it follows from $[X_{12} \diamond A_{11}, P_2] = 0$ that

$$\begin{aligned}\phi([X_{12} \diamond (A_{11} + B_{11}), P_2]) &= \phi([X_{12} \diamond A_{11}, P_2]) + \phi([X_{12} \diamond B_{11}, P_2])\\ &= [\phi(X_{12}) \diamond (A_{11} + B_{11}), P_2] + [X_{12} \diamond (\phi(A_{11}) + \phi(B_{11})), P_2]\\ &+ [X_{12} \diamond (A_{11} + B_{11}), \phi(P_2)]\end{aligned}$$

and

$$\phi([X_{12} \diamond (A_{11} + B_{11}), P_2]) = [\phi(X_{12}) \diamond (A_{11} + B_{11}), P_2] + [X_{12} \diamond \phi(A_{11} + B_{11}), P_2] + [X_{12} \diamond (A_{11} + B_{11}), \phi(P_2)].$$

Then $[X_{12} \diamond T, P_2] = 0$. Hence $P_2 T P_2 = 0$. For $1 \leq k \neq l \leq 2$, it follows from $[P_k \diamond A_{11}, P_l] = 0$ that

$$\begin{aligned}\phi([P_k \diamond (A_{11} + B_{11}), P_l]) &= \phi([P_k \diamond A_{11}, P_l]) + \phi([P_k \diamond B_{11}, P_l])\\ &= [\phi(P_k) \diamond (A_{11} + B_{11}), P_l] + [P_k \diamond (\phi(A_{11}) + \phi(B_{11})), P_l]\\ &+ [P_k \diamond (A_{11} + B_{11}), \phi(P_l)].\end{aligned}$$

On the other hand,

$$\phi([P_k \diamond (A_{11}+B_{11}), P_l]) = [\phi(P_k) \diamond (A_{11}+B_{11}), P_l] + [P_k \diamond \phi(A_{11}+B_{11}), P_l] + [P_k \diamond (A_{11}+B_{11}), \phi(P_l)].$$

Then $[P_k \diamond T, P_l] = 0$. Hence $P_l T P_k = 0$. For any $X_{21} \in \mathcal{A}_{21}$, it follows from Lemma 2.3 and Lemma 2.4 that

$$\begin{aligned}
\phi([X_{21} \diamond (A_{11}+B_{11}), P_1]) &= \phi(X_{21}A_{11}^* - A_{11}X_{21}^* + X_{21}B_{11}^* - B_{11}X_{21}^*) \\
&= \phi(X_{21}A_{11}^* + X_{21}B_{11}^*) + \phi(-A_{11}X_{21}^* - B_{11}X_{21}^*) \\
&= \phi(X_{21}A_{11}^*) + \Phi(X_{21}B_{11}^*) + \phi(-A_{11}X_{21}^*) + \Phi(-B_{11}X_{21}^*) \\
&= \phi(X_{21}A_{11}^* - A_{11}X_{21}^*) + \phi(X_{21}B_{11}^* - B_{11}X_{21}^*) \\
&= [\phi(X_{21}) \diamond (A_{11}+B_{11}), P_1] + [X_{21} \diamond (\phi(A_{11}) + \phi(B_{11})), P_1] \\
&+ [X_{21} \diamond (A_{11}+B_{11}), \phi(P_1)].
\end{aligned}$$

On the other hand,

$$\phi([X_{21} \diamond (A_{11}+B_{11}), P_1]) = [\phi(X_{21}) \diamond (A_{11}+B_{11}), P_1] + [X_{21} \diamond \phi(A_{11}+B_{11}), P_1] + [X_{21} \diamond (A_{11}+B_{11}), \phi(P_1)].$$

Then $[X_{21} \diamond T, P_1] = X_{21}T^*P_1 - P_1TX_{21}^* = 0$, which implies that $P_1TX_{21}^* = 0$. Hence $P_1TP_1 = 0$, and so $T = 0$. Similarly, we can show that (2) holds. □

Lemma 2.6. *ϕ is additive on $\mathcal{A}$.*

Proof. Let $A = \sum_{i,j=1}^2 A_{ij}$, $B = \sum_{i,j=1}^2 B_{ij}$, where $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$. It follows from Lemma 2.3-2.5 that

$$\begin{aligned}
\phi(A+B) &= \phi(\sum_{i,j=1}^2 A_{ij} + \sum_{i,j=1}^2 B_{ij}) = \phi(\sum_{i,j=1}^2 (A_{ij}+B_{ij})) \\
&= \sum_{i,j=1}^2 \phi(A_{ij}+B_{ij}) = \phi(\sum_{i,j=1}^2 A_{ij}) + \phi(\sum_{i,j=1}^2 B_{ij}) = \phi(A) + \phi(B).
\end{aligned}$$

Hence ϕ is additive. □

Lemma 2.7. *$P_1\phi(P_i)P_2 = P_1\phi(P_i)^*P_2$, $P_2\phi(P_i)P_1 = P_2\phi(P_i)^*P_1$, $i \in \{1,2\}$.*

Proof. It follows from Lemma 2.2 and Lemma 2.6 that

$$\begin{aligned}
0 &= \phi([P_1 \diamond P_2, P_1]) = [\phi(P_1) \diamond P_2, P_1] + [P_1 \diamond \phi(P_2), P_1] \\
&= [\phi(P_1)P_2 + P_2\phi(P_1)^*, P_1] + [P_1\phi(P_2)^* + \phi(P_2)P_1, P_1] \\
&= P_2\phi(P_1)^*P_1 - P_1\phi(P_1)P_2 + P_1\phi(P_2)^*P_1 + \phi(P_2)P_1 - P_1\phi(P_2)^* - P_1\phi(P_2)P_1 \\
&= P_2\phi(P_1)^*P_1 - P_1\phi(P_1)P_2 - P_1\phi(P_2)^*P_2 + P_2\phi(P_2)P_1. \\
&= P_2\phi(P_1)^*P_1 - P_1\phi(P_1)P_2 + P_1\phi(P_1)^*P_2 - P_2\phi(P_1)P_1.
\end{aligned}$$

Hence $P_1\phi(P_1)P_2 = P_1\phi(P_1)^*P_2$ and $P_2\phi(P_1)P_1 = P_2\phi(P_1)^*P_1$. Similarly, we can obtain that

$$P_1\phi(P_2)P_2 = P_1\phi(P_2)^*P_2 \text{ and } P_2\phi(P_2)P_1 = P_2\phi(P_2)^*P_1.$$

□

Remark 2.1. Let $T = P_1\phi(P_1)P_2 - P_2\phi(P_1)P_1$. It follows from Lemma 2.7 that $T^* = -T$. Defining a map $\delta : \mathcal{A} \to \mathcal{A}$ by $\delta(A) = \phi(A) - [A, T]$ for all $A \in \mathcal{A}$. By Lemma 2.6 and Lemma 2.7, it is easy to check that δ is an additive map, and satisfies

$$\delta([A \diamond B, C]) = [\delta(A) \diamond B, C] + [A \diamond \delta(B), C] + [A \diamond B, \delta(C)]$$

for any $A, B, C \in \mathcal{M}$. Besides,

$$\delta(P_i) = P_1\delta(P_i)P_1 + P_2\delta(P_i)P_2 \tag{2.1}$$

with $i \in \{1, 2\}$.

Lemma 2.8. *$\delta(\mathrm{i}A) - \mathrm{i}\delta(A) + (\delta(\mathrm{i}I) - \mathrm{i}\delta(I))A \in \mathcal{Z}(\mathcal{A})$ for any $A \in \mathcal{A}_{sa}$.*

Proof. It follows from Lemma 2.2 that

$$0 = \delta([\mathrm{i}A \diamond I, C]) = [\delta(\mathrm{i}A) \diamond I, C] + [\mathrm{i}A \diamond \delta(I), C]$$

and

$$0 = \delta([A \diamond \mathrm{i}I, C]) = [\delta(A) \diamond \mathrm{i}I, C] + [A \diamond \delta(\mathrm{i}I), C]$$

for any $A \in \mathcal{A}_{sa}$, $C \in \mathcal{A}$. Then

$$\delta(\mathrm{i}A) + \delta(\mathrm{i}A)^* - \mathrm{i}(\delta(I) - \delta(I)^*)A \in \mathcal{Z}(\mathcal{A}) \tag{2.2}$$

and

$$\mathrm{i}\delta(A)^* - \mathrm{i}\delta(A) + (\delta(\mathrm{i}I) + \delta(\mathrm{i}I)^*)A \in \mathcal{Z}(\mathcal{A}) \tag{2.3}$$

for any $A \in \mathcal{A}_{sa}$. For any $A \in \mathcal{A}_{sa}$, it follows from Lemma 2.6 and $\mathrm{i}A \diamond \mathrm{i}I = A \diamond I$ that

$$0 = \delta([\mathrm{i}A \diamond \mathrm{i}I - A \diamond I, C]) = [\delta(\mathrm{i}A) \diamond \mathrm{i}I - \delta(A) \diamond I, C] + [\mathrm{i}A \diamond \delta(\mathrm{i}I) - A \diamond \delta(I), C].$$

By the above equation, we can see that

$$\delta(\mathrm{i}A) - \delta(\mathrm{i}A)^* - \mathrm{i}(\delta(A) + \delta(A)^*) - (\delta(\mathrm{i}I)^* - \delta(\mathrm{i}I))A - \mathrm{i}(\delta(I)^* + \delta(I))A \in \mathcal{Z}(\mathcal{A}). \tag{2.4}$$

Further more, by Eq.(2.2), Eq.(2.3) and Eq.(2.4), we can obtain that

$$\delta(\mathrm{i}A) - \mathrm{i}\delta(A) + (\delta(\mathrm{i}I) - \mathrm{i}\delta(I))A \in \mathcal{Z}(\mathcal{A})$$

for any $A \in \mathcal{A}_{sa}$. □

Lemma 2.9. (1) For any $C_{ij} \in \mathcal{A}_{ij}$, $\delta(C_{ij}) \in \mathcal{A}_{ij}$ with $1 \le i \ne j \le 2$;
(2) $\delta(\mathrm{i}P_1) = P_1\delta(\mathrm{i}P_1)P_1$ and $\delta(\mathrm{i}P_2) = P_2\delta(\mathrm{i}P_2)P_2$;
(3) $\delta(I)^* = -\delta(I)$, $\delta(\mathrm{i}A) - \mathrm{i}\delta(A) \in \mathcal{Z}(\mathcal{A})$ for any $A \in \mathcal{A}$.

Proof. (1) For any $C_{12} \in \mathcal{A}_{12}$,

$$\begin{aligned} 2\delta(C_{12}) = \delta([I \diamond P_1, C_{12}]) &= [\delta(I) \diamond P_1, C_{12}] + [I \diamond \delta(P_1), C_{12}] + [I \diamond P_1, \delta(C_{12})] \\ &= (\delta(I) + \delta(I)^*)C_{12} + (\delta(P_1) + \delta(P_1)^*)C_{12} - C_{12}(\delta(P_1)^* + \delta(P_1)) \\ &\quad + 2P_1\delta(C_{12}) - 2\delta(C_{12})P_1. \end{aligned} \tag{2.5}$$

Multiplying Eq.(2.5) on both side by P_1 and P_2, respectively. It follows from Eq.(2.1) that

$$P_1\delta(C_{12})P_1 = 0 \text{ and } P_2\delta(C_{12})P_2 = 0.$$

Multiplying Eq.(2.5) on the left-hand side by P_2 and on the right-hand side by P_1, then

$$P_2\delta(C_{12})P_1 = 0.$$

Hence $\delta(C_{12}) = P_1\delta(C_{12})P_2$. Similarly, for any $C_{21} \in \mathcal{A}_{21}$, we can obtain that $\delta(C_{21}) = P_2\delta(C_{21})P_1$.

(2) Take $A = B = P_2$, $C = \mathrm{i}P_1$, it follows from Eq.(2.1) that

$$\begin{aligned} 0 = \delta([P_2 \diamond P_2, \mathrm{i}P_1]) &= [\delta(P_2) \diamond P_2, \mathrm{i}P_1] + [P_2 \diamond \delta(P_2), \mathrm{i}P_1] \\ &\quad + [P_2 \diamond P_2, \delta(\mathrm{i}P_1)] \\ &= 2(P_2\delta(\mathrm{i}P_1) - \delta(\mathrm{i}P_1)P_2). \end{aligned}$$

Then $P_2\delta(\mathrm{i}P_1)P_1 = 0$ and $P_1\delta(\mathrm{i}P_1)P_2 = 0$. For any $X_{12} \in \mathcal{A}_{12}$, $C \in \mathcal{A}$, it follows from $[\mathrm{i}P_1 \diamond X_{12}, C] = 0$ that

$$\begin{aligned} 0 = \delta([\mathrm{i}P_1 \diamond X_{12}, C]) &= [\delta(\mathrm{i}P_1) \diamond X_{12}, C] \\ &= [\delta(\mathrm{i}P_1)X_{12}^* + X_{12}\delta(\mathrm{i}P_1)^*, C]. \end{aligned}$$

Then

$$\delta(\mathrm{i}P_1)X_{12}^* + X_{12}\delta(\mathrm{i}P_1)^* \in \mathcal{Z}(\mathcal{A}) \tag{2.6}$$

for any $X_{12} \in \mathcal{A}_{12}$. Multiplying Eq.(2.6) on the left-hand side by P_2 and on the right-hand side by P_1, we can obtain that

$$P_2\delta(\mathrm{i}P_1)X_{12}^* = 0.$$

Sine $\mathcal{A}$ is prime, we have $P_2\delta(\mathrm{i}P_1)P_2 = 0$. Hence $\delta(\mathrm{i}P_1) = P_1\delta(\mathrm{i}P_1)P_1$. In the same way, we can obtain that $\delta(\mathrm{i}P_2) = P_2\delta(\mathrm{i}P_2)P_2$.

(3) It follows from Lemma 2.8 that

$$\delta(\mathrm{i}C_{12} + \mathrm{i}C_{12}^*) - \mathrm{i}\delta(C_{12} + C_{12}^*) + (\delta(\mathrm{i}I) - \mathrm{i}\delta(I))(C_{12} + C_{12}^*) \in \mathcal{Z}(\mathcal{A})$$

for any $C_{12} \in \mathcal{A}_{12}$. From the assert (1), we have

$$\delta(\mathrm{i}C_{12}) - \mathrm{i}\delta(C_{12}) + (\delta(\mathrm{i}I) - \mathrm{i}\delta(I))C_{12} = 0. \tag{2.7}$$

Replacing C_{12} by $\mathrm{i}C_{12}$ in Eq.(2.7), then

$$\mathrm{i}\delta(C_{12}) - \delta(\mathrm{i}C_{12}) + (\delta(\mathrm{i}I) - \mathrm{i}\delta(I))C_{12} = 0. \tag{2.8}$$

It follows from Eq.(2.7) and Eq.(2.8) that $\delta(\mathrm{i}I) = \mathrm{i}\delta(I)$. For any $C_{12} \in \mathcal{A}_{12}$, it follows from the assert (2) that

$$\begin{aligned} \delta(\mathrm{i}C_{12}^*) - \delta(\mathrm{i}C_{12}) = \delta([C_{12} \diamond I, \mathrm{i}P_1]) &= [\delta(C_{12}) \diamond I, \mathrm{i}P_1] + [C_{12} \diamond \delta(I), \mathrm{i}P_1] \\ &\quad + [C_{12} \diamond I, \delta(\mathrm{i}P_1)] \\ &= \mathrm{i}\delta(C_{12})^* - \mathrm{i}\delta(C_{12}) + \mathrm{i}\delta(I)C_{12}^* - \mathrm{i}\delta(I)^*C_{12} \\ &\quad + \delta(\mathrm{i}I)C_{12}^* - \delta(\mathrm{i}I)C_{12}. \end{aligned}$$

Thus

$$\delta(\mathrm{i}C_{12}) - \mathrm{i}\delta(C_{12}) - (\delta(\mathrm{i}I) + \mathrm{i}\delta(I)^*)C_{12} = 0. \tag{2.9}$$

It follows from Eq.(2.7) and Eq.(2.9) that $\delta(\mathrm{i}I) = \frac{1}{2}(\mathrm{i}\delta(I) - \mathrm{i}\delta(I)^*)$. Thus $\delta(\mathrm{i}I)^* = \delta(\mathrm{i}I)$, and so $\delta(I)^* = -\delta(I)$. By Lemma 2.8, we have $\delta(\mathrm{i}A) - \mathrm{i}\delta(A) \in \mathcal{Z}(\mathcal{A})$ for any $A \in \mathcal{A}_{sa}$. For any $A \in \mathcal{A}$, write $A = A_1 + \mathrm{i}A_2$, where $A_1, A_2 \in \mathcal{A}_{sa}$, then

$$\begin{aligned} \delta(\mathrm{i}A) - \mathrm{i}\delta(A) &= \delta(\mathrm{i}(A_1 + \mathrm{i}A_2)) - \mathrm{i}\delta(A_1 + \mathrm{i}A_2) \\ &= \delta(\mathrm{i}A_1) - \delta(A_2) - \mathrm{i}\delta(A_1) - \mathrm{i}\delta(\mathrm{i}A_2) \\ &= \delta(\mathrm{i}A_1) - \mathrm{i}\delta(A_1) + \mathrm{i}(\mathrm{i}\delta(A_2) - \delta(\mathrm{i}A_2)) \in \mathcal{Z}(\mathcal{A}). \end{aligned}$$

□

Remark 2.2. Let $\Phi(A) = \delta(A) - \delta(I)A$. Obviously, $\Phi : \mathcal{A} \to \mathcal{A}$ is an additive map and $\Phi(I) = 0$.

The proof of Theorem 2.1. For any $B, C \in \mathcal{A}$, on the one hand,

$$\begin{aligned}\delta([B^* + B, C]) &= \delta([I \diamond B, C]) = [\delta(I) \diamond B, C] + [I \diamond \delta(B), C] + [I \diamond B, \delta(C)] \\ &= [\delta(I)B^* + \delta(I)^*B, C] + [\delta(B)^* + \delta(B), C] \\ &+ [B^* + B, \delta(C)].\end{aligned} \tag{2.10}$$

On the other hand, it follows from Lemma 2.9 that

$$\begin{aligned}\delta([-B^* + B, C]) &= \delta([\mathrm{i}I \diamond B, \mathrm{i}C]) = [\delta(\mathrm{i}I) \diamond B, \mathrm{i}C] + [\mathrm{i}I \diamond \delta(B), \mathrm{i}C] \\ &+ [\mathrm{i}I \diamond B, \delta(\mathrm{i}C)] \\ &= [-\delta(I)B^* + \delta(I)^*B, C] + [-\delta(B)^* + \delta(B), C] \\ &+ [-B^* + B, \delta(C)].\end{aligned} \tag{2.11}$$

Compare Eq.(2.10) and Eq.(2.11), we have

$$\delta([B, C]) = [\delta(I)^*B + \delta(B), C] + [B, \delta(C)] \tag{2.12}$$

for any $B, C \in \mathcal{A}$. For any $A, B, C \in \mathcal{A}$, it follows from Eq.(2.12) that

$$\delta([A \diamond B, C]) = [\delta(I)^*A \diamond B + \delta(A \diamond B), C] + [A \diamond B, \delta(C)].$$

On the other hand,

$$\delta([A \diamond B, C]) = [\delta(A) \diamond B, C] + [A \diamond \delta(B), C] + [A \diamond B, \delta(C)].$$

Thus

$$\delta(I)^*A \diamond B + \delta(A \diamond B) - \delta(A) \diamond B - A \diamond \delta(B) \in \mathcal{Z}(\mathcal{A}). \tag{2.13}$$

Take $B = I$ in Eq.(2.13), we have

$$\delta(A^*) - \delta(A)^* + (\delta(I)^* - \delta(I))A^* \in \mathcal{Z}(\mathcal{A}). \tag{2.14}$$

It follows from Eq.(2.12) and Eq.(2.14) that

$$\Phi([A, B]) = [\Phi(A), B] + [A, \Phi(B)]$$

and

$$\Phi(A^*) - \Phi(A)^* \in \mathcal{Z}(\mathcal{A}) \tag{2.15}$$

for any $A, B \in \mathcal{A}$. By [11, Main Theorem], there exists an additive derivation $\theta : \mathcal{A} \to \mathcal{A}$ such that $\Phi(A) = \theta(A) + \xi(A)$, where $\xi : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ is an additive map vanishing at commutators.

It follows from Lemma 2.9 and Eq.(2.15) that

$$\theta(A_{ij}^*) = \theta(A_{ij})^* \tag{2.16}$$

for any $A_{ij} \in \mathcal{A}_{ij}$ with $1 \le i \ne j \le 2$. For any $A_{11} \in \mathcal{A}_{11}, A_{12} \in \mathcal{A}_{12}$, it follows from Eq. (2.16) that

$$\theta(A_{12}^*)A_{11}^* + A_{12}^*\theta(A_{11}^*) = \theta(A_{12}^*A_{11}^*) = \theta(A_{11}A_{12})^* = \theta(A_{12}^*)A_{11}^* + A_{12}^*\theta(A_{11})^*$$

and

$$\theta(A_{21}^*)A_{11}^* + A_{21}^*\theta(A_{11}^*) = \theta(A_{21}^*A_{11}^*) = \theta(A_{11}A_{21})^* = \theta(A_{21}^*)A_{11}^* + A_{21}^*\theta(A_{11})^*.$$

Thus

$$A_{12}^*(\theta(A_{11}^*) - \theta(A_{11})^*) = 0 \text{ and } A_{21}^*(\theta(A_{11}^*) - \theta(A_{11})^*) = 0. \tag{2.17}$$

Write $T = \theta(A_{11}^*) - \theta(A_{11})^*$. By Eq.(2.17), we have

$$A_{12}^*T = 0 \text{ and } A_{21}^*T = 0.$$

Since $\mathcal{A}$ is prime, we can obtain that $P_1T = 0$ and $P_2T = 0$. Hence $\theta(A_{11}^*) = \theta(A_{11})^*$. In the same way, $\theta(A_{22}^*) = \theta(A_{22})^*$. It follows that $\theta(A^*) = \theta(A)^*$. Since $\delta(A) = \delta(I)A + \theta(A) + \xi(A)$ for all $A \in \mathcal{A}$. For any $A, B, C \in \mathcal{A}$, on the one hand,

$$\delta([A \diamond B, C]) = \delta(I)[A \diamond B, C] + \theta([A \diamond B, C]).$$

On the other hand,

$$\begin{aligned}\delta([A \diamond B, C]) = &[(\delta(I)A + \theta(A) + \xi(A)) \diamond B, C] + [A \diamond (\delta(I)B + \theta(B) + \xi(B)), C]\\ &+ [A \diamond B, \delta(I)C + \theta(C) + \xi(C)]\end{aligned}$$

Thus

$$[(\delta(I)A + \xi(A)) \diamond B, C] + [A \diamond (\delta(I)B + \xi(B)), C] = 0$$

for any $A, B, C \in \mathcal{A}$. It follows from Lemma 2.9 that

$$[\xi(A) \diamond B, C] + [A \diamond \xi(B), C] = 0. \tag{2.18}$$

For any $X_{12} \in \mathcal{A}_{12}$, take $B = X_{12}$, $C = P_1$ in Eq.(2.18), we have

$$[\xi(A) \diamond X_{12}, P_1] = 0,$$

which implies that $\xi(A)X_{12}^* - \xi(A)^*X_{12} = 0$. Thus

$$\begin{cases} \xi(A)X_{12}^* = 0,\\ \xi(A)^*X_{12} = 0.\end{cases}$$

Since $\mathcal{A}$ is prime, we have

$$\begin{cases} \xi(A)P_1 = 0,\\ \xi(A)P_2 = 0.\end{cases}$$

Thus $\xi(A) = 0$ for any $A \in \mathcal{A}$. Since $\phi(A) = \theta(A) + [A, T] + \delta(I)A$, it follows from Lemma 2.9 that there exists an element $\lambda \in \mathcal{Z}_s(\mathcal{A})$ such that $\phi(A) = d(A) + \mathrm{i}\lambda A$, where $d(A) = \theta(A) + [A, T]$ is an additive $*$-derivation. □

Let $\mathcal{M}$ be a factor von Neumann algebra. It is well known that a factor von Neumann algebra $\mathcal{M}$ is prime and its center is $\mathbb{C}I$. As a consequence of Theorem 2.1, we have the following corollary.

Corollary 2.10. *Let $\mathcal{M}$ be a factor von Neumann algebra with* $\dim\mathcal{M} > 1$*, and let $\phi : \mathcal{M} \to \mathcal{M}$ be a nonlinear mixed triple derivable mapping, that is, ϕ satisfies*

$$\phi([A \diamond B, C]) = [\phi(A) \diamond B, C] + [A \diamond \phi(B), C] + [A \diamond B, \phi(C)]$$

for any $A, B, C \in \mathcal{M}$ if and only if there exists a real number λ such that $\phi(A) = d(A) + \mathrm{i}\lambda A$, where $d : \mathcal{M} \to \mathcal{M}$ is an additive $*$-derivation.

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