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Ultra filters and quasi-subalgebras of Sheffer stroke BL-algebras based on the bipolar-valued fuzzy environment

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Abstract. The notion of bipolar-valued filters and quasi-subalgebras of Sheffer stroke BL-algebras are introduced, and several properties are investigated. The relations among between a bipolar-valued fuzzy set, a bipolar-valued fuzzy filter, a bipolar-valued fuzzy quasi-subalgebra and a bipolar-valued fuzzy ultra filer-valued fuzzy ultra filer are studied and necassasy and sufficiendt conditions are given. We prove that every bipolar-valued fuzzy filter is a bipolar-valued fuzzy quasi-subalgebra.

1. Introduction

In the conventional fuzzy sets, the membership degrees of elements range over the closed interval $[0, 1]$. The membership degree depictures the degree of belongingness of elements to a fuzzy set. The membership degree 1 puts forth that an element completely belongs to its corresponding fuzzy set, and the membership degree 0 expresses that an element does not belong to the fuzzy set. The membership degrees on the interval $(0, 1)$ infer the partial membership to the fuzzy set. Occasionally, the membership degree means the satisfaction degree of elements to some property or constraint related a fuzzy set. In the sense of satisfaction degree, the membership degree 0 is appointed to elements which do not carry out some property. The elements with membership degree 0 are usually meant to be having the same characteristics in the fuzzy set representation. Among such elements, some have irrelevant characteristics to the property corresponding to a fuzzy set while the others have contrary characteristics to the property. It is difficult to set out the difference of the irrelevant elements from the contrary elements in fuzzy sets only with the membership degrees ranged on the interval $[0, 1]$. Based on these observations, Lee [7] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. The bipolar-valued fuzzy set has also been widely applied in algebraic structures such as BCH-algebras [15], BCI-algebras [1, 6] and KU-algebras [2]. The notion of m-polar fuzzy sets is a generalization of bipolar fuzzy sets, and it is applied to BCI-algebras in [3].

Go by the name of NAND operations or alternative denial, called Sheffer strokes [4] play a exceptional role in contrast to other fuzzy connectives in classical Boolean logic. Strictly speaking, this operator can

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be engaged to build up a logical formal system by itself absent tenancy of any other logical connective, while no other unary or binary connective carries out this speciality. At about we know, fuzzy set theory advertised by Zadeh [8] brought new applications of multivalued logic and new aspects in investigation of logical connectives. Sheffer stroke (or Sheffer operation) is a single binary logical connective that suffices to define all logical connectives. It has been applied to several algebraic structures such as Boolean algebra, BCK-algebra, BL-algabra, Hilbert algebras, basic algebras, and ortholattices. [11], [17], [14], [10], [12], [5].

The list of acronyms is given in Table 1.

Table 1: List of acronyms

Acronyms	Representation
BVF-set	bipolar-valued fuzzy set
BVFq-subalgebra	bipolar-valued fuzzy quasi-subalgebra
BVF-filter	bipolar-valued fuzzy filter
BVFu-filter	bipolar-valued fuzzy ultra filter

The second section lists the basic information needed to create this paper. In the third section, we define the bipolar-valued fuzzy ultra filter and the bipolar-valued fuzzy quasi-subalgebras filter in Shffer stoke BL-algebras and investigate several properties. We discuss various characterization of them. We show that every bipolar-valued fuzzy set is a bipolar-valued fuzzy ultra quasi-subalgebra, and provide an example to show that its inverse do not hold. The necessary and sufficient conditions among a bipolar-valued fuzzy set, a bipolar-valued fuzzy filter, a bipolar-valued fuzzy quasi-subalgebra and a bipolar-valued fuzzy ultra filer are given.

2. Preliminaries

Definition 2.1 ([4]). *Let $\mathfrak{A} := (A, |)$ be a groupoid. Then the operation "|" is said to be Sheffer operation or Sheffer stroke if it satisfies:*

(s1) $(\forall \mathfrak{z}, \mathfrak{u} \in A)\ (\mathfrak{z}|\mathfrak{u} = \mathfrak{u}|\mathfrak{z})$,

(s2) $(\forall \mathfrak{z}, \mathfrak{u} \in A)\ ((\mathfrak{z}|\mathfrak{z})|(\mathfrak{z}|\mathfrak{u}) = \mathfrak{z})$,

(s3) $(\forall \mathfrak{z}, \mathfrak{u}, \mathfrak{c} \in A)\ (\mathfrak{z}|((\mathfrak{u}|\mathfrak{c})|(\mathfrak{u}|\mathfrak{c})) = ((\mathfrak{z}|\mathfrak{u})|(\mathfrak{z}|\mathfrak{u}))|\mathfrak{c})$,

(s4) $(\forall \mathfrak{z}, \mathfrak{u}, \mathfrak{c} \in A)\ ((\mathfrak{z}|((\mathfrak{z}|\mathfrak{z})|(\mathfrak{u}|\mathfrak{u})))|(\mathfrak{z}|((\mathfrak{z}|\mathfrak{z})|(\mathfrak{u}|\mathfrak{u}))) = \mathfrak{z})$.

Definition 2.2 ([13]). *An algebra $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ of type $(2, 2, 2, 0, 0)$ is called a Sheffer stroke BL-algebra if it satisfies:*

(sBL1) $(W, \vee, \wedge, 0, 1)$ *is a bounded lattice,*

(sBL2) $\mathfrak{W} := (W, |)$ *is a groupoid with the Sheffer stroke "|",*

(sBL3) $\mathfrak{z} \wedge \mathfrak{u} = (\mathfrak{z}|(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u})))|(\mathfrak{z}|(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u})))$,

(sBL4) $(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u})) \vee (\mathfrak{u}|(\mathfrak{z}|\mathfrak{z})) = 1$,

for all $\mathfrak{z}, \mathfrak{u} \in W$, where $1 = 0|0$ and $0 = 1|1$ are the greatest element of W and the least element of W, respetively.

Proposition 2.3 ([13]). *Every Sheffer stroke BL-algebra* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *satisfies:*

$$\mathfrak{z}|((\mathfrak{u}|(\mathfrak{c}|\mathfrak{c}))|(\mathfrak{u}|(\mathfrak{c}|\mathfrak{c}))) = \mathfrak{u}|((\mathfrak{z}|(\mathfrak{c}|\mathfrak{c}))|(\mathfrak{z}|(\mathfrak{c}|\mathfrak{c}))), \tag{1}$$

$$\mathfrak{z}|(\mathfrak{z}|\mathfrak{z}) = 1, \tag{2}$$

$$1|(\mathfrak{z}|\mathfrak{z}) = \mathfrak{z}, \tag{3}$$

$$\mathfrak{z}|(1|1) = 1, \tag{4}$$

$$\mathfrak{z} \leq_W \mathfrak{u} \;\Leftrightarrow\; \mathfrak{z}|(\mathfrak{u}|\mathfrak{u}) = 1, \tag{5}$$

$$\mathfrak{z} \leq_W \mathfrak{u}|(\mathfrak{z}|\mathfrak{z}) \tag{6}$$

$$\mathfrak{z} \leq_W (\mathfrak{z}|\mathfrak{u})|\mathfrak{u}, \tag{7}$$

$$\begin{cases} (\mathfrak{z}|(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u})))|(\mathfrak{z}|(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u}))) \leq_W \mathfrak{z}, \\ (\mathfrak{z}|(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u})))|(\mathfrak{z}|(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u}))) \leq_W \mathfrak{u}, \end{cases} \tag{8}$$

$$\mathfrak{z} \vee \mathfrak{u} = (\mathfrak{z}|(\mathfrak{u}|\mathfrak{u}))|(\mathfrak{u}|\mathfrak{u}), \tag{9}$$

$$(\mathfrak{z}|\mathfrak{u})|(\mathfrak{z}|\mathfrak{u}) \leq_W \mathfrak{c} \;\Leftrightarrow\; \mathfrak{z} \leq_W \mathfrak{u}|(\mathfrak{c}|\mathfrak{c}), \tag{10}$$

$$\mathfrak{z}|((\mathfrak{u}|(\mathfrak{c}|\mathfrak{c}))|(\mathfrak{u}|(\mathfrak{c}|\mathfrak{c}))) = (\mathfrak{z}|(\mathfrak{u}|\mathfrak{u}))|((\mathfrak{z}|(\mathfrak{c}|\mathfrak{c}))|(\mathfrak{z}|(\mathfrak{c}|\mathfrak{c}))), \tag{11}$$

$$(\mathfrak{z}|\mathfrak{u})|(\mathfrak{z}|\mathfrak{u}) \leq_W \mathfrak{c} \;\Leftrightarrow\; \mathfrak{z} \leq_W \mathfrak{u}|(\mathfrak{c}|\mathfrak{c}), \tag{12}$$

$$\mathfrak{z} \leq_W \mathfrak{u} \;\Rightarrow \begin{cases} \mathfrak{c}|(\mathfrak{z}|\mathfrak{z}) \leq_W \mathfrak{c}|(\mathfrak{u}|\mathfrak{u}), \\ (\mathfrak{z}|\mathfrak{c})|(\mathfrak{z}|\mathfrak{c}) \leq_W (\mathfrak{u}|\mathfrak{c})|(\mathfrak{u}|\mathfrak{c}), \\ \mathfrak{u}|(\mathfrak{c}|\mathfrak{c}) \leq_W \mathfrak{z}|(\mathfrak{c}|\mathfrak{c}), \end{cases} \tag{13}$$

$$(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u}))|(\mathfrak{u}|\mathfrak{u}) = (\mathfrak{u}|(\mathfrak{z}|\mathfrak{z}))|(\mathfrak{z}|\mathfrak{z}), \tag{14}$$

$$((\mathfrak{z}|(\mathfrak{u}|\mathfrak{u}))|(\mathfrak{u}|\mathfrak{u}))|(\mathfrak{u}|\mathfrak{u}) = \mathfrak{z}|(\mathfrak{u}|\mathfrak{u}) \tag{15}$$

for all $\mathfrak{z}, \mathfrak{u}, \mathfrak{c} \in W$.

Definition 2.4 ([13]). *Let* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *be a Sheffer stroke BL-algebra. A subset* F *of* W *is called a filter of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *if*

$$(\forall \mathfrak{z}, \mathfrak{u} \in W)(\mathfrak{z}, \mathfrak{u} \in F \;\Rightarrow\; (\mathfrak{z}|\mathfrak{u})|(\mathfrak{z}|\mathfrak{u}) \in F) \tag{16}$$

$$(\forall \mathfrak{z}, \mathfrak{u} \in W)(\mathfrak{z} \in F,\, \mathfrak{z} \leq_W \mathfrak{u} \;\Rightarrow\; \mathfrak{u} \in F). \tag{17}$$

Lemma 2.5 ([13]). *A subset* F *of* W *is a filter of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *if and only if it satisfies:*

$$1 \in F, \tag{18}$$

$$(\forall \mathfrak{z}, \mathfrak{u} \in W)(\mathfrak{z} \in F,\, \mathfrak{z}|(\mathfrak{u}|\mathfrak{u}) \in F \;\Rightarrow\; \mathfrak{u} \in F). \tag{19}$$

Definition 2.6 ([13]). *Let* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *be a Sheffer stroke BL-algebra. An ultra filter of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *is defined to be a filter* F *of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *that satisfies:*

$$(\forall \mathfrak{z}, \mathfrak{u} \in W)(\mathfrak{z} \in F \text{ or } \mathfrak{z}|\mathfrak{z} \in F). \tag{20}$$

Definition 2.7 ([9]). *Let* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *be a Sheffer stroke BL-algebra. A subset* F *of* W *is called a quasi-subalgebra of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *if* $\mathfrak{z}|(\mathfrak{u}|\mathfrak{u}) \in F$ *for all* $\mathfrak{z}, \mathfrak{u} \in F$.

Lemma 2.8 ([13]). *Let* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *be a Sheffer stroke BL-algebra. A subset* F *of* W *is an ultra filter of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *if and only if it is a filter of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *that satisfies:*

$$(\forall \mathfrak{z}, \mathfrak{u} \in W)(\mathfrak{z} \vee \mathfrak{u} \in F \;\Rightarrow\; \mathfrak{z} \in F \text{ or } \mathfrak{u} \in F). \tag{21}$$

Definition 2.9 ([13]). *Let* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *be a Sheffer stroke BL-algebra. A fuzzy set* β *in* W *is called a fuzzy filter of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *if it satisfies:*

$$(\forall \mathfrak{z} \in W)(\beta(1) \geq \beta(\mathfrak{z})), \tag{22}$$

$$(\forall \mathfrak{z}, \mathfrak{u} \in W)(\beta(\mathfrak{u}) \geq \min\{\beta(\mathfrak{z}), \beta(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u}))\}). \tag{23}$$

Definition 2.10 ([13]). *Let $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ be a Sheffer stroke BL-algebra. A fuzzy set β in W is called a fuzzy ultra filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ if it is a fuzzy filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ such that $\beta(\mathfrak{z}) = \beta(1)$ or $\beta(\mathfrak{z}|\mathfrak{z}) = \beta(1)$ for all $\mathfrak{z} \in W$.*

Consider two maps β^- and β^+ on W (; a universe of discourse) as follows:

$$\beta^- : W \to [-1, 0] \text{ and } \beta^+ : W \to [0, 1],$$

respectively. A structure

$$\beta := \{(\mathfrak{z}; \beta^-(\mathfrak{z}), \beta^+(\mathfrak{z})) \mid \mathfrak{z} \in W\}$$

is called a *BVF-set* on W (see [7]), and is will be denoted by simply $\beta := (W; \beta^-, \beta^+)$.

For a BVF-set $\beta := (W; \beta^-, \beta^+)$ in W and $(s, t) \in [-1, 0] \times [0, 1]$, we define

$$L(\beta^-; s) := \{\mathfrak{z} \in W \mid \beta^-(\mathfrak{z}) \leq s\},$$
$$U(\beta^+; t) := \{\mathfrak{z} \in W \mid \beta^+(\mathfrak{z}) \geq t\}$$

which are called the *negative s-cut* and the *positive t-cut* of $\beta := (W; \beta^-, \beta^+)$, respectively.

It is clear that if $s_1 \leq s_2$, then $L(\beta^-; s_1) \subseteq L(\beta^-; s_2)$, and if $t_1 \leq t_2$, then $U(\beta^+; t_1) \supseteq U(\beta^+; t_2)$.

Definition 2.11 ([16]). *A BVF-set $\beta := (W; \beta^-, \beta^+)$ on W is called a BVF-filter of a Sheffer stroke BL-algebra $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ if its negative s-cut and positive t-cut are filters of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ whenever they are nonempty for all $(s, t) \in [-1, 0] \times [0, 1]$.*

Lemma 2.12 ([16]). *A BVF-set $\beta := (W; \beta^-, \beta^+)$ in W is a BVF-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ if and only if it satisfies:*

$$(\forall \mathfrak{z} \in W)\left(\begin{array}{l}\beta^-(1) \leq \beta^-(\mathfrak{z})\\ \beta^+(1) \geq \beta^+(\mathfrak{z})\end{array}\right), \tag{24}$$

$$(\forall \mathfrak{z}, \mathfrak{u} \in W)\left(\begin{array}{l}\beta^-(\mathfrak{u}) \leq \max\{\beta^-(\mathfrak{z}), \beta^-(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u}))\}\\ \beta^+(\mathfrak{u}) \geq \min\{\beta^+(\mathfrak{z}), \beta^+(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u}))\}\end{array}\right), \tag{25}$$

Lemma 2.13 ([16]). *A BVF-set $\beta := (W; \beta^-, \beta^+)$ in W is a BVF-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ if and only if it satisfies:*

$$(\forall \mathfrak{z}, \mathfrak{u} \in W)\left(\mathfrak{z} \leq_W \mathfrak{u} \Rightarrow \left\{\begin{array}{l}\beta^-(\mathfrak{z}) \geq \beta^-(\mathfrak{u})\\ \beta^+(\mathfrak{z}) \leq \beta^+(\mathfrak{u})\end{array}\right.\right), \tag{26}$$

$$(\forall \mathfrak{z}, \mathfrak{u} \in W)\left(\begin{array}{l}\beta^-((\mathfrak{z}|\mathfrak{u})|(\mathfrak{z}|\mathfrak{u})) \leq \max\{\beta^-(\mathfrak{z}), \beta^-(\mathfrak{u})\}\\ \beta^+((\mathfrak{z}|\mathfrak{u})|(\mathfrak{z}|\mathfrak{u})) \geq \min\{\beta^+(\mathfrak{z}), \beta^+(\mathfrak{u})\}\end{array}\right). \tag{27}$$

3. Bipolar-valued fuzzy ultra filters

In what follows, let $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ denote a Sheffer stroke BL-algebra unless otherwise specified.

Definition 3.1. *A BVF-set $\beta := (W; \beta^-, \beta^+)$ on W is called a BVFu-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ if its negative s-cut and positive t-cut are ultra filters of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ whenever they are nonempty for all $(s, t) \in [-1, 0] \times [0, 1]$.*

Definition 3.2. *A BVF-set $\beta := (W; \beta^-, \beta^+)$ on W is called a BVFq-subalgebra of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ if its negative s-cut and positive t-cut are quasi-subalgebras of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ whenever they are nonempty for all $(s, t) \in [-1, 0] \times [0, 1]$.*

Figure 1: Hasse Diagram

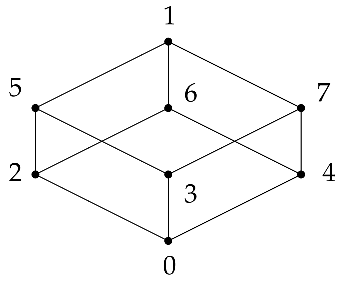

Table 2: Cayley table of the Sheffer stroke "|"

|	0	2	3	4	5	6	7	1
0	1	1	1	1	1	1	1	1
2	1	7	1	1	7	7	1	7
3	1	1	6	1	6	1	6	6
4	1	1	1	5	1	5	5	5
5	1	7	6	1	4	7	6	4
6	1	7	1	5	7	3	5	3
7	1	1	6	5	6	5	2	2
1	1	7	6	5	4	3	2	1

Example 3.3. *Let* $W := \{0, 1, 2, 3, 4, 5, 6, 7\}$. *The Hasse diagram, the Sheffer stroke "|" and binary operations "$\vee$" and "$\wedge$" on* W *are given by Figure 1, Table 2, Table 3 and Table 4, respectively.*

Then $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *is a Sheffer stroke BL-algebra (see [13]). Let* $\beta := (W; \beta^-, \beta^+)$ *and* $\gamma := (W; \gamma^-, \gamma^+)$ *be BVF-sets in* W *given by Table 5.*

The negative s-cut and the positive t-cut of $\beta := (W; \beta^-, \beta^+)$ *are calculated as follows:*

$$L(\beta^-; s) = \begin{cases} \emptyset & \text{if } s \in [-1, -0.72), \\ \{1\} & \text{if } s \in [-0.72, -0.62), \\ \{0, 1\} & \text{if } s \in [-0.62, -0.58), \\ W & \text{if } s \in [-0.58, 0], \end{cases}$$

Table 3: Cayley table of the binary operation "$\vee$"

$\vee$	0	2	3	4	5	6	7	1
0	0	2	3	4	5	6	7	1
2	2	2	5	6	5	6	1	1
3	3	5	3	7	5	1	7	1
4	4	6	7	4	1	6	7	1
5	5	5	5	1	5	1	1	1
6	6	6	1	6	1	6	1	1
7	7	1	7	7	1	1	7	1
1	1	1	1	1	1	1	1	1

Table 4: Cayley table of the binary operation "$\wedge$"

$\wedge$	0	2	3	4	5	6	7	1
0	0	0	0	0	0	0	0	0
2	0	2	0	0	2	2	0	2
3	0	0	3	0	3	0	3	3
4	0	0	0	4	0	4	4	4
5	0	2	3	0	5	2	3	5
6	0	2	0	4	2	6	4	6
7	0	0	3	4	3	4	7	7
1	0	2	3	4	5	6	7	1

Table 5: Tabular representation of β and γ

W	$\beta^-(x)$	$\beta^+(x)$	$\gamma^-(x)$	$\gamma^+(x)$
0	−0.62	0.77	−0.63	0.36
2	−0.58	0.48	−0.63	0.97
3	−0.58	0.48	−0.63	0.36
4	−0.58	0.48	−0.69	0.36
5	−0.58	0.48	−0.63	0.97
6	−0.58	0.48	−0.69	0.97
7	−0.58	0.48	−0.69	0.36
1	−0.72	0.86	−0.69	0.97

and

$$U(\beta^+; t) = \begin{cases} W & \text{if } t \in [0, 0.48], \\ \{0, 1\} & \text{if } t \in (0.48, 0.77], \\ \{1\} & \text{if } t \in (0.77, 0.86], \\ \emptyset & \text{if } t \in (0.86, 1], \end{cases}$$

which are quasi-subalgebras of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$*. Hence* $\beta := (W; \beta^-, \beta^+)$ *is a BVFq-subalgebra of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$*. Also, the negative s-cut and the positive t-cut of* $\gamma := (W; \gamma^-, \gamma^+)$ *are calculated as follows:*

$$L(\gamma^-; s) = \begin{cases} \emptyset & \text{if } s \in [-1, -0.69), \\ \{1, 4, 6, 7\} & \text{if } s \in [-0.69, -0.63), \\ W & \text{if } s \in [-0.63, 0], \end{cases}$$

and

$$U(\gamma^+; t) = \begin{cases} W & \text{if } t \in [0, 0.36], \\ \{1, 2, 5, 6\} & \text{if } t \in (0.36, 0.97], \\ \emptyset & \text{if } t \in (0.97, 1], \end{cases}$$

which are ultra filters of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$*. Therefore* $\gamma := (W; \gamma^-, \gamma^+)$ *is a BVFu-filter of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$*.*

Theorem 3.4. *A BVF-set* $\beta := (W; \beta^-, \beta^+)$ *in* W *is a BVFq-subalgebra of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *if and only if it satisfies:*

$$(\forall y, \mathfrak{b} \in W) \begin{pmatrix} \beta^-(y|(\mathfrak{b}|\mathfrak{b})) \leq \max\{\beta^-(y), \beta^-(\mathfrak{b})\} \\ \beta^+(y|(\mathfrak{b}|\mathfrak{b})) \geq \min\{\beta^+(y), \beta^+(\mathfrak{b})\} \end{pmatrix}. \tag{28}$$

Proof. Let $\beta := (W; \beta^-, \beta^+)$ be a BVFq-subalgebra of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. If (28) is not valid, then

$$\beta^-(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u})) > \max\{\beta^-(\mathfrak{z}), \beta^-(\mathfrak{u})\} \text{ or } \beta^+(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u})) < \min\{\beta^+(\mathfrak{z}), \beta^+(\mathfrak{u})\}$$

for some $\mathfrak{z}, \mathfrak{u} \in W$. If $\beta^-(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u})) > \max\{\beta^-(\mathfrak{z}), \beta^-(\mathfrak{u})\}$, then $\mathfrak{z}, \mathfrak{u} \in L(\beta^-; s)$ and $\mathfrak{z}|(\mathfrak{u}|\mathfrak{u}) \notin L(\beta^-, s)$ for $s := \max\{\beta^-(\mathfrak{z}); \beta^-(\mathfrak{u})\}$. If

$$\beta^+(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u})) < \min\{\beta^+(\mathfrak{z}), \beta^+(\mathfrak{u})\},$$

then $\mathfrak{z}, \mathfrak{u} \in U(\beta^+; t)$ and $\mathfrak{z}|(\mathfrak{u}|\mathfrak{u}) \notin U(\beta^+, t)$ for $t := \min\{\beta^+(\mathfrak{z}), \beta^+(\mathfrak{u})\}$. This is a contradiction, and so (28) is valid.

Conversely, assume that $\beta := (W; \beta^-, \beta^+)$ satisfies (28). Let $(s, t) \in [-1, 0] \times [0, 1]$ be such that $L(\beta^-, s) \neq \emptyset \neq U(\beta^+, t)$. Let $y, \mathfrak{b} \in L(\beta^-, s) \cap U(\beta^+, t)$. Then

$$\beta^-(y|(\mathfrak{b}|\mathfrak{b})) \leq \max\{\beta^-(y), \beta^-(\mathfrak{b})\} \leq s$$

and

$$\beta^+(y|(\mathfrak{b}|\mathfrak{b})) \geq \min\{\beta^+(y), \beta^+(\mathfrak{b})\} \geq t$$

by (28), and so $y|(\mathfrak{b}|\mathfrak{b}) \in L(\beta^-, s) \cap U(\beta^+, t)$. Hence $L(\beta^-, s)$ and $U(\beta^+, t)$ are quasi-subalgebras of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$, and therefore $\beta := (W; \beta^-, \beta^+)$ is a BVFq-subalgebra of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. □

Theorem 3.5. *Every BVF-filter is a BVFq-subalgebra.*

Proof. Let $\beta := (W; \beta^-, \beta^+)$ be a BVF-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ and let $y, \mathfrak{b} \in W$. Then

$$\begin{aligned} &((y|\mathfrak{b})|(y|\mathfrak{b}))|((y|(\mathfrak{b}|\mathfrak{b}))|(y|(\mathfrak{b}|\mathfrak{b}))) \\ &\overset{(s3)}{=} y|((\mathfrak{b}|((y|(\mathfrak{b}|\mathfrak{b}))|(y|(\mathfrak{b}|\mathfrak{b}))))|(\mathfrak{b}|((y|(\mathfrak{b}|\mathfrak{b}))|(y|(\mathfrak{b}|\mathfrak{b}))))) \\ &\overset{(1)}{=} y|((y|((\mathfrak{b}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|(\mathfrak{b}|\mathfrak{b}))))|(y|((\mathfrak{b}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|(\mathfrak{b}|\mathfrak{b}))))) \\ &\overset{(2)}{=} y|((y|(1|1))|(y|(1|1))) \\ &\overset{(4)}{=} 1, \end{aligned}$$

and so $(y|\mathfrak{b})|(y|\mathfrak{b}) \leq_W y|(\mathfrak{b}|\mathfrak{b})$ by (5). It follows from (26) and (27) that

$$\beta^-(y|(\mathfrak{b}|\mathfrak{b})) \leq \beta^-((y|\mathfrak{b})|(y|\mathfrak{b})) \leq \max\{\beta^-(y), \beta^-(\mathfrak{b})\}$$

and

$$\beta^+(y|(\mathfrak{b}|\mathfrak{b})) \geq \beta^+((y|\mathfrak{b})|(y|\mathfrak{b})) \geq \min\{\beta^+(y), \beta^+(\mathfrak{b})\}.$$

Therefore $\beta := (W; \beta^-, \beta^+)$ is a BVFq-subalgebra of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ by Theorem 3.4. □

The converse of Theorem 3.5 may not be true as seen in the example below.

Example 3.6. *Consider the BVFq-subalgebra $\beta := (W; \beta^-, \beta^+)$ in Example 3.3. We can observe that $0 \leq_W 4$, but $\beta^-(0) = -0.62 < -0.58 = \beta^-(4)$ and/or $\beta^+(0) = 0.77 > 0.48 = \beta^+(4)$. Hence $\beta := (W; \beta^-, \beta^+)$ does not satisfy the condition (26), and therefore $\beta := (W; \beta^-, \beta^+)$ is not a BVF-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$.*

It is clear that every BVFu-filter is a BVF-filter. But the converse is not ture in general as seen in the following example.

Example 3.7. *Consider the Sheffer stroke BL-algebra $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ in Example 3.3 and let $\beta := (W; \beta^-, \beta^+)$ be a BVF-set in W given given by Table 6.*

Table 6: Tabular representation of $\beta := (W; \beta^-, \beta^+)$

W	$\beta^-(x)$	$\beta^+(x)$
0	−0.61	0.29
2	−0.61	0.29
3	−0.61	0.29
4	−0.61	0.29
5	−0.72	0.29
6	−0.61	0.29
7	−0.61	0.62
1	−0.72	0.62

Then $\beta := (W; \beta^-, \beta^+)$ *is a BVF-filter of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. *But it is not a BVFu-filter of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *since the negative s-cut and the positive t-cut are not ultra filters of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$, *where the negative s-cut and the positive t-cut are given, respectively, as follows:*

$$L(\beta^-; s) = \begin{cases} \emptyset & \text{if } s \in [-1, -0.72), \\ \{1, 5\} & \text{if } s \in [-0.72, -0.61), \\ W & \text{if } s \in [-0.61, 0], \end{cases}$$

and

$$U(\beta^+; t) = \begin{cases} W & \text{if } t \in [0, 0.29], \\ \{1, 7\} & \text{if } t \in (0.29, 0.62], \\ \emptyset & \text{if } t \in (0.62, 1]. \end{cases}$$

We provide conditions for the BVF-filter to be the BVFu-filter.

Theorem 3.8. *A BVF-set* $\beta := (W; \beta^-, \beta^+)$ *in* W *is a BVFu-filter of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *if and only if it is a BVF-filter of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *that satisfies:*

$$(\forall y \in W) \left(\begin{array}{l} \beta^-(y) \neq \beta^-(1) \Rightarrow \beta^-(y|y) = \beta^-(1) \\ \beta^+(y) \neq \beta^+(1) \Rightarrow \beta^+(y|y) = \beta^+(1) \end{array} \right). \tag{29}$$

Proof. Assume that $\beta := (W; \beta^-, \beta^+)$ is a BVFu-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. Then it is clear that $\beta := (W; \beta^-, \beta^+)$ is a BVF-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. Let $y \in W$ be such that $\beta^-(y) \neq \beta^-(1)$ and $\beta^+(y) \neq \beta^+(1)$. Then $\beta^-(y) > \beta^-(1)$ and $\beta^+(y) < \beta^+(1)$ by (24), and so $y \notin L(\beta^-; s) \cap U(\beta^+; t)$ for $s := \beta^-(1)$ and $t := \beta^+(1)$. It follows that $y|y \notin L(\beta^-; s) \cap U(\beta^+; t)$. Hence $\beta^-(y|y) \leq s = \beta^-(1)$ and $\beta^+(y|y) \geq t = \beta^+(1)$, and therefore $\beta^-(y|y) = \beta^-(1)$ and $\beta^+(y|y) = \beta^+(1)$.

Conversely, let $\beta := (W; \beta^-, \beta^+)$ be a BVF-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ that satisfies (29). Then $L(\beta^-; s)$ and $U(\beta^+; t)$ are filters of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ whenever they are nonempty for all $(s, t) \in [-1, 0] \times [0, 1]$. Suppose that $y \notin L(\beta^-; s) \cap U(\beta^+; t)$ for all $y \in W$ and $(s, t) \in [-1, 0] \times [0, 1]$. If $y \in L(\beta^-; \beta^-(1)) \cap U(\beta^+; \beta^+(1))$, then $y \in L(\beta^-; s) \cap U(\beta^+; t)$, and so $L(\beta^-; s)$ and $U(\beta^+; t)$ are ultra filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. If $y \notin L(\beta^-; \beta^-(1)) \cap U(\beta^+; \beta^+(1))$, then $y \notin L(\beta^-; \beta^-(1))$ or $y \notin U(\beta^+; \beta^+(1))$. If $y \notin L(\beta^-; \beta^-(1))$, then $y \in L(\beta^-; s) \setminus L(\beta^-; \beta^-(1))$ and so $\beta^-(y) \neq \beta^-(1)$. If $y \notin U(\beta^+; \beta^+(1))$, then $y \in U(\beta^+; t) \setminus U(\beta^+; \beta^+(1))$ and thus $\beta^+(y) \neq \beta^+(1)$. It follows from (29) that $\beta^-(y|y) = \beta^-(1)$ and $\beta^+(y|y) = \beta^+(1)$. Hence $y|y \in L(\beta^-; \beta^-(1)) \subseteq L(\beta^-; s)$ and $y|y \in U(\beta^+; \beta^+(1)) \subseteq U(\beta^+; t)$. Therefore $L(\beta^-; s)$ and $U(\beta^+; t)$ are ultra filters of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. Consequently, $\beta := (W; \beta^-, \beta^+)$ is a BVFq-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. □

Theorem 3.9. *A BVF-set* $\beta := (W; \beta^-, \beta^+)$ *in* W *is a BVFu-filter of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *if and only if it is a BVF-filter of* $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ *that satisfies:*

$$(\forall y, \mathfrak{b} \in W) \left(\begin{array}{l} \beta^-(y) \neq \beta^-(1) \Rightarrow \beta^-(y|(\mathfrak{b}|\mathfrak{b})) = \beta^-(1) \\ \beta^+(y) \neq \beta^+(1) \Rightarrow \beta^+(y|(\mathfrak{b}|\mathfrak{b})) = \beta^+(1) \end{array} \right). \tag{30}$$

Proof. Assume that $\beta := (W; \beta^-, \beta^+)$ is a BVFu-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. Then $\beta := (W; \beta^-, \beta^+)$ is a BVF-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. Let $y, \mathfrak{b} \in W$ be such that $\beta^-(y) \neq \beta^-(1)$ and $\beta^+(y) \neq \beta^+(1)$. Then $\beta^-(y|y) = \beta^-(1)$ and $\beta^+(y|y) = \beta^+(1)$ by (29). Since

$$y|y \overset{(6)}{\leq_W} (\mathfrak{b}|\mathfrak{b})|((y|y)|(y|y)) \overset{(s1),(s2)}{=} y|(\mathfrak{b}|\mathfrak{b}),$$

we have $(y|y)|((y|(\mathfrak{b}|\mathfrak{b}))|(y|(\mathfrak{b}|\mathfrak{b}))) = 1$ for all $y, \mathfrak{b} \in W$ by (5). Using (25) and Theorem 3.8 leads to

$$\begin{aligned}\beta^-(y|(\mathfrak{b}|\mathfrak{b})) &\leq \max\{\beta^-(y|y), \beta^-((y|y)|((y|(\mathfrak{b}|\mathfrak{b}))|(y|(\mathfrak{b}|\mathfrak{b}))))\}\\ &= \max\{\beta^-(1), \beta^-(1)\} = \beta^-(1)\end{aligned}$$

and

$$\begin{aligned}\beta^+(y|(\mathfrak{b}|\mathfrak{b})) &\geq \min\{\beta^+(y|y), \beta^+((y|y)|((y|(\mathfrak{b}|\mathfrak{b}))|(y|(\mathfrak{b}|\mathfrak{b}))))\}\\ &= \min\{\beta^+(1), \beta^+(1)\} = \beta^+(1),\end{aligned}$$

and hence $\beta^-(y|(\mathfrak{b}|\mathfrak{b})) = \beta^-(1)$ and $\beta^+(y|(\mathfrak{b}|\mathfrak{b})) = \beta^+(1)$ for all $y, \mathfrak{b} \in W$.

Conversely, let $\beta := (W; \beta^-, \beta^+)$ be a BVF-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ that satisfies (30). Suppose that $\beta^-(y) \neq \beta^-(1)$ and $\beta^+(y) \neq \beta^+(1)$ for all $y \in W$. In particular, $\beta^-(1|1) \neq \beta^-(1)$ and $\beta^+(1|1) \neq \beta^+(1)$. It follows from (s1), (s2), (3) and (30) that

$$\begin{aligned}\beta^-(y|y) &= \beta^-(1|((y|y)|(y|y))) = \beta^-(y|1)\\ &= \beta^-(y|((1|1)|(1|1))) = \beta^-(1)\end{aligned}$$

and

$$\begin{aligned}\beta^+(y|y) &= \beta^+(1|((y|y)|(y|y))) = \beta^+(y|1)\\ &= \beta^+(y|((1|1)|(1|1))) = \beta^+(1).\end{aligned}$$

Therefore, $\beta := (W; \beta^-, \beta^+)$ is a BVFu-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ by Theorem 3.8. □

Theorem 3.10. *A BVF-set $\beta := (W; \beta^-, \beta^+)$ in W is a BVFu-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ if and only if it is a BVF-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ that satisfies:*

$$(\forall y, \mathfrak{b} \in W)\left(\begin{array}{l}\beta^-(y \vee \mathfrak{b}) \geq \min\{\beta^-(y), \beta^-(\mathfrak{b})\}\\ \beta^+(y \vee \mathfrak{b}) \leq \max\{\beta^+(y), \beta^+(\mathfrak{b})\}\end{array}\right). \tag{31}$$

Proof. Let $\beta := (W; \beta^-, \beta^+)$ be a BVFu-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. Then $\beta := (W; \beta^-, \beta^+)$ is a BVF-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. If $\beta^-(\mathfrak{z}) = \beta^-(1)$ for $\mathfrak{z} \in W$, then $\min\{\beta^-(\mathfrak{z}), \beta^-(\mathfrak{u})\} = \beta^-(1) \leq \beta^-(\mathfrak{z} \vee \mathfrak{u})$ for all $\mathfrak{u} \in W$ If $\beta^+(y) = \beta^+(1)$ for $y \in W$, then $\max\{\beta^+(y), \beta^+(\mathfrak{b})\} = \beta^+(1) \geq \beta^+(y \vee \mathfrak{b})$ for all $\mathfrak{b} \in W$ If $\beta^-(\mathfrak{z}) \neq \beta^-(1)$ and $\beta^+(y) \neq \beta^+(1)$ for all $(\mathfrak{z}, y) \in W \times W$, then $\beta^-(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u})) = \beta^-(1)$ and $\beta^+(y|(\mathfrak{b}|\mathfrak{b})) = \beta^+(1)$ by Theorem 3.9. It follows from Lemma 2.12 and (9) that

$$\begin{aligned}\beta^-(\mathfrak{z} \vee \mathfrak{u}) &= \max\{\beta^-(1), \beta^-(\mathfrak{z} \vee \mathfrak{u})\}\\ &= \max\{\beta^-(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u})), \beta^-((\mathfrak{z}|(\mathfrak{u}|\mathfrak{u}))|(\mathfrak{u}|\mathfrak{u}))\}\\ &\geq \beta^-(\mathfrak{u}) \geq \min\{\beta^-(\mathfrak{z}), \beta^-(\mathfrak{u})\}\end{aligned}$$

and

$$\begin{aligned}\beta^+(y \vee \mathfrak{b}) &= \min\{\beta^+(1), \beta^-(y \vee \mathfrak{b})\}\\ &= \min\{\beta^+(y|(\mathfrak{b}|\mathfrak{b})), \beta^+((y|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}))\}\\ &\leq \beta^+(\mathfrak{b}) \leq \max\{\beta^+(y), \beta^+(\mathfrak{b})\}\end{aligned}$$

Thus (31) is valid.

Conversely, let $\beta := (W; \beta^-, \beta^+)$ be a BVF-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ that satisfies (31). Then

$$\begin{aligned}\beta^-(1) &= \beta^-(y|(y|y)) \\ &= \beta^-(y|(((y|y)|(y|y))|((y|y)|(y|y))) \\ &= \beta^-(y \vee (y|y)) \\ &\geq \min\{\beta^-(y), \beta^-(y|y)\}\end{aligned}$$

and

$$\begin{aligned}\beta^+(1) &= \beta^+(y|(y|y)) \\ &= \beta^+(y|(((y|y)|(y|y))|((y|y)|(y|y))) \\ &= \beta^+(y \vee (y|y)) \\ &\leq \max\{\beta^+(y), \beta^+(y|y)\}\end{aligned}$$

for all $y \in W$ by (2), (s2), (9) and (31). Hence $\beta^-(y|y) = \beta^-(1)$ and $\beta^+(y|y) = \beta^+(1)$ for all $y \in W$ with $\beta^-(y) \neq \beta^-(1)$ and $\beta^+(y) \neq \beta^+(1)$. Therefore, $\beta := (W; \beta^-, \beta^+)$ is a BVFu-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ by Theorem 3.8. □

For every elements $w_-, w_+ \in W$, we consider the sets:

$$\begin{aligned}W_\beta^{w_-} &:= \{y \in W \mid \beta^-(y) \leq \beta^-(w_-)\}, \\ W_\beta^{w_+} &:= \{y \in W \mid \beta^+(y) \geq \beta^+(w_+)\}.\end{aligned}$$

It is clear that $w_- \in W_\beta^{w_-}$ and $w_+ \in W_\beta^{w_+}$.

Lemma 3.11 ([16]). *If $\beta := (W; \beta^-, \beta^+)$ is a BVF-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$, then the sets $W_\beta^{w_-}$ and $W_\beta^{w_+}$ are filters of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ for all $w_-, w_+ \in W$.*

Theorem 3.12. *If $\beta := (W; \beta^-, \beta^+)$ is a BVFu-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$, then the sets $W_\beta^{w_-}$ and $W_\beta^{w_+}$ are ultra filters of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ for all $w_-, w_+ \in W$.*

Proof. Let $\beta := (W; \beta^-, \beta^+)$ be a BVFu-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. Then $\beta := (W; \beta^-, \beta^+)$ is a BVF-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$, and so the sets $W_\beta^{w_-}$ and $W_\beta^{w_+}$ are filters of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ for all $w_-, w_+ \in W$ by Lemma 3.11. Suppose that $(\mathfrak{z}, y) \notin W_\beta^{w_-} \times W_\beta^{w_+}$ for every $(\mathfrak{z}, y) \in W \times W$. Then $\beta^-(\mathfrak{z}) \neq \beta^-(1)$ and $\beta^+(y) \neq \beta^+(1)$, which imply from Theorem 3.8 that $\beta^-(\mathfrak{z}|\mathfrak{z}) = \beta^-(1) \leq \beta^-(w_-)$ and $\beta^+(y|y) = \beta^+(1) \geq \beta^+(w_+)$. Hence $(\mathfrak{z}|\mathfrak{z}, y|y) \in W_\beta^{w_-} \times W_\beta^{w_+}$, and therefore $W_\beta^{w_-}$ and $W_\beta^{w_+}$ are ultra filters of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ for all $w_-, w_+ \in W$. □

Theorem 3.13. *A BVF-set $\beta := (W; \beta^-, \beta^+)$ in W is a BVFu-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ if and only if the fuzzy sets β_c^- and β^+ are fuzzy ultra filters of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$, where $\beta_c^- : W \to [0, 1],\ y \mapsto 1 - \beta^-(y)$.*

Proof. Assume that $\beta := (W; \beta^-, \beta^+)$ is a BVFu-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. It is clear that β^+ is a fuzzy filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. For every $y, \mathfrak{b} \in W$, we have $\beta_c^-(1) = 1 - \beta^-(1) \geq 1 - \beta^-(y) = \beta_c^-(y)$ and

$$\begin{aligned}\beta_c^-(\mathfrak{b}) &= 1 - \beta^-(\mathfrak{b}) \geq 1 - \max\{\beta^-(y), \beta^-(y|(\mathfrak{b}|\mathfrak{b}))\} \\ &= \min\{1 - \beta^-(y), 1 - \beta^-(y|(\mathfrak{b}|\mathfrak{b}))\} \\ &= \min\{\beta_c^-(y), \beta_c^-(\mathfrak{b})\}.\end{aligned}$$

Hence β_c^- is a fuzzy filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. Let $(\mathfrak{z}, y) \in W \times W$ be such that $\beta_c^-(\mathfrak{z}) \neq \beta_c^-(1)$ and $\beta^+(y) \neq \beta^+(1)$. Then $1 - \beta^-(\mathfrak{z}) = \beta_c^-(\mathfrak{z}) \neq \beta_c^-(1) = 1 - \beta^-(1)$, and so $\beta^-(\mathfrak{z}) \neq \beta^-(1)$. Using Theorem 3.8 leads to

$\beta^-(\mathfrak{z}|\mathfrak{z}) = \beta^-(1)$, and so $\beta_c^-(\mathfrak{z}|\mathfrak{z}) = 1 - \beta^-(\mathfrak{z}|\mathfrak{z}) = 1 - \beta^-(1) = \beta_c^-(1)$, and $\beta^+(y|y) = \beta^+(1)$. Therefore, β_c^- and β^+ are fuzzy ultra filters of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$.

Conversely, suppose that the fuzzy sets β_c^- and β^+ are fuzzy ultra filters of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. Then $1 - \beta^-(1) = \beta_c^-(1) \geq \beta_c^-(y) = 1 - \beta^-(y)$ and

$$\begin{aligned} 1 - \beta^-(\mathfrak{b}) = \beta_c^-(\mathfrak{b}) &\geq \min\{\beta_c^-(y), \beta_c^-(y|(\mathfrak{b}|\mathfrak{b}))\} \\ &= \min\{1 - \beta^-(y), 1 - \beta^-(y|(\mathfrak{b}|\mathfrak{b}))\} \\ &= 1 - \max\{\beta^-(y), \beta^-(y|(\mathfrak{b}|\mathfrak{b}))\} \end{aligned}$$

for all $y, \mathfrak{b} \in W$. It follows that $\beta^-(1) \leq \beta^-(y)$ and

$$\beta^-(\mathfrak{b}) \leq \max\{\beta^-(y), \beta^-(y|(\mathfrak{b}|\mathfrak{b}))\}$$

for all $y, \mathfrak{b} \in W$. Hence $\beta := (W; \beta^-, \beta^+)$ is a BVF-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ by Lemma 2.12. Let $(\mathfrak{z}, y) \in W \times W$ be such that $\beta^-(\mathfrak{z}) \neq \beta^-(1)$ and $\beta^+(y) \neq \beta^+(1)$. Then $\beta_c^-(\mathfrak{z}) = 1 - \beta^-(\mathfrak{z}) \neq 1 - \beta^-(1) = \beta_c^-(1)$. Since β_c^- and β^+ are fuzzy ultra filters of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$, it follows that $\beta_c^-(\mathfrak{z}|\mathfrak{z}) = \beta_c^-(1)$, and so $\beta^-(\mathfrak{z}|\mathfrak{z}) = \beta^-(1)$, and $\beta^+(y|y) = \beta^+(1)$. Consequently, $\beta := (W; \beta^-, \beta^+)$ is a BVFu-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ by Theorem 3.8. □

Theorem 3.14. *For a nonempty subset F of W, consider the BVF-set $\beta_F := (W; \beta_F^-, \beta_F^+)$ be a BVF-set in W defined as follows:*

$$\beta_F^- : W \to [-1, 0],\ \mathfrak{z} \mapsto \begin{cases} s^- & \text{if } \mathfrak{z} \in F, \\ t^- & \text{otherwise,} \end{cases}$$

and

$$\beta_F^+ : W \to [0, 1],\ y \mapsto \begin{cases} s^+ & \text{if } y \in F, \\ t^+ & \text{otherwise,} \end{cases}$$

where $s^- < t^-$ in $[-1, 0]$ and $s^+ > t^+$ in $[0, 1]$. Then $\beta_F := (W; \beta_F^-, \beta_F^+)$ is a BVFu-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ if and only if F is an ultra filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. Moreover, we have $F = \{y \in W \mid \beta_F^-(y) = \beta_F^-(1), \beta_F^+(y) = \beta_F^+(1)\}$.

Proof. Let $\beta_F := (W; \beta_F^-, \beta_F^+)$ be a BVFu-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. Then $\beta_F^-(1) = s^-$ and $\beta_F^+(1) = s^+$, and thus $1 \in F$. Let $(\mathfrak{z}, \mathfrak{u}), (y, \mathfrak{b}) \in W \times W$ be such that $(\mathfrak{z}, y) \in F \times F$ and $(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u}), y|(\mathfrak{b}|\mathfrak{b})) \in F \times F$. Then $\beta_F^-(\mathfrak{z}) = s^- = \beta_F^-(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u}))$ and $\beta_F^+(y) = s^+ = \beta_F^+(y|(\mathfrak{b}|\mathfrak{b}))$. Hence

$$\beta_F^-(\mathfrak{u}) \leq \max\{\beta_F^-(\mathfrak{z}), \beta_F^-(\mathfrak{z}|(\mathfrak{u}|\mathfrak{u}))\} = s^-$$

and

$$\beta_F^+(\mathfrak{b}) \geq \min\{\beta_F^+(y), \beta_F^+(y|(\mathfrak{b}|\mathfrak{b}))\} = s^+$$

by (25), and so $\beta_F^-(\mathfrak{u}) = s^-$ and $\beta_F^+(\mathfrak{b}) = s^+$. This shows that $(\mathfrak{u}, \mathfrak{b}) \in F \times F$. Thus F is a filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. Suppose that $\mathfrak{u}, \mathfrak{b}, \mathfrak{c}, z \notin F$ for all $(\mathfrak{u}, \mathfrak{b}), (\mathfrak{c}, z) \in W \times W$. Then $\beta_F^-(\mathfrak{u}) = t^- = \beta_F^-(\mathfrak{c})$ and $\beta_F^+(\mathfrak{b}) = t^+ = \beta_F^+(z)$. It follows from Theorem 3.10 that

$$\beta_F^-(\mathfrak{u} \vee \mathfrak{c}) \geq \min\{\beta_F^-(\mathfrak{u}), \beta_F^-(\mathfrak{c})\} = t^-$$

and $\beta_F^+(\mathfrak{b} \vee z) \leq \max\{\beta_F^+(\mathfrak{b}), \beta_F^+(z)\} = t^+$. Thus $\beta_F^-(\mathfrak{u} \vee \mathfrak{c}) = t^-$ and $\beta_F^+(\mathfrak{b} \vee z) = t^+$, which shows that $(\mathfrak{u} \vee \mathfrak{c}, \mathfrak{b} \vee z) \notin F \times F$. Using Lemma 2.8, we conclude that F is an ultra filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$.

Conversely, suppose that F is an ultra filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. Then the negative s-cut and the positive t-cut of $\beta_F := (W; \beta_F^-, \beta_F^+)$ are calculated as follows:

$$L(\beta_F^-; s) = \begin{cases} \emptyset & \text{if } s \in [-1, s^-), \\ F & \text{if } s \in [s^-, t^-), \\ W & \text{if } s \in [t^-, 0], \end{cases}$$

and

$$U(\beta_F^+;t) = \begin{cases} W & \text{if } t \in [0, t^+], \\ F & \text{if } t \in (t^+, s^+], \\ \emptyset & \text{if } t \in (s^+, 1], \end{cases}$$

which are ultra filters of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$ whenever they are nonempty. Therefore, $\beta_F := (W; \beta_F^-, \beta_F^+)$ is a BVFu-filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$. Since F is a filter of $\mathfrak{W} := (W, \vee, \wedge, |, 0, 1)$, we have

$$\begin{aligned} F &= \{y \in W \mid y \in F\} \\ &= \{y \in W \mid \beta_F^- = s^-, \beta_F^+ = s^+\} \\ &= \{y \in W \mid \beta_F^- = \beta_F^-(1), \beta_F^+ = \beta_F^+(1)\}. \end{aligned}$$

This completes the proof. □

4. Conclusion

The defination of the Sheffer stroke BG-algebra and some of features about filter, ultra filter, fuzzy filter and quasisubalgebra are studied. In this paper we first define BVFu-filter and BVFq-subalgebra of Sheffer stroke BG-algebra. After these definitions, we give some conditions when BVF-set becomes BVFq-subalgebra and BVFufilter of Sheffer stroke BG-algebra. Also we show that every BVF-filter is a BVFq-subalgebra. Finally we present some results about BVFu-filter and fuzzy ultra filters. The ideas and results of this paper will be applied to various types of logical algebras in the future to derive meaningful results and to further examine the relationships between them.

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