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A variation of continuity in n-normed spaces

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Abstract. In this study, we examine the s-th forward difference sequence in an n-normed space X, which tends to zero and is inspired by the consecutive terms of a sequence approaching zero. Functions that transform sequences satisfying this condition into sequences that also satisfy it are called s-ward continuous functions. Inclusion theorems related to this kind of continuity and uniform continuity are also considered. Additionally, we investigate the concept of s-ward compactness of a subset of X via s-quasi-Cauchy sequences. It turns out that the uniform limit of a sequence of s-ward continuous functions is also s-ward continuous, and the set of s-ward continuous functions is a closed subset of the set of continuous functions.

1. Introduction and preliminaries

Although some evaluations were initially made regarding the axioms of an abstract n-dimensional metric, the main developments concerning the definition of the 2-metric, 2-normed spaces and their topological properties were described by Gähler [8]. Subsequently these concepts were extended to the most generalized case of n-metric and n-normed spaces, where n is an arbitrary natural number, by Gähler[9]. Shortly after the introduction of the concept of an n-normed space, the concept of a 2-inner product space was also defined in [5]. Afterwards, many authors made impressive improvements in n-normed spaces and 2-inner product spaces ([6, 10–13, 15, 17–19]). The notion of an n-normed space was conceived by considering whether there exists a problem where the n-norm topology is effective while the norm topology is not. As an application of the concept of an n-norm, we can examine cases where a term in the definition of the n-norm reflects changes in shape; in such instances, the n-norm represents the associated volume of the corresponding surface. Suppose that, for any particular output, one requires n-inputs, with one main input and the remaining (n-1)-inputs as dummy inputs needed to complete the operation. This concept may find applications in various scientific areas.

Definition 1.1. An n-norm on a real vector space X of dimension d, where $2 \leq n \leq d$, is a real valued function $\|., ..., .\|$ on X^n that satisfies the following conditions:

1. $\|x_1, x_2, ..., x_n\| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,
2. $\|x_1, x_2, ..., x_n\| = \|x_{i_1}, ..., x_{i_n}\|$ for every permutation $(i_1, ..., i_n)$ of $(1, ..., n)$,
3. $\|x_1, x_2, ..., kx_n\| = |k|\|x_1, x_2, ..., x_n\|$ for any real number k,

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4. $\|y+z,x_1,...,x_{n-1}\| \leq \|y,x_1,...,x_{n-1}\| + \|z,x_1,...,x_{n-1}\|$.

A set X equipped with an n-norm $\|.,...,.\|$ is referred to as an n-normed space.

Definition 1.2. A sequence (x_k) is said to converge to an $l \in X$ in an n-normed space X if, for each $\epsilon > 0$, there exists a positive integer N such that for every $k \geq N$

$$\|x_k - l, u_1, \ldots, u_{n-1}\| < \epsilon, \quad \forall u_1, \ldots, u_{n-1} \in X.$$

Definition 1.3. A sequence (x_k) is called a Cauchy sequence if, for each $\epsilon > 0$, there exists a positive integer N such that for every $k, m \geq N$

$$\|x_k - x_m, u_1, \ldots, u_{n-1}\| < \epsilon, \quad \forall u_1, \ldots, u_{n-1} \in X.$$

If every Cauchy sequence in X converges to an element of X, we call X complete, and if X is complete, then it is called an n-Banach space.

Recently, the notion of quasi-Cauchy sequences was introduced in [2]. The distance between consecutive terms of a sequence tending to zero is expressed by Burton and Coleman through the quasi-Cauchy sequence. Building on this concept, various types of continuity were defined in [3, 4, 7]. The aim of this research is to generalize the notions of a quasi-Cauchy sequence and ward continuity of a function to the concepts of an s-quasi-Cauchy sequence and s-ward continuity of a function in an n-normed space, for any fixed positive integer s. Additionally, the paper presents interesting theorems related to ordinary continuity, uniform continuity, compactness, and s-ward continuity. The results not only extend those of [7] to an n-normed space but also introduce new findings in 2-normed spaces as a special case for $n = 2$.

2. Main results

In this paper, X represents a first countable n-normed space with an n-norm denoted by $\|.,...,.\|$, $\mathbb{R}$ denotes the set of all real numbers, and s represents a fixed positive integer.

Definition 2.1. A sequence (x_k) of points in X is s-quasi-Cauchy if, for all $u_1, u_2, ..., u_{n-1} \in X$, it satisfies

$$\lim_{k\to\infty} \|\Delta_s x_k, u_1, u_2, ..., u_{n-1}\| = 0,$$

where $\Delta_s x_k = x_{k+s} - x_k$ for each positive integer k.

If one sets $s = 1$, the sequence reverts to ordinary quasi-Cauchy sequences. Additionally, utilizing the equality

$$x_{k+s} - x_k = x_{k+s} - x_{k+s-1} + x_{k+s-1} - x_{k+s-2} ... - x_{k+2} + x_{k+2} - x_{k+1} + x_{k+1} - x_k,$$

we observe that any quasi-Cauchy sequence is s-quasi-Cauchy. However, the converse is not necessarily true.

Every Cauchy sequence is s-quasi-Cauchy, as is any convergent sequence. Additionally, a sequence of partial sums of a convergent series is s-quasi-Cauchy. One observes that the set $\Delta_s(X)$, the set of s-quasi-Cauchy sequences in X, forms a vector space. If (x_k) and (y_k) are s-quasi-Cauchy sequences in X such that

$$\lim_{k\to\infty} \|\Delta_s x_k, u_1, u_2, ..., u_{n-1}\| = 0 \text{ and } \lim_{k\to\infty} \|\Delta_s y_k, u_1, u_2, ..., u_{n-1}\| = 0$$

for all $u_1, u_2, ..., u_{n-1} \in X$, then

$$\lim_{k\to\infty} \|\Delta_s(x_k + y_k), u_1, u_2, ..., u_{n-1}\| \leq \lim_{k\to\infty} \|\Delta_s x_k, u_1, u_2, ..., u_{n-1}\| + \lim_{k\to\infty} \|\Delta_s y_k, u_1, u_2, ..., u_{n-1}\| = 0.$$

Thus, the sum of two s-quasi-Cauchy sequences is again s-quasi-Cauchy. It is clear that (ax_k) is an s-quasi-Cauchy sequence in X for any constant $a \in \mathbb{R}$.

Definition 2.2. A subset A of X is termed s-ward compact if every sequence in the set A possesses an s-quasi-Cauchy subsequence.

If a set A is an s-ward compact subset of X, then any subset of A is also s-ward compact. Furthermore, any ward compact subset of X is s-ward compact. The union of a finite number of s-ward compact subsets of X is s-ward compact. Additionally, any sequentially compact subset of X is s-ward compact.

For each real number $\alpha > 0$, an α-ball with center a in X is defined as

$$B_\alpha(a, x_1, ..., x_{n-1}) = \{x \in X : \|a - x, x_1 - x, ..., x_{n-1} - x\| < \alpha\}$$

for $x_1, ..., x_{n-1} \in X$. The family of all sets $W_i(a) = B_{\alpha_i}(a, x_{i_1}, ..., x_{i_{(n-1)}})$, where $i = 1, 2, ..,$ forms an open basis at a. Let β_{n-1} be the collection of linearly independent sets B with $n - 1$ elements. For $B \in \beta_{n-1}$, the mapping

$$p_B(x) = \|x, x_1, ..., x_{n-1}\|$$

for $x \in X,\ \ x_1, ..., x_{n-1} \in B$ defines a seminorm on X, and the collection $\{p_B : B \in \beta_{n-1}\}$ of seminorms makes X a locally convex topological vector space. For each $x \in X$, different from zero, there exist $x_1, ..., x_{n-1} \in B$ such that $x, x_1, x_2, ..., x_{n-1}$ are linearly independent, so $\|x, x_1, ..., x_{n-1}\| \neq 0$, which ensures that X is a Hausdorff space. A neighborhood of the origin for this topology is in the form of a finite intersection

$$\bigcap_{i=1}^{n} \{x \in X : \|x, x_{i_1} - x, ..., x_{i_{(n-1)}} - x\| < \epsilon\},$$

where $\epsilon > 0$.

Now, the following theorem characterizes total boundedness, not only for n-normed spaces but also for 2-normed spaces. It extends the results for quasi-Cauchy sequences given in [7] for 2-normed valued sequences to n-normed valued s-quasi-Cauchy sequences, where setting $s = 1$ recovers earlier results for 2-normed spaces. It is worth noting that Theorem 3 in [4] can not be obtained simply by putting $n = 1$ in the n-normed space to obtain a normed space, which would be awkward. In contrast, the following theorem is interesting for studying a new space.

Lemma 2.3. *If a subset of X is totally bounded, then every sequence in A contains an s-quasi-Cauchy subsequence.*

Proof. Let A be totally bounded, and consider any sequence (x_n) in A. Since A is totally bounded, it is covered by a finite number of balls in X with diameter less than 1. Let A_1 be one of these sets covering A with diameter less than 1, x_{n_1} be an element of the sequence (x_n) that lies in A_1. As A_1 is totally bounded, it is covered by a finite number of balls with diameter less than 1/2. Choose of these balls and denote it as A_2. This ball must contain x_n for infinitely many values of n. Choose a positive integer n_2 such that $n_2 > n_1$ and $x_{n_2} \in A_2$. Since $A_2 \subset A_1$, it follows that $x_{n_2} \in A_1$. Continue this process iteratively. At the k-th step, choose a ball A_k in A_{k-1} of diameter less than $1/k$ that contains x_n for infinitely many n. Choose $n_k > n_{k-1}$, for any positive integer k, such that $x_{n_k} \in A_k$. So, $x_{n_k}, x_{n_{k+1}}, ..., x_{n_{k+s}}, ...$ lie in A_k. The diameter of A_k is less than $1/k$, so the distance between any two terms x_{n_k} and $x_{n_{k+s}}$ for $s > 0$ is less than $1/k$. As k increases, the diameter of A_k decreases, and therefore, the distance betwen terms in the subsequence becomes arbitrarily small as k increasess. Therefore, (x_{n_k}) is an s-quasi-Cauchy subsequence of (x_n). □

Theorem 2.4. *A subset of X is totally bounded if and only if it is s-ward compact for every positive integer s.*

Proof. If A is totally bounded, then, according to Lemma 2.3, every sequence in A possesses an s-quasi-Cauchy subsequence. Consequently, the set A is s-ward compact for any given positive integer s. Assume A is not a totally bounded set. Choose any $x_1 \in A$ and $\alpha > 0$. Since A is not totally bounded, the neighborhood of a point x_1 in A, denoted by $B_\alpha(x_1, u_1^1, ..., u_{n-1}^1) = \{y \in A; \|x_1 - y, u_1^1 - y, ..., u_{n-1}^1 - y\| < \alpha\}$, is not equal to A. Consequently, there exists $x_2 \in A$ such that $x_2 \notin B_\alpha(x_1, u_1^1, ..., u_{n-1}^1)$, implying, $\|x_1 - x_2, u_1^1 - x_2, ..., u_{n-1}^1 - x_2\| \geq \alpha$. As A is not totally bounded, the union of the neighborhoods $B_\alpha(x_1, u_1^1, ..., u_{n-1}^1) \cup B_\alpha(x_2, u_1^2, ..., u_{n-1}^2)$

is not equal to A, where $B_\alpha(x_2, u_1^2, ..., u_{n-1}^2)$ is the neighborhood of a point x_2 in A. Continuining this procedure, we obtain a sequence (x_k) of points in A such that $x_{k+s} \notin \bigcup_{i=1}^{k+s-1} B_\alpha(x_i, u_1^i, ..., u_{n-1}^i)$. Consequently, $\|x_{k+s} - x_k, u_1^i - x_k, ..., u_{n-1}^i - x_k\| \geq \alpha$, for all nonzero $u_1^i, ..., u_{n-1}^i$ in A, where $i = 1, ..., k+s-1$. As a result, the sequence (x_k) has no s-quasi-Cauchy subsequence for any positive integer s, contradicting the assumption that A is s-ward compact for every positive integer s. Therefore, if A is not totally bounded, it can not be s-ward compact for some positive integers. □

Definition 2.5. A function f is termed s-ward continuous on a subset A of X if

$$lim_{k\to\infty}\|\Delta_s x_k, u_1, u_2, ..., u_{n-1}\| = 0$$

is satisfied, for all $u_1, u_2, ..., u_{n-1} \in X$, then

$$lim_{k\to\infty}\|\Delta_s f(x_k), f(u_1), f(u_2), ..., f(u_{n-1})\| = 0.$$

Theorem 2.6. *Any function f that is s-ward continuous function on a subset A of X is continuous on A.*

Proof. Let f be s-ward continuous on $A \subset X$, and consider any sequence (x_k) in A converging to l, that is

$$lim_{k\to\infty}\|x_k - l, u_1, u_2, ..., u_{n-1}\| = 0$$

for all $u_1, u_2, ..., u_{n-1} \in X$. Now, let's construct a new sequence using certain terms from (x_k):

$$(t_m) = (x_1, ..., x_1, l, ..., l, x_2, ..., x_2, l, ..., l, ..., x_n, ..., x_n, l, ...l, ...)$$

where the same terms are repeated s-times. Since every convergent sequences is Cauchy, and moreover, any Cauchy sequence is s-quasi-Cauchy, it follows that

$$lim_{m\to\infty}\|\Delta_s t_m, u_1, u_2, ..., u_{n-1}\| = lim_{m\to\infty}\|t_{m+s} - t_m, u_1, u_2, ..., u_{n-1}\| = 0$$

where either

$$lim_{m\to\infty}\|t_{m+s} - l, u_1, u_2, ..., u_{n-1}\| = 0$$

or

$$lim_{m\to\infty}\|l - t_m, u_1, u_2, ..., u_{n-1}\| = 0$$

for every $u_1, u_2, ..., u_{n-1}$. This result implies that (t_m) is an s-quasi Cauchy sequence. Now, since the function f is assumed to be s-ward continuous, utilizing this assumption yields

$$\begin{aligned} &lim_{m\to\infty}\|\Delta_s f(t_m), f(u_1), f(u_2), ..., f(u_{n-1})\| \\ &= lim_{m\to\infty}\|f(t_{m+s}) - f(t_m), f(u_1), f(u_2), ..., f(u_{n-1})\| = 0 \end{aligned}$$

where either

$$lim_{m\to\infty}\|f(t_{m+s}) - f(l), f(u_1), f(u_2), ..., f(u_{n-1})\| = 0$$

or

$$lim_{m\to\infty}\|f(l) - f(t_m), f(u_1), f(u_2), ..., f(u_{n-1})\| = 0.$$

Hence, $(f(x_k))$ converges to $f(l)$. □

Since the sum of two s-ward continuous functions on A is s-ward continuous, and cf is s-ward continuous for any constant real number c, the set of s-ward continuous functions on A forms a vector subspace of the vector space of all continuous function on A.

Theorem 2.7. *Every s-ward continuous function on $A \subset X$ is also ward continuous on A.*

Proof. Assume that (x_k) is a quasi-Cauchy sequence in A, and f is any s-ward continuous function on A. If $s = 1$, the result is obvious. For $s > 1$, consider a sequence

$$(t_m) = (\underbrace{x_1, x_1, ..., x_1}_{s-times}, \underbrace{x_2, x_2, ..., x_2}_{s-times}, ..., \underbrace{x_n, x_n, ..., x_n}_{s-times}, ...),$$

that is, s-quasi-Cauchy, meaning

$$lim_{m\to\infty} \|\Delta_s t_m, u_1, u_2, ..., u_{n-1}\| = 0.$$

We then have

$$lim_{m\to\infty} \|\Delta_s f(t_m), f(u_1), f(u_2), ..., f(u_{n-1})\| = 0$$

by utilizing the s-ward continuity of the function f. Therefore,

$$lim_{m\to\infty} \|\Delta f(t_m), f(u_1), f(u_2), ..., f(u_{n-1})\| = 0.$$

Thus, the s-ward continuity of the function f implies the ward continuity of f on $A \subset X$. □

Theorem 2.8. *The image of an s-ward compact subset of X under an s-ward continuous function is s-ward compact.*

Proof. Assume that f is an s-ward continuous function, and A is an s-ward compact subset of X. Choose a sequence t as $t = (t_k) \subset f(A)$, where $(t_k) = (f(x_k))$ with $x_k \in A$. Since A is s-ward compact, there exists a subsequence (x_m) of (x_k) such that

$$\lim_{m\to\infty} \|\Delta_s x_m, u_1, u_2, ..., u_{n-1}\| = 0$$

for all $u_1, u_2, ..., u_{n-1} \in X$. Utilizing the s-ward continuity of f, we have

$$\lim_{m\to\infty} \|\Delta_s f(x_m), f(u_1), f(u_2), ..., f(u_{n-1})\| = 0.$$

So, there exists an s-quasi-Cauchy subsequence $(f(x_m))$ of t. This result implies that the subset $f(A) \subset X$ is s-ward compact. □

The s-ward continuous image of any compact subset of X is compact. This result follows directly from Theorem 2.6.

Theorem 2.9. *If a function f is uniformly continuous on a subset A of X, then it is also s-ward continuous on A.*

Proof. Let f be a uniformly continuous function on A, and consider the sequence (x_k) an s-quasi-Cauchy sequence in A. The aim is to prove that the sequence $(f(x_k))$ is also an s-quasi-Cauchy sequence in A. Take any $\varepsilon > 0$. There exists a $\delta > 0$ such that if

$$\|x - y, u_1, u_2, ..., u_{n-1}\| < \delta, \text{ then } \|f(x) - f(y), f(u_1), f(u_2), ..., f(u_{n-1})\| < \epsilon.$$

For this $\delta > 0$, there exists an $N = N(\delta)$ such that

$$\|\Delta_s x_k, u_1, u_2, ..., u_{n-1}\| < \delta$$

for every $u_1, u_2, ..., u_{n-1} \in X$ whenever $k > N$. The uniform continuity of f on A for every $k > N$ implies

$$\|\Delta_s f(x_k), f(u_1), f(u_2), ..., f(u_{n-1})\| < \varepsilon$$

for every $f(u_1), f(u_2), ..., f(u_{n-1}) \in X$. Consequently, the sequence $(f(x_k))$ is s-quasi-Cauchy, demonstrating that the function f is s-ward continuous. □

Theorem 2.10. *The uniform limit of a sequence of s-ward continuous functions is also s-ward continuous.*

Proof. Let (f_t) be a sequence of s-ward continuous functions uniformly converging to a function f. Consider an s-quasi-Cauchy sequence (x_k) in A, and choose any $\varepsilon > 0$. There exists an integer $N \in Z^+$ such that

$$\|f_t(x) - f(x), f(u_1), f(u_2), ..., f(u_{n-1})\| < \frac{\varepsilon}{3}$$

for every $x \in A$, and all $f(u_1), f(u_2), ..., f(u_{n-1}) \in X$ whenever $t \geq N$. Utilizing the s-ward continuity of f_N, there is a positive integer $N_1(\varepsilon) > N$ such that

$$\|\Delta_s f_t(x_k), f(u_1), f(u_2), ..., f(u_{n-1})\| < \frac{\varepsilon}{3}$$

for every $t \geq N_1$. Now, for $t \geq N_1$, we have

$$\begin{aligned} \|\Delta_s f(x_k), f(u_1), f(u_2), ..., f(u_{n-1})\| &= \|f(x_{k+s}) - f(x_k), f(u_1), f(u_2), ..., f(u_{n-1})\| \\ \leq \|f(x_{k+s}) - f_N(x_{k+s}), f(u_1), f(u_2), ..., f(u_{n-1})\| &+ \|\Delta_s f_N(x_k), f(u_1), f(u_2), ..., f(u_{n-1})\| \\ +\|f_N(x_k) - f(x_k), f(u_1), f(u_2), ..., f(u_{n-1})\| &< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Therefore, the function f is s-ward continuous on A. □

Theorem 2.11. *The collection of the s-ward continuous functions on $A \subset X$ forms a closed subset of the collection of all continuous functions on A.*

Proof. Let E be the collection of all s-ward continuous functions on $A \subset X$, and $\overline{E}$ be the closure of E. $\overline{E}$ is defined such that for every $x \in X$, there exists $x_k \in E$ with $\lim_{k\to\infty} x_k = x$, and E is closed if $E = \overline{E}$. It is obvious that $E \subseteq \overline{E}$. Let f be any element of the set of all closure points of E, which means there exists a sequence of points f_t in E such that

$$\lim_{t\to\infty} \|f_t - f, f(u_1), f(u_2), ..., f(u_{n-1})\| = 0$$

for all $f(u_1), f(u_2), ..., f(u_{n-1}) \in X$, and f_t is a s-ward continuous. Choose the sequence (x_k) as any s-quasi-Cauchy sequence. Since (f_t) converges to f, for every $\varepsilon > 0$ and $x \in E$, there is any N such that for every $t \geq N$,

$$\|f(x) - f_t(x), f(u_1), f(u_2), ..., f(u_{n-1})\| < \frac{\varepsilon}{3}.$$

As f_N is s-ward continuous, $N_1 > N$ exists such that for all $t \geq N_1$,

$$\|\Delta_s f_N(x_k), f(u_1), f(u_2), ..., f(u_{n-1})\| < \frac{\varepsilon}{3}.$$

Hence, for all $t \geq N_1$,

$$\begin{aligned} \|\Delta_s f(x_k), f(u_1), f(u_2), ..., f(u_{n-1})\| &= \|f(x_{k+s}) - f(x_k), f(u_1), f(u_2), ..., f(u_{n-1})\| \\ \leq \|f(x_{k+s}) - f_N(x_{k+s}), f(u_1), f(u_2), ..., f(u_{n-1})\| &+ \|f(x_k) - f_N(x_k), f(u_1), f(u_2), ..., f(u_{n-1})\| \\ +\|\Delta_s f_N(x_k), f(u_1), f(u_2), ..., f(u_{n-1})\| &< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since the function f is s-ward continuous in E, then $E = \overline{E}$, using Theorem 2.6, and it concludes the proof. □

3. Conclusion

This paper investigates the generalization of the notions of quasi-Cauchy sequences and ward continuous functions to the concepts of s-quasi-Cauchy sequences and s-ward continuous functions in n-normed spaces. Additionally, intriguing interesting inclusion theorems related to ordinary continuity, uniform continuity, s-ward continuity, and s-ward compactness are established. The paper establishes that the uniform limit of a sequence of s-ward continuous functions is s-ward continuous, and the set of s-ward continuous functions forms a closed subset of the set of continuous functions. We recommend further research on s-quasi-Cauchy sequences of points and fuzzy functions in an n-normed fuzzy space as potential avenues for further studies. However, due to structural differences, the methods of proof may differ from those presented in this study (see [16], [1]). Additionally, we suggest investigating s-quasi-Cauchy sequences of double sequences in n-normed spaces as another potential area for further study (see [20], [14]).

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