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A note on the delay nonlinear parabolic differential equations

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Abstract. In this paper, the initial boundary value problem (IBVP) for nonlinear delay differential equations (DEs) in a Banach space with strongly unbounded operators is studied. The theorem on the existence and uniqueness of a bounded solution (BS) of this problem is established. The application of the main theorem to nonlinear delay parabolic equations is provided. Theorems on the existence and uniqueness of a bounded solution of the initial boundary value problems for three types of nonlinear delay parabolic equations are established. The first and second order of accuracy difference schemes (FSOADSs) for the solution of one dimensional nonlinear parabolic equation with time delay are presented. Finally, certain numerical experiments are given to confirm the agreement between experimental and theoretical results and to make clear how effective the proposed approach is.

1. Introduction

In general, the inclusion of an unbounded delay term in DEs makes it difficult to analyse these types of equations. Additionally, there are a couple of works for which analytical solutions are provided. Because of this reason, the studies on numerical approaches compensate for the dearth of theoretical research. Particularly, one of the primary techniques employed in this field is the finite difference method.

Lu [13] investigates monotone iterative schemes for finite-difference solutions of reaction-diffusion systems with time delays and provides improved iterative schemes using the upper-lower solutions approach with the Gauss-Seidel or the Jacobi method.

The initial value problem (IVP) for linear delay parabolic differential equations (DPDEs) was studied by Ashyralyev and Sobolevskii [8]; they provide a sufficient condition for the stability of the solution of this problem and obtain the stability estimates in Hölder norms. Different types of problems involving linear DPDEs were investigated by Ashyralyev and Ağırseven [1]-[6]. They provide convergence and stability theorems.

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Finally, the existence and uniqueness of a BS of nonlinear delay parabolic equation was established by Ashyralyev, Agirseven and Ceylan in [7]. They provide sufficient conditions for the existence of a unique BS of nonlinear delay parabolic equation.

It is clear that several initial-boundary value problems for nonlinear delay parabolic equations can be reduced to an initial-value problem for the differential equation

$$\begin{cases} \frac{du}{dt} + Au(t) = g(t, u(t), u(t-\omega)), t \in [0, \infty), \\ u(t) = \varphi(t), t \in [-\omega, 0] \end{cases} \tag{1}$$

in an arbitrary Banach space E with the unbounded operators $A(t)$ in E with dense domains $D(A(t)) \subset E$. Suppose that for each $t \in [0, \infty)$ the operator $-A(t)$ generates an analytic semi-group $\exp\{-sA(t)\}(s \geq 0)$ with exponentially decreasing norm when $s \to +\infty$, i.e. the following estimates

$$\left\|\exp(-sA(t))\right\|_{E\to E}, \left\|sA(t)\exp(-sA(t))\right\|_{E\to E} \leq Me^{-\delta s}(s > 0) \tag{2}$$

hold for some $M \in [1, +\infty)$, $\delta \in (0, +\infty)$. From this inequality it follows the operator $A^{-1}(t)$ exists and bounded and hence $A(t)$ is closed in $E_1 \subset E$, such that $A(t) : D(A(t)) \to E$ and $D(A(t)) = D(A(0))$ for $0 \leq t < \infty$.

Assume that the operator $A(t)A^{-1}(s)$ is Holder continuous in t in the uniform operator topology for each fixed s, that is,

$$\left\|[A(t) - A(\tau)]A^{-1}(s)\right\|_{E\to E} \leq M|t-\tau|^{\varepsilon}, 0 < \varepsilon \leq 1, \tag{3}$$

where M and ε are positive constants independent of t, s and τ for $0 \leq t, s, \tau < \infty$.

An operator-valued function $V(t, y)$, defined and strongly continuous jointly in t, y for $0 \leq y < t < \infty$, is called a fundamental solution of (1) if

(1) the operator $V(t, y)$ is strongly continuous in t and y for $0 \leq y < t < \infty$,

(2) the following identity holds:

$$V(t, y) = V(t, \tau)V(\tau, y), V(y, y) = I \tag{4}$$

for $0 \leq y \leq \tau \leq t < \infty$, where, I is the identity operator,

(3) the operator $V(t, y)$ maps the region D into itself. The operator $U(t, y) = A(t)V(t, y)A^{-1}(y)$ is bounded and strongly continuous in t and y for $0 \leq y < t < \infty$,

(4) on the region D the operator $V(t, y)$ is differentiable relative to t and y, while

$$V_t(t, y) + A(t)V(t, y) = 0, \tag{5}$$

and

$$V_y(t, y) - V(t, y)A(y) = 0, \tag{6}$$

(5) the subsequent estimates hold:

$$\|V(t, y)\|_{E\to E} \leq Pe^{-\delta(t-y)}, \quad t \geq y \geq 0 \tag{7}$$

for some $\delta \in [0, \infty)$ and $P \in [1, \infty)$.

A function $u(t)$ is called a solution of problem (1) if the conditions below are satisfied:

1. $u(t)$ is continuously differentiable on $[-\omega, \infty)$.

2. The element $u(t) \in D(A(t))$, $\forall t \in [-\omega, \infty)$, and the function $A(t)u(t)$ is continuous on $[-\omega, \infty)$.
3. $u(t)$ satisfies the equation and the initial condition (1).

In the present work, we aim at providing sufficient conditions for the existence of a unique BS of problem (1).

The main theorem on the existence and uniqueness of a BS of problem (1) is established for a nonlinear DPDE. The application of the main theorem for three types of nonlinear DPDEs is illustrated. It is precisely difficult to obtain the solution of nonlinear problems. Consequently, FSOADSs for the solution of nonlinear one dimensional DPDE are shown. Numerical results are provided. It should be noted that in past publications [9]-[27], BSs of nonlinear parabolic and hyperbolic partial differential equations (PHPDEs) with or without delay have been investigated. However, due to the generality of the strategy used in this research, a larger class of nonlinear parabolic equations can be treated.

2. Existence and uniqueness theorem

We reduced problem (1) into an integral equation of the form

$$u(t) = V(t, m\omega)u(m\omega) + \int_{m\omega}^{t} V(t, y)g(y, u(y), u(y-\omega))dy,$$

$$m\omega \leq t \leq (m+1)\,\omega, m = 0, 1, ..., u(t) = \varphi(t), -\omega \leq t \leq 0$$

in $[-\omega, \infty) \times E$ and using successive approximations, obtained recursive formula for the solution of problem (1) is

$$\begin{aligned} u_i(t) &= V(t, m\omega)u_i(m\omega) + \int_{m\omega}^{t} V(t, y)g(y, u_{i-1}(y), u_i(y-\omega))dy, \\ u_0(t) &= V(t, m\omega)u(m\omega), m\omega \leq t \leq (m+1)\omega, m = 0, 1, ..., \\ & i = 1, 2, ..., u(t) = \varphi(t), -\omega \leq t \leq 0. \end{aligned} \tag{8}$$

Theorem 2.1. *Assume the hypotheses below:*

1. $\varphi : [-\omega, 0] \times E \longrightarrow E$ *be continuous function and*
$$\|\varphi(t)\|_E \leq M. \tag{9}$$
2. $g : [0, \infty) \times E \times E \longrightarrow E$ *be bounded and continuous function, i.e.;*
$$\|g(t, u, v)\|_E \leq \bar{M} \tag{10}$$
and with respect to z, the Lipschitz condition holds uniformly
$$\|g(t, v, z) - g(t, u, z)\|_E \leq L\|v - u\|_E, \tag{11}$$

where $L, M, \bar{M}$ are positive constants. Then problem (1) has a unique bounded solution in $[0, \infty) \times E$.

Proof. Using the interval $t \in [0, \omega]$, problem (1) can be written as

$$\frac{du}{dt} + A(t)u(t) = g(t, u(t), \varphi(t-\omega)), u(0) = \varphi(0)$$

which in an equivalent integral form, becomes

$$u(t) = V(t, 0)\varphi(0) + \int_{0}^{t} V(t, y)g(y, u(y), \varphi(y-\omega))dy. \tag{12}$$

In accordance with the recursive approximation approach (8), we get

$$u_i(t) = V(t,0)\varphi(0) + \int_0^t V(t,y)g(y,u_{i-1}(y),\varphi(y-\omega))dy, i = 1,2,.... \tag{13}$$

Therefore,

$$u(t) = u_0(t) + \sum_{i=1}^{\infty}(u_i(t) - u_{i-1}(t)), \tag{14}$$

where

$$u_0(t) = V(t,0)\varphi(0).$$

From (7) and (9), we obtain

$$\|u_0(t)\|_E = \|V(t,0)\|\|\varphi(0)\|_E \leq MP.$$

Using formula (13) along with estimates (7) and (10), we get

$$\|u_1(t) - u_0(t)\|_E \leq \int_0^t \|V(t,y)\|\|g(y,u_0,\varphi(y-\omega))\|_E dy \leq \bar{M}Pt.$$

By the triangle inequality, we have

$$\|u_1(t)\|_E \leq MP + \bar{M}Pt.$$

Applying formula (13) along with estimates (11),(7) and (10), we obtain

$$\|u_2(t) - u_1(t)\|_E \leq \int_0^t \|V(t,y)\|\|g(y,u_1(y),\varphi(y-\omega)) - g(y,u_0(y),\varphi(y-\omega))\|_E dy$$

$$\leq LP\int_0^t \|u_1(y) - u_0(y)\|_E dy \leq LP^2\bar{M}\int_0^t y dy = \frac{\bar{M}}{L}\frac{(PLt)^2}{2!}.$$

Then, by the triangle inequality, we have

$$\|u_2(t)\|_E \leq MP + \bar{M}Pt + \frac{\bar{M}}{L}\frac{(PLt)^2}{2!}.$$

Let

$$\|u_i(t) - u_{i-1}(t)\|_E \leq \frac{\bar{M}}{L}\frac{(LPt)^i}{i!}.$$

Then, we obtain

$$\|u_{i+1}(t) - u_i(t)\|_E \leq \int_0^t \|V(t,y)\|\|g(y,u_i(y),\varphi(y-\omega)) - g(y,u_{i-1}(y),\varphi(y-\omega))\|_E dy$$

$$\leq P\int_0^t L\|u_i(y) - u_{i-1}(y)\|_E dy \leq P\int_0^t L\frac{\bar{M}}{L}\frac{(LPy)^i}{i!}dy = \frac{\bar{M}}{L}\frac{(LPt)^{i+1}}{(i+1)!}.$$

Consequently, for any $i, i \geq 1$, we have that

$$\|u_{i+1}(t) - u_i(t)\|_E \leq \frac{\bar{M}}{L}\frac{(LPt)^{i+1}}{(i+1)!}$$

and

$$\|u_{i+1}(t)\|_E \leq PM + \bar{M}Pt + \frac{\bar{M}}{L}\frac{(PLt)^2}{2!} + ... + \frac{\bar{M}}{L}\frac{(LPt)^{i+1}}{(i+1)!}$$

by mathematical induction. It is implied by that and formula (14)

$$\|u(t)\|_E \leq \|u_0(t)\|_E + \sum_{i=1}^{\infty}\|u_i(t) - u_{i-1}(t)\|_E \leq MP + \sum_{i=1}^{\infty}\frac{\bar{M}}{L}\frac{(LPt)^i}{i!}$$

$$\leq MP + \frac{\bar{M}}{L}e^{LPt}, 0 \leq t \leq \omega$$

which shows a solution of problem (1) exists and is bounded in $[0, \omega] \times E$.
Next, for $t \in [\omega, 2\omega]$, note that $0 \leq t - \omega \leq \omega$. We denote that

$$\varphi_1(t) = u(t - \omega), t \in [\omega, 2\omega].$$

and suppose that problem (1) has a BS in $[\omega, 2\omega] \times E$. Replacing t and $t - \omega$ and assuming that

$$\|g(t, u_0(t), \varphi_1(t))\|_E \leq \bar{M}_1$$

and

$$\|\varphi_1(t)\|_E \leq M_1.$$

Hence,

$$u_0(t) = V(t, \omega)\varphi_1(\omega),$$

$$u_i(t) = V(t, \omega)\varphi_1(\omega) + \int_{\omega}^{t} V(t, y)g(y, u_{i-1}(y), \varphi_1(y))dy, i = 1, 2, ...$$

In the same way, for any $i, i \geq 1$, we have

$$\|u_{i+1}(t) - u_i(t)\|_E \leq \frac{\bar{M}_1}{L}\frac{(LPt)^{i+1}}{(i+1)!},$$

and

$$\|u_{i+1}(t)\|_E \leq PM_1 + \bar{M}_1Pt + \frac{\bar{M}_1}{L}\frac{(LPt)^2}{2!} + ... + \frac{\bar{M}_1}{L}\frac{(LPt)^{i+1}}{(i+1)!}.$$

Then it follows that

$$\|u(t)\|_E \leq M_1P + \frac{\bar{M}_1}{L}e^{LP(t-\omega)}, \omega \leq t \leq 2\omega.$$

This proves a solution of problem (1) exists and is bounded in $[\omega, 2\omega] \times E$.
In the same procedure, we can obtain that

$$\|u(t)\|_E \leq M_mP + \frac{\bar{M}_m}{L}e^{LP(t-m\omega)}, m\omega \leq t \leq (m+1)\omega,$$

where M_m and $\bar{M}_m$ are bounded. This proves the existence of a BS of problem (1) in $[m\omega, (m+1)\,\omega] \times E$. The function $u(t)$ constructed for problem (1) has a BS in $[0, \infty) \times E$.

We shall now prove that this solution of problem (1) is unique. Assume that problem (1) has a BS $v(t)$ and that $v(t) \neq u(t)$. We denote $w(t) = v(t) - u(t)$. Hence for $w(t)$, we obtain that

$$\begin{cases} \frac{dw}{dt} + A(t)w(t) = g(t, v(t), v(t-\omega)) - g(t, u(t), u(t-\omega)), t \in (0, \infty), \\ w(t) = 0, t \in [-\omega, 0]. \end{cases}$$

We consider $0 \leq t \leq \omega$. As $v(t-\omega) = u(t-\omega) = \varphi(t-\omega)$, we get

$$\begin{cases} \frac{dw}{dt} + A(t)w(t) = g(t, v(t), \varphi(t-\omega)) - g(t, u(t), \varphi(t-\omega)), t \in (0, \infty), \\ w(t) = 0, t \in [-\omega, 0]. \end{cases}$$

Therefore,

$$w(t) = \int_0^t V(t, y) \left[g(y, v(y), \varphi(y-\omega)) - g(y, u(y), \varphi(y-\omega)) \right] dy$$

Applying estimates (7) and (10), we get

$$\|w(t)\|_E \leq \int_0^t \|V(t, s)\| \, \|g(y, v(y), \varphi(y-\omega)) - g(y, u(\omega), \varphi(y-\omega))\|_E dy$$

$$\leq PL \int_0^t \|v(y) - u(y)\|_E dy \leq PL \int_0^t \|w(y)\|_E dy.$$

By means of integral inequality, we obtain

$$\|w(t)\|_E \leq 0.$$

This implies that, $w(t) = 0$ which proves that the solution of problem (1) is unique and bounded in $[0, \omega] \times E$. Using similar procedure and mathematical induction, we can prove that the solution of problem (1) is unique and bounded in $[0, \infty) \times E$. □

Remark 2.2. The approach used in the current study also makes it possible to prove, under certain assumptions, that there exists a unique bounded solution of the IVP for nonlinear parabolic equations

$$\begin{cases} \frac{du}{dt} + A(t)u(t) = g(t, B(t)u(t), B(t)u(t-\omega)), t \in [0, \infty), \\ u(t) = \varphi(t), t \in [-\omega, 0] \end{cases} \tag{15}$$

in an arbitrary Banach space E with unbounded operators $A(t)$ and $B(t)$ with dense domains $D(A(t)) \subset D(B(t))$.

3. Applications

First, we consider the IBVP for non-linear one dimensional DPDEs with Dirichlet condition

$$\begin{cases} u_t(t, x) - a(t, x)u_{xx}(t, x) + \delta u(t, x) = g(t, x, u(t, x), u(t-\omega, x)), \\ t \in (0, \infty), x \in (0, b), \\ u(t, x) = \varphi(t, x), \varphi(t, 0) = \varphi(t, b) = 0, t \in [-\omega, 0], x \in [0, b], \\ u(t, 0) = u(t, b) = 0, t \in [0, \infty), \end{cases} \tag{16}$$

where $\varphi(t, x), a(t, x)$ are given sufficiently smooth functions (SSF) and $\delta > 0$ is the sufficiently large number. Assume that $a(t, x) \geq a > 0$.

Theorem 3.1. *Assume the hypotheses below:*

i $\varphi : [-\omega, 0] \times C[0,b] \to C[0,b]$ *be continuous function and*

$$\left\|\varphi(t,.)\right\|_{C[0,b]} \leq M. \tag{17}$$

ii $g : (0,\infty) \times (0,b) \times C[0,b] \times C[0,b] \to C[0,b]$ *be bounded and continuous function, i.e.;*

$$\left\|g(t,.,u,v))\right\|_{C[0,b]} \leq \overline{M} \tag{18}$$

and with respect to z, the Lipschitz condition holds uniformly

$$\left\|g(t,.,u,z) - g(t,.,v,z)\right\|_{C[0,b]} \leq L \left\|u - v\right\|_{C[0,b]}, \tag{19}$$

where, $L, M, \overline{M}$ *are positive constants. Then problem (16) has a unique BS in* $[0,\infty) \times C[0,b]$.

The proof of Theorem 3.1 is based on the abstract Theorem 2.1, on the strong positivity of a differential operator A^x in $C[0,b]$ according to the following formula:

$$A^x(t)v(x) = -a(t,x)\frac{d^2v(x)}{dx^2} + \delta v(x) \tag{20}$$

with domain $D(A^x(0)) = \left\{v \in C^{(2)}[0,b] : v(0) = v(b) = 0\right\}$ [19] and on the estimate

$$\left\|V(t,y)\right\|_{C[0,b]\to C[0,b]} \leq M_1, \quad t \geq y \geq 0. \tag{21}$$

Second, we consider the IBVP for nonlinear one dimensional DPDEs with nonlocal conditions

$$\begin{cases} u_t(t,x) - a(t,x)u_{xx}(t,x) + \delta u(t,x) = g(t,x,u(t,x),u(t-\omega,x)), \\ t \in (0,\infty), x \in (0,b), \\ u(t,x) = \varphi(t,x), \varphi(t,0) = \varphi(t,b), \varphi_x(t,0) = \varphi_x(t,b), \\ t \in [-\omega,0], x \in [0,b], \\ u(t,0) = u(t,b), u_x(t,0) = u_x(t,b), t \in [0,\infty), \end{cases} \tag{22}$$

where $\varphi(t,x), a(t,x)$ are SSF given and $\delta > 0$ is the sufficiently large number. Assume that $a(t,x) \geq a > 0$.

Theorem 3.2. *Suppose that the assumptions (17), (18), and (19) hold. Then problem (22) has a unique BS in* $[0,\infty) \times C[0,b]$.

The proof of Theorem 3.2 is based on the abstract Theorem 2.1, on the strong positivity of a differential operator A^x in $C[0,b]$ according to the following formula:

$$A^x(t)v(x) = -a(t,x)\frac{d^2v(x)}{dx^2} + \delta v(x) \tag{23}$$

with domain $D(A^x(0)) = \left\{v \in C^{(2)}[0,b] : v(0) = v(b), v'(0) = v'(b)\right\}$[2] and on estimate (21).

Third, we consider the initial value problem on the range

$$\{0 \leq t < \infty, x = (x_1, \cdots, x_n) \in R^n, r = (r_1, \cdots, r_n)\}$$

for $2m$-th order multidimensional nonlinear DPDEs

$$\begin{cases} u_t(t,x) + \sum\limits_{|r|=2m} a_r(t,x) u_{x_1^{r_1}\ldots x_n^{r_n}}(t,x) + \delta u(t,x) \\ = g(t,x,u(t,x),u(t-\omega,x)), t \in (0,\infty), x \in R^n, \\ u(t,x) = \varphi(t,x), t \in [-\omega,0], x \in R^n, \end{cases} \tag{24}$$

where $a_r(t,x)$ and $\varphi(t,x)$ are given SSFs and $\delta > 0$ is the sufficiently large number. We will suppose that the symbol $[\xi = (\xi_1, \cdots, \xi_n) \in R^n]$ and $|r| = r_1 + \ldots + r_n$,

$$A^x(t,\xi) = \sum_{|r|=2m} a_r(t,x) (i\xi_1)^{r_1} \ldots (i\xi_n)^{r_n}$$

of the differential operator of the form

$$A_1^x(t) = \sum_{|r|=2m} a_r(t,x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} \tag{25}$$

acting on functions defined on the space R^n, the inequalities are satisfied:

$$0 < M_1 |\xi|^{2m} \le (-1)^m A^x(t,\xi) \le M_2 |\xi|^{2m} < \infty$$

for $\xi \neq 0$, where $|\xi| = \left(|\xi_1|^2 + \cdots + |\xi_n|^2\right)^{\frac{1}{2}}$. We can reduce the initial value problem (24) to the initial value problem (1) in Banach space $E = C(R^n)$ with a strongly positive operator $A^x(t) = A_1^x(t) + \delta I$ defined by (25) [23]-[24]. The corollary below follows from the abstract Theorem 2.1.

Theorem 3.3. *Assume the hypotheses below:*

i $\varphi : [-\omega, 0] \times C(R^n) \to C(R^n)$ *be bounded and continuous function and*

$$\left\|\varphi(t,.)\right\|_{C(R^n)} \le M.$$

ii $g : (0,\infty) \times C(R^n) \times C(R^n) \to C(R^n)$ *be bounded and continuous function, i.e.;*

$$\left\|g(t,.,u,v))\right\|_{C(R^n)} \le \overline{M},$$

and with respect to z, the Lipschitz condition holds uniformly

$$\left\|g(t,.,v,z) - g(t,.,u,z)\right\|_{C(R^n)} \le L \left\|v-u\right\|_{C(R^n)},$$

where $L, M, \overline{M}$ *are positive constants. Then problem (24) has a unique bounded solution in* $[0,\infty) \times C(R^n)$.

The proof of Theorem 3.3 is based on the abstract Theorem 2.1, on the strong positivity of a differential operator $A^x(t)$ in $C(R^n)$ according to the formula (25), and on the estimate

$$\|V(t,y)\|_{C(R^n) \to C(R^n)} \le M_3, \quad t \ge y \ge 0.$$

4. Numerical results

Generally speaking, nonlinear problems cannot be solved precisely. Therefore the FSOADSs for the solution of nonlinear one-dimensional DPDE are presented. Numerical results are given. Consider the

IBVP

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = u(t,x)\left[u([t-1],x)\cos x - \frac{\partial u([t-1],x)}{\partial x}\sin x\right], \\ t \in (0,\infty),\ x \in (0,\pi), \\ u(0,x) = \sin x, x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0,\ t \in [0,\infty) \end{cases} \tag{26}$$

for the nonlinear delay parabolic equation. Here $[\cdot]$ is notation of integer function. The exact solution (ES) of this test example is $u(t,x) = e^{-t}\sin x$.

We get the following iterative first order of accuracy difference scheme (FOADS) for the approximate solution (AS) of the IBVP (26)

$$\begin{cases} \frac{{}_m u_n^k - {}_m u_n^{k-1}}{\tau} - \frac{{}_m u_{n+1}^k - 2{}_m u_n^k + {}_m u_{n-1}^k}{h^2} - {}_{m-1}u_n^k {}_m u_n^{[k-N]}\cos x_n + {}_{m-1}u_n^k \frac{{}_m u_{n+1}^{[k-N]} - {}_m u_{n-1}^{[k-N]}}{2h}\sin x_n = 0, \\ t_k = k\tau, x_n = nh, k \in \overline{1,\infty}, n \in \overline{1, M-1}, \\ {}_m u_n^0 = \sin x_n, x_n = nh, n \in \overline{0,M}, \\ {}_m u_0^k = {}_m u_M^k = 0, k \in \overline{0,\infty} \end{cases} \tag{27}$$

for the nonlinear delay parabolic equation.
Here m denotes the iteration number and an initial guess ${}_0 u_n^k, k \in \overline{0,N}, n \in \overline{0,M}$ is to be made. For solving difference scheme (27),we follow the numerical steps given below. The algorithm is as follows for $k \in \overline{0,N}$,$n \in \overline{0,M}$:

1. $m = 1$,
2. ${}_{m-1}u_n^k$ is known,
3. ${}_m u_n^k$ is calculated,
4. if the max absolute error between ${}_{m-1}u_n^k$ and ${}_m u_n^k$ is greater than the given tolerance value, take $m = m+1$ and go to step 2. Otherwise, terminate the iteration process and take ${}_m u_n^k$ as the result of the given problem.

We write (27) in matrix form

$$A_m U^k + B_m U^{k-1} = R\varphi({}_{m-1}u^k, {}_m u^{k-N}), k \in \overline{1,N},$$

$${}_m U^0 = \{\sin x_n\}_{n=0}^M, \tag{28}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ a & c & a & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & a & c & a & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & a & c & a & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & c & . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & a & c & a \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$a=-\frac{1}{h^2}, \ , \ b=\frac{1}{\tau}, c=\frac{1}{\tau}+\frac{2}{h^2},$$

R is identity matrix of size $M+1$,$_m u_n^k = e^{-t_k}\sin x_n$ for $k\in\overline{-N,0}$ and $\varphi({}_{m-1}u^k, {}_m u^{k-N})$,U^s are $(M+1)\times 1$ column vectors as

$$\varphi({}_{m-1}u^k, {}_m u^{k-N}) = \begin{bmatrix} 0 \\ \varphi_1^k \\ \vdots \\ \varphi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1} ,_m U^s = \begin{bmatrix} {}_mU_0^s \\ {}_mU_1^s \\ \vdots \\ {}_mU_{M-1}^s \\ {}_mU_M^s \end{bmatrix}_{(M+1)\times 1} , \ s=k, \ k\pm 1,$$

where

$$\varphi_n^k = {}_{m-1}u_n^k {}_m u_n^{[k-N]}\cos x_n - {}_{m-1}u_n^k \frac{{}_m u_{n+1}^{[k-N]} - {}_m u_{n-1}^{[k-N]}}{2h}\sin x_n, \ n\in\overline{1,M-1}.$$

So, we have the first order difference equation with respect to k with matrix coefficients. From (28) it follows that

$${}_mU^k = -A^{-1}B_mU^{k-1} + A^{-1}R\varphi^k, k\in\overline{pN+1,(p+1)N}, p=0,1,2,...,$$

$${}_mU^0 = \{\sin x_n\}_{n=0}^M . \tag{29}$$

Additionally, using the SOADS for the AS of problem (26), we obtain the following system of equations

$$\begin{cases} \frac{{}_m u_n^k - {}_m u_n^{k-1}}{\tau} - \frac{{}_m u_{n+1}^k - 2{}_m u_n^k + {}_m u_{n-1}^k}{h^2} + \tau \frac{{}_m u_{n+2}^k - 4{}_m u_{n+1}^k + 6{}_m u_n^k - 4{}_m u_{n-1}^k + {}_m u_{n-2}^k}{2h^4} \\ = \frac{1}{2}{}_m u_n^k \left[{}_{m-1}u_n^{k-N} \cos x_n - \frac{{}_{m-1}u_{n+1}^{k-N} - {}_{m-1}u_{n-1}^{k-N}}{2h} \sin x_n \right] \\ + \frac{1}{2}{}_m u_n^{k-1} \left[{}_{m-1}u_n^{k-1-N} \cos x_n - \frac{{}_{m-1}u_{n+1}^{k-1-N} - {}_{m-1}u_{n-1}^{k-1-N}}{2h} \sin x_n \right] \\ - \frac{\tau}{4}{}_m u_{n+1}^k \frac{{}_{m-1}u_{n+1}^{k-N} \cos x_{n+1} - \frac{{}_{m-1}u_{n+2}^{k-N} - {}_{m-1}u_n^{k-N}}{2h} \sin x_{n+1}}{h^2} \\ - \frac{\tau}{4}{}_m u_n^k \frac{-2{}_{m-1}u_n^{k-N} \cos x_n + 2\frac{{}_{m-1}u_{n+1}^{k-N} - {}_{m-1}u_{n-1}^{k-N}}{2h} \sin x_n}{h^2} \\ - \frac{\tau}{4}{}_m u_{n-1}^k \frac{{}_{m-1}u_{n-1}^{k-N} \cos x_{n-1} - \frac{{}_{m-1}u_n^{k-N} - {}_{m-1}u_{n-2}^{k-N}}{2h} \sin x_{n-1}}{h^2} \\ - \frac{\tau}{4}{}_m u_{n+1}^{k-1} \frac{{}_{m-1}u_{n+1}^{k-1-N} \cos x_{n+1} - \frac{{}_{m-1}u_{n+2}^{k-1-N} - {}_{m-1}u_n^{k-1-N}}{2h} \sin x_{n+1}}{h^2} \\ - \frac{\tau}{4}{}_m u_n^{k-1} \frac{-2{}_{m-1}u_n^{k-1-N} \cos x_n + 2\frac{{}_{m-1}u_{n+1}^{k-1-N} - {}_{m-1}u_{n-1}^{k-1-N}}{2h} \sin x_n}{h^2} \\ - \frac{\tau}{4}{}_m u_{n-1}^{k-1} \frac{{}_{m-1}u_{n-1}^{k-1-N} \cos x_{n-1} - \frac{{}_{m-1}u_n^{k-1-N} - {}_{m-1}u_{n-2}^{k-1-N}}{2h} \sin x_{n-1}}{h^2}, \\ t_k = k\tau, x_n = nh, k \in \overline{1, N-1}, n \in \overline{2, M-2}, \\ {}_m u_n^0 = \sin x_n, n \in \overline{0, M}, {}_m u_0^k = {}_m u_M^k = 0, k \in \overline{0, \infty}, \\ {}_m u_3^k = 4{}_m u_2^k - 5{}_m u_1^k, {}_m u_{M-3}^k = 4{}_m u_{M-2}^k - 5{}_m u_{M-1}^k, k \in \overline{0, \infty}. \end{cases} \tag{30}$$

We obtain again $(M+1) \times (M+1)$ SLEs and we reformat them into matrix form (28), where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 1 \\ e & f & g & f & e & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & e & f & g & f & e & . & . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & e & f & g & f & e \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & -1 & 4 & -5 & 0 \\ 0 & -5 & 4 & -1 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & l & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here

$$e = \frac{\tau}{2h^4}, f = -\frac{1}{h^2} - \frac{2\tau}{h^4},$$

$$g = \frac{1}{\tau} + \frac{2}{h^2} + \frac{3\tau}{h^4}, l = -\frac{1}{\tau},$$

$$\varphi({}_{m-1}u^k, {}_m u^{k-N}) = \begin{bmatrix} 0 \\ 0 \\ \varphi_2^k \\ \vdots \\ \varphi_{M-2}^k \\ 0 \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

where

$$\varphi_n^k = \frac{1}{2}{}_m u_n^k \left[{}_{m-1}u_n^{k-N}\cos x_n - \frac{{}_{m-1}u_{n+1}^{k-N} - {}_{m-1}u_{n-1}^{k-N}}{2h}\sin x_n \right]$$

$$+\frac{1}{2}{}_m u_n^{k-1} \left[{}_{m-1}u_n^{k-1-N}\cos x_n - \frac{{}_{m-1}u_{n+1}^{k-1-N} - {}_{m-1}u_{n-1}^{k-1-N}}{2h}\sin x_n \right]$$

$$-\frac{\tau}{4}{}_m u_{n+1}^k \frac{{}_{m-1}u_{n+1}^{k-N}\cos x_{n+1} - \frac{{}_{m-1}u_{n+2}^{k-N} - {}_{m-1}u_{n}^{k-N}}{2h}\sin x_{n+1}}{h^2} - \frac{\tau}{4}{}_m u_n^k \frac{-2{}_{m-1}u_n^{k-N}\cos x_n + 2\frac{{}_{m-1}u_{n+1}^{k-N} - {}_{m-1}u_{n-1}^{k-N}}{2h}\sin x_n}{h^2}$$

$$-\frac{\tau}{4}{}_m u_{n-1}^k \frac{{}_{m-1}u_{n-1}^{k-N}\cos x_{n-1} - \frac{{}_{m-1}u_{n}^{k-N} - {}_{m-1}u_{n-2}^{k-N}}{2h}\sin x_{n-1}}{h^2} - \frac{\tau}{4}{}_m u_{n+1}^{k-1} \frac{{}_{m-1}u_{n+1}^{k-1-N}\cos x_{n+1} - \frac{{}_{m-1}u_{n+2}^{k-1-N} - {}_{m-1}u_{n}^{k-1-N}}{2h}\sin x_{n+1}}{h^2}$$

$$-\frac{\tau}{4}{}_m u_n^{k-1} \frac{-2{}_{m-1}u_n^{k-1-N}\cos x_n + 2\frac{{}_{m-1}u_{n+1}^{k-1-N} - {}_{m-1}u_{n-1}^{k-1-N}}{2h}\sin x_n}{h^2} - \frac{\tau}{4}{}_m u_{n-1}^{k-1} \frac{{}_{m-1}u_{n-1}^{k-1-N}\cos x_{n-1} - \frac{{}_{m-1}u_{n}^{k-1-N} - {}_{m-1}u_{n-2}^{k-1-N}}{2h}\sin x_{n-1}}{h^2}$$

for $n \in \overline{2, M-2}$. Hence, we have a second order difference equation with respect to k matrix coefficients. Applying (28), we can obtain the solution of this difference scheme. In computations for both first and second order of accuracy difference schemes, the initial guess is chosen as ${}_0u_n^k = e^{-t_k}\sin x_n$ and when the maximum errors between two consecutive results of iterative difference schemes (27) and (30) become less than 10^{-8}, the iterative process is terminated.

We provide numerical results for various values of M and N and the numerical solutions of these difference schemes are represented by u_n^k at (t_k, x_n). Table 1, Table 2, and Table 3 are constructed for $N = M = 30, 60, 120$ in $t \in [n, n+1]$, $n = 0, 1, 2$, respectively. The errors are calculated using the following formula.

$${}_m\left(E_M^N\right)_p = \max_{\substack{pN+1\le k\le (p+1)N, p=0,1,\ldots, \\ 1\le n\le M-1}} \left| u(t_k, x_n) - {}_m u_n^k \right|. \tag{31}$$

To finish iteration process it was used condition

$$\max_{\substack{pN+1\le k\le (p+1)N, p=0,1,\ldots, \\ 1\le n\le M-1}} \left| {}_m u_n^k - {}_{m-1}u_n^{k,} \right| < 10^{-8} \tag{32}$$

in each subinterval.

Table 1. Error comparison between difference schemes (27) and (30) in $t \in [0,1]$ (Number of iterations=m)

Method	$M = N = 30$	$M = N = 60$	$M = N = 120$
(27)	$6.3783 \times 10^{-3}, m = 2$	$3.1279 \times 10^{-2}, m = 2$	$1.5485 \times 10^{-3}, m = 2$
(30)	$4.5864 \times 10^{-4}, m = 3$	$1.1212 \times 10^{-4}, m = 3$	$2.7577 \times 10^{-5}, m = 2$

Table 2. Error comparison between difference schemes (27) and (30) in $t \in [1,2]$ (Number of iterations=m)

Method	$M = N = 30$	$M = N = 60$	$M = N = 120$
(27)	$2.3464 \times 10^{-3}, m = 3$	$1.5070 \times 10^{-3}, m = 3$	$5.6964 \times 10^{-4}, m = 2$
(30)	$1.6358 \times 10^{-4}, m = 3$	$4.2149 \times 10^{-5}, m = 2$	$1.0698 \times 10^{-5}, m = 2$

Table 3. Error comparison between difference schemes (27) and (30) in $t \in [2,3]$ (Number of iterations=m)

Method	$M = N = 30$	$M = N = 60$	$M = N = 120$
(27)	$8.6321 \times 10^{-4}, m = 3$	$4.2332 \times 10^{-4}, m = 2$	$2.0956 \times 10^{-4}, m = 2$
(30)	$5.3201 \times 10^{-5}, m = 2$	$1.3581 \times 10^{-5}, m = 2$	$3.4122 \times 10^{-6}, m = 2$

We also consider the IBVP

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + \sin(u(t,x)) \\ = u(t,x)\left[2u([t-1],x)\cos 2x - \frac{\partial u([t-1],x)}{\partial x}\sin 2x\right] + f(t,x), \\ \\ u(t,0) = u(t,\pi), u_x(t,0) = u_x(t,\pi), \ t \in [0,\infty) \end{cases} \tag{33}$$

for the nonlinear delay PDE with nonlocal conditions where $f(t,x) = \sin\left(e^{-4t}\sin 2x\right)$. The ES of this test example is $u(t,x) = e^{-4t}\sin 2x$.

We get the following FOADS for the AS of the IBVP (33)

$$\begin{cases} \frac{{}_m u_n^k - {}_m u_n^{k-1}}{\tau} - \frac{{}_m u_{n+1}^k - 2{}_m u_n^k + {}_m u_{n-1}^k}{h^2} - 2{}_{m-1}u_n^k {}_m u_n^{[k-N]}\cos 2x_n \\ +{}_{m-1}u_n^k \frac{{}_m u_{n+1}^{[k-N]} - {}_m u_{n-1}^{[k-N]}}{2h}\sin 2x_n = \sin\left({}_m u_n^k\right) + f(t_k, x_n), \\ \\ t_k = k\tau, x_n = nh, k \in \overline{1,\infty}, n \in \overline{1, M-1}, \\ \\ {}_m u_n^0 = \sin 2x_n, x_n = nh, n \in \overline{0,M}, \\ \\ {}_m u_0^k = {}_m u_M^k, {}_m u_1^k - {}_m u_0^k = {}_m u_M^k - {}_m u_{M-1}^k, k \in \overline{pN+1, (p+1)N}, p = 0, 1, ... \end{cases} \tag{34}$$

We write (34) in matrix form

$$A_m U^k + B_m U^{k-1} = R\theta, k \in \overline{pN+1,(p+1)N}, p = 0,1,...,$$

$$_mU^0 = \{\sin 2x_n\}_{n=0}^M, \tag{35}$$

where

$$_mU^k = \left\{_m u_n^k\right\}_{n=0}^M, \theta_n^k = \sin\left(_m u_n^k\right) + f(t_k, x_n),$$

$$n = 0,...,M, k \in \overline{pN+1,(p+1)N}, p = 0,1,...,$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . & . & 0 & 0 & 0 & 1 \\ a & c_1^k & a & 0 & 0 & 0 & . & . & 0 & 0 & 0 & 0 \\ 0 & a & c_2^k & a & 0 & 0 & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & a & c_3^k & a & 0 & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & c_4^k & a & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & 0 & a & c_{M-1}^k & a \\ 1 & -1 & 0 & 0 & 0 & 0 & . & . & 0 & 0 & -1 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & l & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & l & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . & l & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & l & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$a = -\tfrac{1}{h^2}, l = -\tfrac{1}{\tau},$$
$$c_n^k = \tfrac{1}{\tau} + \tfrac{2}{h^2} - 2u_n^{[k-N]}\cos 2x_n + \tfrac{u_{n+1}^{[k-N]} - u_{n-1}^{[k-N]}}{2h}\sin 2x_n,$$

and R is identity matrix of size $M+1$, θ is zero matrix with $(M+1)\times 1$ dimension. So, we have the first order difference equation with respect to k with matrix coefficients. From (35) it follows that

$$_mU^k = -A^{-1}B_m U^{k-1} + A^{-1}R\theta^k, k \in \overline{pN+1,(p+1)N}, p = 0,1,...,$$

$$_mU^0 = \{\sin 2x_n\}_{n=0}^M. \tag{36}$$

Furthermore, using the SOADS for the AS of problem (26), we obtain the following system of equations

$$\begin{cases} \frac{{}_m u_n^k - {}_m u_n^{k-1}}{\tau} - \frac{{}_m u_{n+1}^k - 2{}_m u_n^k + {}_m u_{n-1}^k}{h^2} + \tau \frac{{}_m u_{n+2}^k - 4{}_m u_{n+1}^k + 6{}_m u_n^k - 4{}_m u_{n-1}^k + {}_m u_{n-2}^k}{2h^4} \\ -\frac{1}{2}\left[{}_m u_n^k {}_{m-1}u_n^{[k-N]} \cos 2x_n - {}_m u_n^k \frac{{}_{m-1}u_{n+1}^{[k-N]} - {}_{m-1}u_{n-1}^{[k-N]}}{2h} \sin 2x_n\right] \\ -\frac{1}{2}\left[{}_m u_n^{k-1} {}_{m-1}u_n^{[k-1-N]} \cos 2x_n - {}_m u_n^{k-1} \frac{{}_{m-1}u_{n+1}^{[k-1-N]} - {}_{m-1}u_{n-1}^{[k-1-N]}}{2h} \sin 2x_n\right] \\ +\frac{\tau}{4} \frac{{}_m u_{n+1}^k {}_{m-1}u_{n+1}^{[k-N]} \cos 2x_{n+1} - {}_m u_{n+1}^k \frac{{}_{m-1}u_{n+2}^{[k-N]} - {}_{m-1}u_n^{[k-N]}}{2h} \sin 2x_{n+1}}{h^2} \\ +\frac{\tau}{4} \frac{-2{}_m u_n^k {}_{m-1}u_n^{[k-N]} \cos 2x_n + 2{}_m u_n^k \frac{{}_{m-1}u_{n+1}^{[k-N]} - {}_{m-1}u_{n-1}^{[k-N]}}{2h} \sin 2x_n}{h^2} \\ +\frac{\tau}{4} \frac{{}_m u_{n-1}^k {}_{m-1}u_{n-1}^{[k-N]} \cos 2x_{n-1} - {}_m u_{n-1}^k \frac{{}_{m-1}u_n^{[k-N]} - {}_{m-1}u_{n-2}^{[k-N]}}{2h} \sin 2x_{n-1}}{h^2} \\ +\frac{\tau}{4} \frac{{}_m u_{n-1}^{k-1} {}_{m-1}u_{n+1}^{[k-1-N]} \cos 2x_{n+1} - {}_m u_{n+1}^{k-1} \frac{{}_{m-1}u_{n+2}^{[k-1-N]} - {}_{m-1}u_n^{[k-1-N]}}{2h} \sin 2x_{n+1}}{h^2} \\ +\frac{\tau}{4} \frac{-2{}_m u_n^{k-1} {}_{m-1}u_{n+1}^{[k-1-N]} \cos 2x_n + 2{}_m u_n^{k-1} \frac{{}_{m-1}u_{n+1}^{[k-1-N]} - {}_{m-1}u_{n-1}^{[k-1-N]}}{2h} \sin 2x_{n-1}}{h^2} \\ +\frac{\tau}{4} \frac{{}_m u_{n-1}^{k-1} {}_{m-1}u_{n+1}^{[k-1-N]} \cos 2x_{n-1} - {}_m u_{n-1}^{k-1} \frac{{}_{m-1}u_n^{[k-1-N]} - {}_{m-1}u_{n-2}^{[k-1-N]}}{2h} \sin 2x_{n-1}}{h^2} + \sin\left({}_m u_n^k\right) = f(t_k, x_n), \\ t_k = k\tau, x_n = nh, k \in \overline{1, N-1}, n \in \overline{2, M-2}, \\ {}_m u_n^0 = \sin 2x_n, n \in \overline{0, M}, \\ {}_m u_0^k = {}_m u_M^k, -{}_m u_2^k + 4{}_m u_1^k - 3{}_m u_0^k = 3{}_m u_M^k - 4{}_m u_{M-1}^k + {}_m u_{M-2}^k, \\ -{}_m u_3^k + 4{}_m u_2^k - 5{}_m u_1^k + 2{}_m u_0^k = 2{}_m u_M^k - 5{}_m u_{M-1}^k + 4{}_m u_{M-2}^k - {}_m u_{M-3}^k, \\ -3{}_m u_4^k + 14{}_m u_3^k - 24{}_m u_2^k + 18{}_m u_1^k - 5{}_m u_0^k = 5{}_m u_M^k - 18{}_m u_{M-1}^k + 24{}_m u_{M-2}^k - 14{}_m u_{M-3}^k + 3{}_m u_{M-4}^k, \\ k \in \overline{pN+1, (p+1)N}, p = 0, 1, ... \end{cases} \tag{37}$$

We obtain another $(M+1) \times (M+1)$ SLEs they are then rewritten in matrix form (35), where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 & -1 \\ 2 & -5 & 4 & -1 & 0 & . & 0 & 1 & -4 & 5 & -2 \\ e & f_3^k & g_3^k & w_3^k & e & . & 0 & 0 & 0 & 0 & 0 \\ 0 & e & f_4^k & g_4^k & w_4^k & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & f_{M-2}^k & g_{M-2}^k & w_{M-2}^k & e & 0 \\ 0 & 0 & 0 & 0 & 0 & . & e & f_{M-1}^k & g_{M-1}^k & w_{M-1}^k & e \\ -3 & 4 & -1 & 0 & 0 & . & 0 & 0 & -1 & 4 & -3 \\ -5 & 18 & -24 & 14 & -3 & . & 3 & -14 & 24 & -18 & 5 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 & 0 \\ 0 & z_3^k & l_3^k & m_3^k & 0 & 0 & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_3^k & l_3^k & m_3^k & 0 & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_4^k & l_4^k & m_4^k & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ .0 & 0. & 0 & 0 & 0 & 0 & . & z_{M-2}^k & l_{M-2}^k & m_{M-2}^k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & . & 0 & z_{M-1}^k & l_{M-1}^k & m_{M-1}^k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0. & . & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here

$$
\begin{aligned}
f_n^k &= -\tfrac{1}{h^2} - \tfrac{2\tau}{h^4} + \tfrac{\tau}{2h^2} u_{n-1}^{[k-N]} \cos 2x_{n-1} - \tfrac{\tau}{8h^3} u_n^{[k-N]} \sin 2x_{n-1} + \tfrac{\tau}{8h^3} u_{n-2}^{[k-N]} \sin 2x_{n-1}, \\
e &= \tfrac{\tau}{2h^4}, g_n^k = \tfrac{1}{\tau} + \tfrac{2}{h^2} + \tfrac{3\tau}{h^4} - u_n^{[k-N]} \cos 2x_n + \tfrac{1}{4h} u_{n+1}^{[k-N]} \sin 2x_n \\
&- \tfrac{1}{4h} u_{n-1}^{[k-N]} \sin 2x_n - \tfrac{\tau}{h^2} u_n^{[k-N]} \cos 2x_n + \tfrac{\tau}{4h^3} u_{n+1}^{[k-N]} \sin 2x_n - \tfrac{\tau}{4h^3} u_{n-1}^{[k-N]} \sin 2x_n, \\
w_n^k &= -\tfrac{1}{h^2} - \tfrac{2\tau}{h^4} + \tfrac{\tau}{2h^2} u_{n+1}^{[k-N]} \cos 2x_{n+1} - \tfrac{\tau}{8h^3} u_{n+2}^{[k-N]} \sin 2x_{n+1} + \tfrac{\tau}{8h^3} u_n^{[k-N]} \sin 2x_{n+1}, \\
z_n^k &= \tfrac{\tau}{2h^2} u_{n-1}^{[k-N]} \cos 2x_{n-1} - \tfrac{\tau}{8h^3} u_n^{[k-N]} \sin 2x_{n-1} + \tfrac{\tau}{8h^3} u_{n-2}^{[k-N]} \sin 2x_{n-1}, \\
l_n^k &= -\tfrac{1}{\tau} - u_n^{[k-N]} \cos 2x_n + \tfrac{1}{4h} u_{n+1}^{[k-N]} \sin 2x_n - \tfrac{1}{4h} u_{n-1}^{[k-N]} \sin 2x_n \\
&- \tfrac{\tau}{h^2} u_n^{[k-N]} \cos 2x_n + \tfrac{\tau}{4h^3} u_{n+1}^{[k-N]} \sin 2x_n - \tfrac{\tau}{4h^3} u_{n-1}^{[k-N]} \sin 2x_n, \\
m_n^k &= \tfrac{\tau}{2h^2} u_{n+1}^{[k-N]} \cos 2x_{n+1} - \tfrac{\tau}{8h^3} u_{n+2}^{[k-N]} \sin 2x_{n+1} + \tfrac{\tau}{8h^3} u_n^{[k-N]} \sin 2x_{n+1}.
\end{aligned}
$$

We provide numerical results for a range of values of M and N and u_n^k represent the numerical solutions of these difference schemes at (t_k, x_n). Table 4, Table 5, and Table 6 are constructed for $M = N = 30, 60, 120$ in $t \in [p, p+1]$, $p = 0, 1, 2$, and the errors are computed by the formulas (31) and (32).

Table 4. Error comparison between difference schemes (34) and (37) in $t \in [0, 1]$ (Number of iterations=m)

Method	$M = N = 30$	$M = N = 60$	$M = N = 120$
(34)	$2.4431 \times 10^{-2}, m = 2$	$1.2259 \times 10^{-2}, m = 2$	$6.1304 \times 10^{-3}, m = 2$
(37)	$2.0589 \times 10^{-3}, m = 8$	$5.4628 \times 10^{-4}, m = 8$	$1.3865 \times 10^{-4}, m = 7$

Table 5. Error comparison between difference schemes (34) and (37) in $t \in [1, 2]$ (Number of iterations=m)

Method	$M = N = 30$	$M = N = 60$	$M = N = 120$
(34)	$5.3731 \times 10^{-3}, m = 9$	$2.5664 \times 10^{-3}, m = 8$	$1.2517 \times 10^{-3}, m = 8$
(37)	$3.0514 \times 10^{-4}, m = 8$	$7.5756 \times 10^{-5}, m = 7$	$1.9241 \times 10^{-5}, m = 6$

Table 6. Error comparison between difference schemes (34) and (37) in $t \in [2, 3]$ (Number of iterations=m)

Method	$M = N = 30$	$M = N = 60$	$M = N = 120$
(34)	$1.0838 \times 10^{-4}, m = 7$	$4.9176 \times 10^{-5}, m = 6$	$2.3435 \times 10^{-5}, m = 6$
(37)	$8.1588 \times 10^{-6}, m = 7$	$2.0085 \times 10^{-6}, m = 5$	$4.8130 \times 10^{-7}, m = 3$

As we doubled the values of N and M each time, beginning with $M = N = 30$. In the FOADSs (27) and (34) in Tables 1-6 respectively, the errors decrease roughly by a proportion of 1/2, while in the SOADSs (30) and (37) in Tables 1-6 respectively, the errors decrease roughly by a proportion of 1/4. Errors shown in the tables demonstrate the consistency of the different schemes and the reliability of the findings. Accordingly, the SOADS increases faster than the FOADS. These numerical experiments back up the theoretical claims as shown in the tables. With more grid points, the maximum errors can be reduced.

In this manuscript, the abbreviations used are as follows:

IBVP	initial boundary value problem
IVP	initial value problem
BS	bounded solution
DE	differential equation
DPDE	delay parabolic differential equation
SSF	sufficiently smooth functions
FSOADS	first and second-Order of accuracy difference scheme
PHPDE	parabolic and hyperbolic partial differential equation
ES	exact solution
FOADS	first order of accuracy difference scheme
SOADS	second order of accuracy difference scheme
AS	approximate solution
SLEs	system of linear equations

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