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Examination of the topological equivalence of some dynamical systems on the Box fractal

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Abstract. In this paper, we examine dynamical systems obtained with the same expanding and different numbers of folding transformations on the Box fractal (B). We express these dynamical systems through the addresses of points by using the terms of $\{0, 1, 2, 3, 4\}$. We then compute and compare the periodic points of the dynamical systems. Finally, we examine these systems in the sense of topological equivalence and we investigate the chaos conditions for the dynamical systems.

1. Introduction

In the literature, it has been seen that fractals, which are popular shapes because of their different features, have been frequently included in the many studies for a long time [1, 5, 9, 10, 13, 14, 17]. Especially, due to the characteristic of self-similarity, the examination of the properties of many structures on the fractals can be more systematic and comprehensible. Some maps defined on the fractals, which have many remarkable and interesting features, are well known in the literature. For instance, the Tent map and Horse-shoe map are favoured and distinctive examples, which are defined on the Cantor set (C) and $C \times C$ respectively (see in [7]). Since, it can be easier to examine the properties of the dynamical systems, to define these systems on the fractals can be more significant and sensible. There are many studies about dynamical systems defined on many fractal sets such as Sierpinski gasket, Sierpinki tetrahedron, Cantor set, Cantor dust, Box fractal etc. (see in [2–4, 6, 18]). It can be seen that to express these dynamical systems with the help of the code representations of the points provides many conveniences in computing of periodic points, investigating the topological conjugacy and other properties. Therefore, defining an intrinsic metric formula via the addresses of points on the related self-similar set is a remarkable matter.

In [4], two dynamical systems are defined on the code sets of the Box fractal. Moreover, there is an intrinsic metric formula, which is defined on B via code representations in [16]. Thanks to this useful metric, it is shown that the dynamical systems defined in [4] are chaotic in the sense of Devaney. However, there is no study about examining the topological conjugacy of any dynamical systems on the Box fractal. As a result of this, in this study we propose to construct some dynamical systems on this self-similar set B by using different number of folding mappings. We notice that in some cases, the number of folding mappings and at which levels these mappings take place can effect the number of the periodic points of

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order n. These dynamical systems are defined via code representations of points to facilitate the analysis of chaos conditions and topological equivalence.

In the present paper, we first give some necessary definitions and notions. Then, in order to obtain different dynamical systems, we construct the functions F, G and T by using the same expanding and different number of folding mappings on the Box fractal. One can clearly see the effects of these functions on the code sets of B in Figure 2, Figure 3 and Figure 4 respectively. Then, we express these dynamical systems via code representations in Proposition 2.1, Proposition 2.3 and Proposition 2.4. This new expression enables us to determine whether these dynamical systems are chaotic or not with the help of the intrinsic metric formula in Theorem 1.4. In addition, we find the periodic points with basic computations, we thus get a result that these systems are not topologically equivalent. Finally, we also give an example of a dynamical system which is topologically conjugate to $\{B; G\}$ (see in Figure 5) and we find a conjugacy map between these systems in Lemma 3.5. Through this map, we also compute the periodic points of $\{B; G'\}$ using the periodic points of $\{B; G\}$.

First of all we give some fundamental notions.

Definition 1.1. Let (X, d) be a metric space and $f : X \to X$ be a transformation. Then f is called a dynamical system on X which is represented by $\{X; f\}$.

Definition 1.2. Two dynamical systems $\{X; f\}$ and $\{Y; g\}$ are topologically conjugate (equivalent), if there exists a homeomorphism $h : X \to Y$ such that $g = h \circ f \circ h^{-1}$ (or equivalently $h(f(x)) = g(h(x))$ for every $x \in X$) (see [10]).

Definition 1.3. A dynamical system $\{X; f\}$ is called chaotic in the sense of Devaney, if it satisfies density of periodic points, sensitivity dependence on initial conditions and topological transitivity.

Density of periodic points: If there is a periodic point of f, which is sufficiently close to any point of X, the periodic points of f are dense on X.

Sensitivity dependence on initial conditions: If there exists an $\epsilon > 0$ such that for any $x \in X$ and any ball $B(x, \delta)$, there exists $y \in B(x, \delta)$ and an integer $n \geq 0$ satisfying that $d(f^n(x), f^n(y)) > \epsilon$, then we call that $\{X; f\}$ has sensitive dependence on initial conditions.

Topological transitivity: If for any non-empty open subsets $U, V \subset X$ there exists an integer n such that $U \cap f^n(V) \neq \emptyset$, then $\{X; f\}$ is said to have topological transitivity (see [12]).

Throughout this paper, we express the dynamical systems by using the code representations of the points. Thus, we now define the code sets on Box fractal and the code representations of the points on B :

It is known that according to Hutchinson theory [15], Box fractal on $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ can be constructed by using the following contraction mappings, where $w_i : \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 0, 1, 2, 3, 4)$ such that

$$\begin{aligned}
w_0(x, y) &= \left(\frac{x}{3}, \frac{y}{3}\right), \\
w_1(x, y) &= \left(\frac{x}{3} + \frac{1}{3}, \frac{y}{3} + \frac{1}{3}\right), \\
w_2(x, y) &= \left(\frac{x}{3} - \frac{1}{3}, \frac{y}{3} + \frac{1}{3}\right), \\
w_3(x, y) &= \left(\frac{x}{3} - \frac{1}{3}, \frac{y}{3} - \frac{1}{3}\right), \\
w_4(x, y) &= \left(\frac{x}{3} + \frac{1}{3}, \frac{y}{3} - \frac{1}{3}\right).
\end{aligned}$$

In this case B is called as the attractor of the iterated function system (IFS) $\{\mathbb{R}^2; w_0, w_1, w_2, w_3, w_4\}$ that is $B = \bigcup_{i=0}^{4} \omega_i(B)$. By the contraction mappings w_0, w_1, w_2, w_3 and w_4 the middle part of B denoted by B_0, the upper-right part of B denoted by B_1, the upper-left part of B denoted by B_2, the lower-left part of B denoted by B_3 and the lower-right part of B denoted by B_4 are obtained respectively (see Figure 1).

It is seen that from the above figure, the union of these code sets B_0, B_1, B_2, B_3, B_4 forms the Box fractal (B).

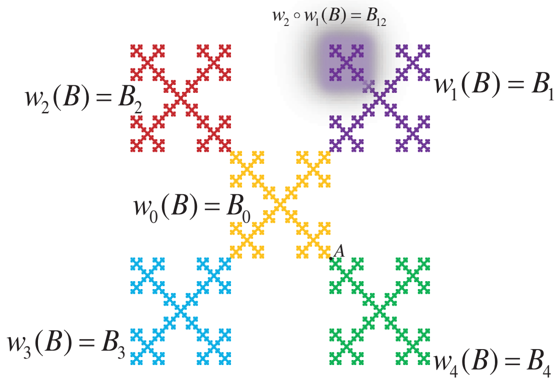

Figure 1: Box fractal with contraction mappings

Let $\sigma = x_1x_2\dots x_n$, for all $x_i \in \{0,1,2,3,4\}$ for $i = 1,2,\dots,n$. Then any subset at level n, B_σ could be defined as $B_\sigma = w_\sigma(B)$, where $w_\sigma = w_{x_n} \circ \dots \circ w_{x_2} \circ w_{x_1}$, (for details see Figure 1). It is obvious that the middle part of B_σ is $B_{\sigma 0}$, the upper-right part of B_σ is $B_{\sigma 1}$, the upper-left part of B_σ is $B_{\sigma 2}$, the lower-left part of B_σ is $B_{\sigma 3}$ and the lower-right part of B_σ is $B_{\sigma 4}$.

On the other hand, there is a relationship between the sets B_{x_1}, $B_{x_1x_2}$, $B_{x_1x_2x_3}, \dots$ as follows

$$B_{x_1} \supset B_{x_1x_2} \supset B_{x_1x_2x_3} \supset \dots \supset B_{x_1x_2\dots x_n} \supset \dots.$$

Then by the Cantor intersection theorem,

$$\bigcap_{n=1}^{\infty} B_\sigma = \{X\}$$

is a point on B, say X. The address (code representation) of the point X is represented by the sequence $x_1x_2\dots x_n\dots$ (see [16]).

B_σ is also a part of B. It is quite apparent that the union of

$$B_{\sigma\omega} = \{\sigma\omega x_{n+2}x_{n+3}x_{n+4}\dots | \omega, x_i \in \{0,1,2,3,4\} \text{ and } i = n+2, n+3, \dots\}$$

forms B_σ.

Note that for $i = 0$ and $j \neq 0$, the set $B_{\sigma i} \cap B_{\sigma j}$ has only one point. Let $\{X\} = B_{\sigma\omega} \cap B_{\sigma 0}$, then both the nested sets

$$B_\sigma \supset B_{\sigma\omega} \supset B_{\sigma\omega\omega'} \supset B_{\sigma\omega\omega'\omega'} \supset B_{\sigma\omega\omega'\omega'\omega'} \supset \dots$$

and

$$B_\sigma \supset B_{\sigma 0} \supset B_{\sigma 0\omega} \supset B_{\sigma 0\omega\omega} \supset B_{\sigma 0\omega\omega\omega} \supset \dots$$

contains the junction point, X. That means X is represented by

$$\sigma\omega\omega'\omega'\omega' \dots \text{ and } \sigma 0\omega\omega\omega \dots$$

where

$$\omega' = \begin{cases} 3, & \omega = 1 \\ 4, & \omega = 2 \\ 1, & \omega = 3 \\ 2, & \omega = 4. \end{cases}$$

Here, ω' is named as the conjugate of ω. The point A in Figure 1 has two code representations which are $42222\ldots$ and $04444\ldots$.

In the present paper, we also use the intrinsic metric formula on the Box fractal in [16] via the code representations of the points as follows:

Theorem 1.4. ([16]) *Let* $x_1x_2x_3\ldots x_{k-1}x_kx_{k+1}\ldots$ *and* $y_1y_2y_3\ldots y_{k-1}y_ky_{k+1}\ldots$ *be two code representations, respectively, of the points* $X \in B$ *and* $Y \in B$ *such that* $x_i = y_i$ *for* $i = 1,2,3,\ldots k-1$ *and* $x_k \neq y_k$. *Then the function* $d_{box} : B \times B \to \mathbb{R}^+ \cup \{0\}$ *such that*

$$d_{box}(X,Y) = \begin{cases} \frac{\sqrt{2}}{3^k} + D_X + D_Y, & x_k \neq 0 \neq y_k \\ D_{\tilde{X}} + D_Y, & x_k = 0,\ y_k \neq 0 \\ D_X + D_{\tilde{Y}}, & x_k \neq 0,\ y_k = 0 \end{cases}$$

where $\tilde{X}$ *and* $\tilde{Y}$ *have the code representations* $x_1x_2\ldots x_{k-1}y'_kx_{k+1}\ldots$ *and* $y_1y_2\ldots y_{k-1}x'_ky_{k+1}\ldots$ *respectively, determines the strictly intrinsic metric on the Box fractal* B.

Here, D_X *is computed as follows:*

$$D_X = \sum_{n=k+1}^{\infty} |A_n|$$

where

$$|A_n| = \begin{cases} 0, & x_n = t_n \\ \frac{\sqrt{2}}{3^n}, & x_n = 0 \\ \frac{2\sqrt{2}}{3^n}, & otherwise \end{cases}$$

such that

$$t_n = \begin{cases} t_{n-1}, & x_n = t_{n-1} \text{ or } x_n = 0 \\ x'_n, & otherwise \end{cases}$$

and $t_k = x'_k$.

2. The dynamical systems $\{B; F\}$, $\{B; G\}$ and $\{B; T\}$

We now intend to define dynamical systems on B via an expanding and different number of folding mappings. These systems are defined on B, restricted on the square $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$. To define the first dynamical system on B, considering the geometrical structure of the Box fractal, we firstly choose one expanding and 4 folding mappings as given below:

$f_i : \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 1,2,3,4,5)$

$$\begin{aligned} f_1(x,y) &= (3x, 3y), \\ f_2(x,y) &= \left(-\frac{1}{2}\left|x+y-1\right| + \frac{1}{2}(x-y-1)+1, -\frac{1}{2}\left|x+y-1\right| - \frac{1}{2}(x-y-1)\right), \\ f_3(x,y) &= \left(\frac{1}{2}\left|x-y+1\right| + \frac{1}{2}(x+y+1)-1, -\frac{1}{2}\left|x-y+1\right| + \frac{1}{2}(x+y+1)\right), \\ f_4(x,y) &= \left(\frac{1}{2}\left|x+y+1\right| + \frac{1}{2}(x-y+1)-1, \frac{1}{2}\left|x+y+1\right| - \frac{1}{2}(x-y+1)\right), \\ f_5(x,y) &= \left(-\frac{1}{2}\left|x-y-1\right| + \frac{1}{2}(x+y-1)+1, \frac{1}{2}\left|x-y-1\right| + \frac{1}{2}(x+y-1)\right). \end{aligned}$$

We construct the function F, which is composed by f_1, f_2, f_3, f_4, f_5 on B as follows

$$F = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1. \tag{1}$$

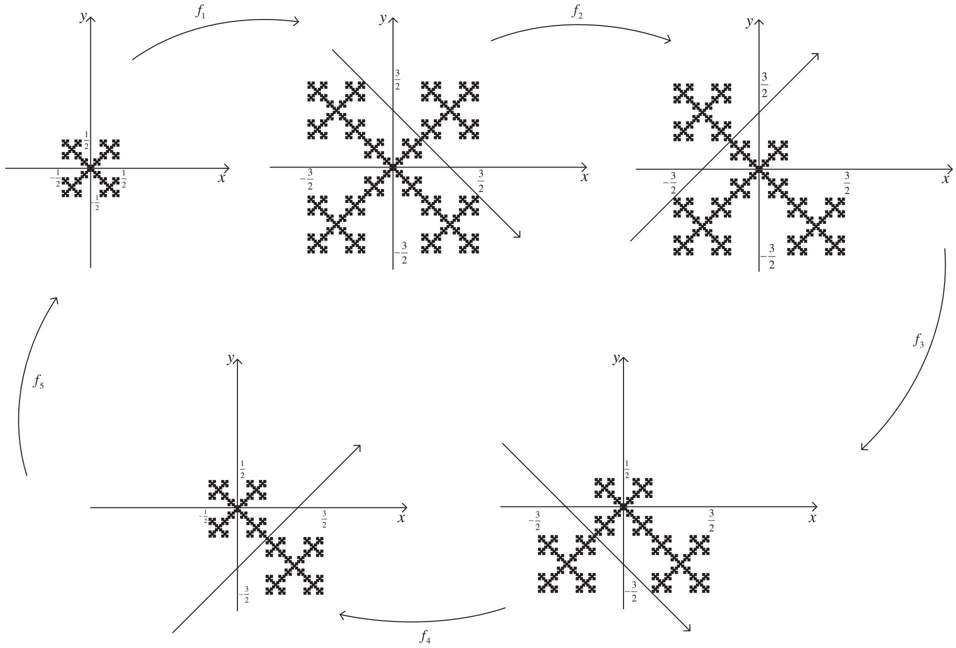

Figure 2: The dynamical system $\{B; F\}$ on the Box fractal

Then the dynamical system defined in this way on the Box fractal is denoted by $\{B; F\}$. We can see the effects of these functions geometrically in the Figure 2.

Another dynamical system with an expanding and 5 folding transformations, is defined as $\{B; G\}$, where G is the composition function of the following mappings: $g_i : \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 1, 2, 3, 4, 5, 6)$

$$\begin{aligned}
g_1(x, y) &= (3x, 3y),\\
g_2(x, y) &= \left(-\frac{1}{2}\left|x + y - 2\right| + \frac{1}{2}(x - y - 2) + 2, -\frac{1}{2}\left|x + y - 2\right| - \frac{1}{2}(x - y - 2)\right),\\
g_3(x, y) &= \left(-\frac{1}{2}\left|x + y - 1\right| + \frac{1}{2}(x - y - 1) + 1, -\frac{1}{2}\left|x + y - 1\right| - \frac{1}{2}(x - y - 1)\right),\\
g_4(x, y) &= \left(\frac{1}{2}\left|x - y + 1\right| + \frac{1}{2}(x + y + 1) - 1, -\frac{1}{2}\left|x - y + 1\right| + \frac{1}{2}(x + y + 1)\right),\\
g_5(x, y) &= \left(\frac{1}{2}\left|x + y + 1\right| + \frac{1}{2}(x - y + 1) - 1, \frac{1}{2}\left|x + y + 1\right| - \frac{1}{2}(x - y + 1)\right),\\
g_6(x, y) &= \left(-\frac{1}{2}\left|x - y - 1\right| + \frac{1}{2}(x + y - 1) + 1, \frac{1}{2}\left|x - y - 1\right| + \frac{1}{2}(x + y - 1)\right)
\end{aligned}$$

such that

$$G = g_6 \circ g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1. \tag{2}$$

Note that the first mappings of both F and G are expanding transformations that expand the points three times. The other transformations are folding mappings, indeed G has one more different mapping from F, which is a folding transformation g_2 that moves the points from the upper hand side of the line $y = -x + 2$ to the lower hand side. Both f_2, f_3, f_4, f_5 and g_3, g_4, g_5, g_6 are folding mappings with respect to the lines $y = -x + 1$, $y = x + 1$, $y = -x - 1$, and $y = x - 1$ respectively.

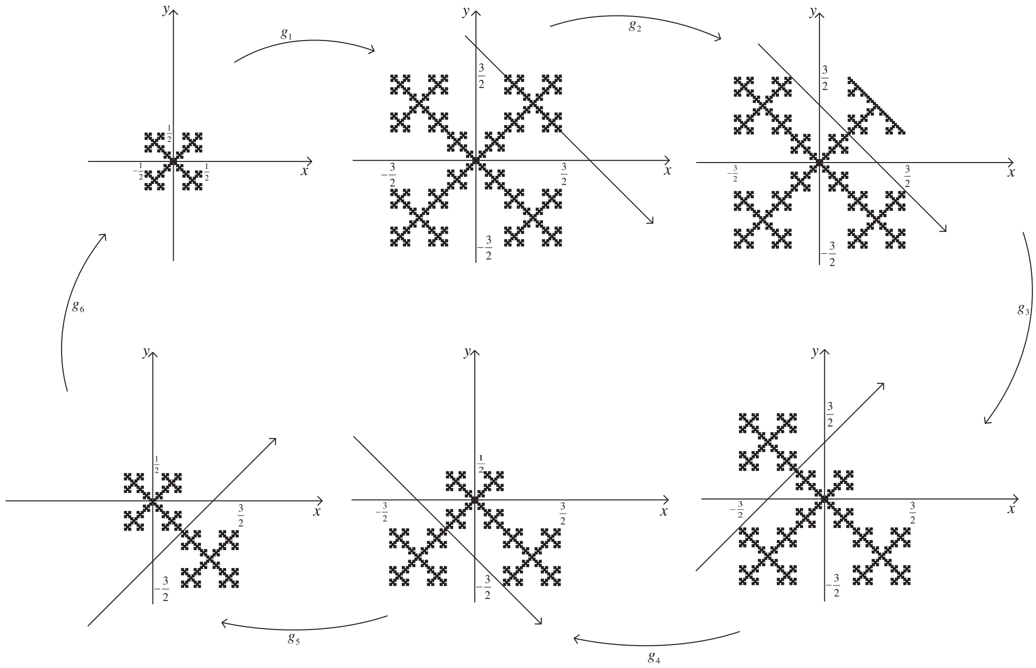

Figure 3: The effect of G on the Box fractal

We define the last dynamical system on the Box fractal $\{B, T\}$ with the following mappings $t_i : \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 1, 2, 3, 4, 5, 6, 7, 8, 9)$

$$
\begin{aligned}
t_1(x,y) &= (3x, 3y),\\
t_2(x,y) &= \left(-\frac{1}{2}\left|x+y-2\right| + \frac{1}{2}(x-y-2)+2, -\frac{1}{2}\left|x+y-2\right| - \frac{1}{2}(x-y-2)\right),\\
t_3(x,y) &= \left(-\frac{1}{2}\left|x+y-1\right| + \frac{1}{2}(x-y-1)+1, -\frac{1}{2}\left|x+y-1\right| - \frac{1}{2}(x-y-1)\right),\\
t_4(x,y) &= \left(\frac{1}{2}\left|x-y+2\right| + \frac{1}{2}(x+y+2)-2, -\frac{1}{2}\left|x-y+2\right| + \frac{1}{2}(x+y+2)\right),\\
t_5(x,y) &= \left(\frac{1}{2}\left|x-y+1\right| + \frac{1}{2}(x+y+1)-1, -\frac{1}{2}\left|x-y+1\right| + \frac{1}{2}(x+y+1)\right),\\
t_6(x,y) &= \left(-\frac{1}{2}\left|x-y-2\right| + \frac{1}{2}(x+y-2)+2, \frac{1}{2}\left|x-y-2\right| + \frac{1}{2}(x+y-2)\right),\\
t_7(x,y) &= \left(-\frac{1}{2}\left|x-y-1\right| + \frac{1}{2}(x+y-1)+1, \frac{1}{2}\left|x-y-1\right| + \frac{1}{2}(x+y-1)\right),\\
t_8(x,y) &= \left(\frac{1}{2}\left|x+y+2\right| - \frac{1}{2}(-x+y-2)-2, \frac{1}{2}\left|x+y+2\right| + \frac{1}{2}(-x+y-2)\right),\\
t_9(x,y) &= \left(\frac{1}{2}\left|x+y+1\right| + \frac{1}{2}(x-y+1)-1, \frac{1}{2}\left|x+y+1\right| - \frac{1}{2}(x-y+1)\right).
\end{aligned}
$$

such that

$$T = t_9 \circ t_8 \circ t_7 \circ t_6 \circ t_5 \circ t_4 \circ t_3 \circ t_2 \circ t_1. \tag{3}$$

T has 3 more different mappings from G, which are folding transformations t_4, t_6 and t_8. Here, t_4 moves the points above the line $y = x + 2$ to the below side. t_6 and t_8 move the points below the lines $y = x - 2$ and $y = -x - 2$ to the above side respectively.

Since using the composition functions F, G and T require too many extra and compelling processes, we state these dynamical systems via the addresses of the points on B in Proposition 2.1, Proposition 2.3 and Proposition 2.4 given below.

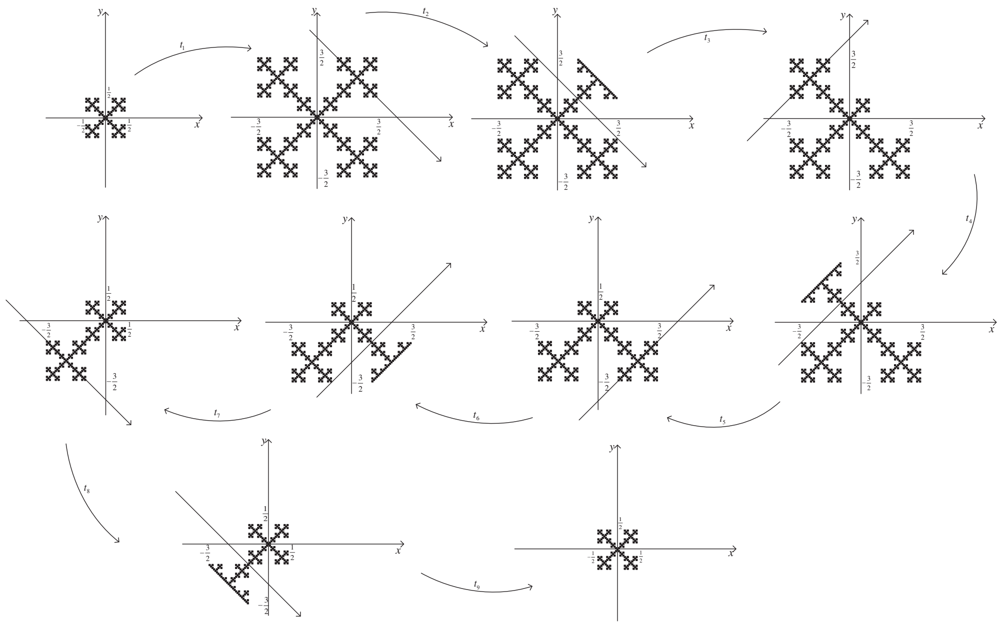

Figure 4: The dynamical system $\{B; T\}$ on the Box fractal

Proposition 2.1. *Let the points $X, Y \in B$ be represented by $x_1x_2x_3\ldots$ and $y_1y_2y_3\ldots$ respectively where $x_i, y_i \in \{0, 1, 2, 3, 4\}$ for $i = 1, 2, 3, \ldots$. Then the function $F : B \to B$ defined in (1) is stated as:*
if $x_1 = 0$, then

$$F(x_1x_2x_3\ldots) = x_2x_3x_4\ldots$$

if $x_1 = 1$ or $x_1 = 3$, then

$$F(x_1x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 1, & x_{i+1} = 3 \\ 2, & x_{i+1} = 2 \\ 3, & x_{i+1} = 1 \\ 4, & x_{i+1} = 4 \end{cases} \quad (i \geq 1)$$

if $x_1 = 2$ or $x_1 = 4$, then

$$F(x_1x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 1, & x_{i+1} = 1 \\ 2, & x_{i+1} = 4 \\ 3, & x_{i+1} = 3 \\ 4, & x_{i+1} = 2 \end{cases} \quad (i \geq 1)$$

which implies $\{B; F\}$ *is a dynamical system.*

Proof. We want to show that F, defined by via representations, is well-defined on B. If $X \in B$ has a unique address, then $F(X)$ is also stated by a unique address. If X is represented by $\sigma\omega\omega'\omega'\omega'\ldots$ or $\sigma 0\omega\omega\omega\ldots$ where $\sigma = x_1x_2x_3\ldots x_n$, then $F(X)$ can also be stated by two different code representations. Therefore, we investigate the images of the points, whose code representations are one of the $0\overline{1}, 0\overline{2}, 0\overline{3}, 0\overline{4}, 1\overline{3}, 2\overline{4}, 3\overline{1}, 4\overline{2}$, $00\overline{1}, 00\overline{2}, 00\overline{3}, 00\overline{4}, 01\overline{3}, 02\overline{4}, 03\overline{1}, 04\overline{2}, 10\overline{1}, 10\overline{2}, 10\overline{3}, 10\overline{4}, 11\overline{3}, 12\overline{4}, 13\overline{1}, 14\overline{2}, 20\overline{1}, 20\overline{2}, 20\overline{3}, 20\overline{4}, 21\overline{3}, 22\overline{4}, 23\overline{1}, 24\overline{2}$, $30\overline{1}, 30\overline{2}, 30\overline{3}, 30\overline{4}, 31\overline{3}, 32\overline{4}, 33\overline{1}, 34\overline{2}, 40\overline{1}, 40\overline{2}, 40\overline{3}, 40\overline{4}, 41\overline{3}, 42\overline{4}, 43\overline{1}, 44\overline{2}$.

Under the function F, the images of these points are obtained as

$$\begin{array}{llll} F(0\overline{1}) = \overline{1}, & F(0\overline{2}) = \overline{2}, & F(0\overline{3}) = \overline{3}, & F(0\overline{4}) = \overline{4}, \\ F(1\overline{3}) = \overline{1}, & F(2\overline{4}) = \overline{2}, & F(3\overline{1}) = \overline{3}, & F(4\overline{2}) = \overline{4}, \end{array}$$

$$\begin{array}{llll} F(00\overline{1}) = 0\overline{1}, & F(00\overline{2}) = 0\overline{2}, & F(00\overline{3}) = 0\overline{3}, & F(00\overline{4}) = 0\overline{4}, \\ F(01\overline{3}) = 1\overline{3}, & F(02\overline{4}) = 2\overline{4}, & F(03\overline{1}) = 3\overline{1}, & F(04\overline{2}) = 4\overline{2}, \\ F(10\overline{1}) = 0\overline{3}, & F(10\overline{2}) = 0\overline{2}, & F(10\overline{3}) = 0\overline{1}, & F(10\overline{4}) = 0\overline{4}, \\ F(11\overline{3}) = 3\overline{1}, & F(12\overline{4}) = 2\overline{4}, & F(13\overline{1}) = 1\overline{3}, & F(14\overline{2}) = 4\overline{2}, \\ F(20\overline{1}) = 0\overline{1}, & F(20\overline{2}) = 0\overline{4}, & F(20\overline{3}) = 0\overline{3}, & F(20\overline{4}) = 0\overline{2}, \\ F(21\overline{3}) = 1\overline{3}, & F(22\overline{4}) = 4\overline{2}, & F(23\overline{1}) = 3\overline{1}, & F(24\overline{2}) = 2\overline{4}, \\ F(30\overline{1}) = 0\overline{3}, & F(30\overline{2}) = 0\overline{2}, & F(30\overline{3}) = 0\overline{1}, & F(30\overline{4}) = 0\overline{4}, \\ F(31\overline{3}) = 3\overline{1}, & F(32\overline{4}) = 2\overline{4}, & F(33\overline{1}) = 1\overline{3}, & F(34\overline{2}) = 4\overline{2}, \\ F(40\overline{1}) = 0\overline{1}, & F(40\overline{2}) = 0\overline{4}, & F(40\overline{3}) = 0\overline{3}, & F(40\overline{4}) = 0\overline{2}, \\ F(41\overline{3}) = 1\overline{3}, & F(42\overline{4}) = 4\overline{2}, & F(43\overline{1}) = 3\overline{1}, & F(44\overline{2}) = 2\overline{4}. \end{array}$$

As a result, we can easily conclude that for any points in the form $x_1x_2x_3\ldots x_n0\overline{1}$ and $x_1x_2x_3\ldots x_n1\overline{3}$, $x_1x_2x_3\ldots x_n0\overline{2}$ and $x_1x_2x_3\ldots x_n2\overline{4}$, $x_1x_2x_3\ldots x_n0\overline{3}$ and $x_1x_2x_3\ldots x_n3\overline{1}$, $x_1x_2x_3\ldots x_n0\overline{4}$ and $x_1x_2x_3\ldots x_n4\overline{2}$, under F correspond to the same points on B respectively. □

Remark 2.2. By changing the order of the folding maps in F, it is possible to get new topologically equivalent dynamical systems via the conjugacy map $I(x)$. That is why, the order of these folding mappings does not affect the function F for this structure, i.e. $f_2 \circ f_3 = f_3 \circ f_2$. Therefore, there is an identity map as a conjugacy $H = I$ between F and F' such that $H(F(X)) = F'(H(X))$ where

$$F' = f_5 \circ f_4 \circ f_2 \circ f_3 \circ f_1.$$

Proposition 2.3. *Let the points* $X, Y \in B$ *be* $x_1x_2x_3\ldots$ *and* $y_1y_2y_3\ldots$ *respectively, where* $x_i, y_i \in \{0,1,2,3,4\}$ *for* $i = 1,2,3,\ldots$. *Then the function* $G : B \to B$ *defined in (2) is stated as:*

if $x_1 = 0$, *then*

$$G(x_1x_2x_3\ldots) = x_2x_3x_4\ldots.$$

if $x_1 = 2$ *or* $x_1 = 4$, *then*

$$G(x_1x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 1, & x_{i+1} = 1 \\ 4, & x_{i+1} = 2 \quad (i \geq 1) \\ 3, & x_{i+1} = 3 \\ 2, & x_{i+1} = 4 \end{cases}$$

if $x_1 = 3$, *then*

$$G(x_1x_2x_3\dots) = y_1y_2y_3\dots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 3, & x_{i+1} = 1 \\ 2, & x_{i+1} = 2 \\ 1, & x_{i+1} = 3 \\ 4, & x_{i+1} = 4 \end{cases} \quad (i \geq 1)$$

if $x_1 = 1$, $x_i = 0,2,4$ $(i = 2,3,\dots,k)$ *and* $x_{k+1} = 1$ *or* $x_1 = x_2 = 1$, *then*

$$G(x_1x_2\dots x_kx_{k+1}x_{k+2}x_{k+3}\dots) = x_2x_3x_4\dots,$$

if $x_1 = 1$, $x_2 = 3$ *or* $x_1 = 1$, $x_i = 0,2,4$ $(i = 2,3,\dots,k)$ *and* $x_{k+1} = 3$, *then*

$$G(x_1x_2x_3\dots) = y_1y_2y_3\dots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 3, & x_{i+1} = 1 \\ 2, & x_{i+1} = 2 \\ 1, & x_{i+1} = 3 \\ 4, & x_{i+1} = 4 \end{cases} \quad (i \geq 1),$$

which implies $\{B; G\}$ *is a dynamical system.*

Proof. If $X \in B$ is represented by a unique address, then there is only one code representation of $G(X)$. If X is stated by two different addresses, then the code representations of $G(X)$ must indicate the same point. In order to show that the function G is well-defined on the code set of B, we check the image of the points which is represented by two different code representations. Under the function G, the images of these points are

$$\begin{array}{llll} G(0\overline{1}) = \overline{1}, & G(0\overline{2}) = \overline{2}, & G(0\overline{3}) = \overline{3}, & G(0\overline{4}) = \overline{4}, \\ G(1\overline{3}) = \overline{1}, & G(2\overline{4}) = \overline{2}, & G(3\overline{1}) = \overline{3}, & G(4\overline{2}) = \overline{4}, \end{array}$$

$$\begin{array}{llll} G(00\overline{1}) = 0\overline{1}, & G(00\overline{2}) = 0\overline{2}, & G(00\overline{3}) = 0\overline{3}, & G(00\overline{4}) = 0\overline{4}, \\ G(01\overline{3}) = 1\overline{3}, & G(02\overline{4}) = 2\overline{4}, & G(03\overline{1}) = 3\overline{1}, & G(04\overline{2}) = 4\overline{2}, \\ \\ G(10\overline{1}) = 0\overline{1}, & G(10\overline{2}) = 0\overline{2}, & G(10\overline{3}) = 0\overline{1}, & G(10\overline{4}) = 0\overline{4}, \\ G(11\overline{3}) = 1\overline{3}, & G(12\overline{4}) = 2\overline{4}, & G(13\overline{1}) = 1\overline{3}, & G(14\overline{2}) = 4\overline{2}, \\ \\ G(20\overline{1}) = 0\overline{1}, & G(20\overline{2}) = 0\overline{4}, & G(20\overline{3}) = 0\overline{3}, & G(20\overline{4}) = 0\overline{2}, \\ G(21\overline{3}) = 1\overline{3}, & G(22\overline{4}) = 4\overline{2}, & G(23\overline{1}) = 3\overline{1}, & G(24\overline{2}) = 2\overline{4}, \\ \\ G(30\overline{1}) = 0\overline{3}, & G(30\overline{2}) = 0\overline{2}, & G(30\overline{3}) = 0\overline{1}, & G(30\overline{4}) = 0\overline{4}, \\ G(31\overline{3}) = 3\overline{1}, & G(32\overline{4}) = 2\overline{4}, & G(33\overline{1}) = 1\overline{3}, & G(34\overline{2}) = 4\overline{2}, \\ \\ G(40\overline{1}) = 0\overline{1}, & G(40\overline{2}) = 0\overline{4}, & G(40\overline{3}) = 0\overline{3}, & G(40\overline{4}) = 0\overline{2}, \\ G(41\overline{3}) = 1\overline{3}, & G(42\overline{4}) = 4\overline{2}, & G(43\overline{1}) = 3\overline{1}, & G(44\overline{2}) = 2\overline{4}. \end{array}$$

In general, it is seen that the images of the points, which are represented by both $x_1x_2x_3\dots x_n0\overline{1}$ and $x_1x_2x_3\dots x_n1\overline{3}$, $x_1x_2x_3\dots x_n0\overline{2}$ and $x_1x_2x_3\dots x_n2\overline{4}$, $x_1x_2x_3\dots x_n0\overline{3}$ and $x_1x_2x_3\dots x_n3\overline{1}$, $x_1x_2x_3\dots x_n0\overline{4}$ and $x_1x_2x_3\dots x_n4\overline{2}$, indicate the same points on B respectively. □

Proposition 2.4. *Let the points $X, Y \in B$ be represented by $x_1x_2x_3\ldots$ and $y_1y_2y_3\ldots$ respectively, where $x_i, y_i \in \{0,1,2,3,4\}$ for $i = 1,2,3,\ldots$. Then the function $T : B \to B$ defined in (3) is stated as:*

if $x_1 = 0$, then

$$T(x_1x_2x_3\ldots) = x_2x_3x_4\ldots,$$

if $x_1 = 1$, $x_i = 0,2,4$ $(i = 2,3,\ldots,k)$ and $x_{k+1} = 1$ or $x_1 = x_2 = 1$, then

$$T(x_1x_2\ldots x_kx_{k+1}x_{k+2}x_{k+3}\ldots) = x_2x_3x_4\ldots,$$

if $x_1 = 1$, $x_i = 0,2,4$ $(i = 2,3,\ldots,k)$ and $x_{k+1} = 3$ or $x_1 = 1$, $x_2 = 3$, then

$$T(x_1x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 3, & x_{i+1} = 1 \\ 2, & x_{i+1} = 2 \quad (i \geq 1) \\ 1, & x_{i+1} = 3 \\ 4, & x_{i+1} = 4 \end{cases}$$

if $x_1 = 2$, $x_i = 0,1,3$ $(i = 2,3,\ldots,k)$ and $x_{k+1} = 2$ or $x_1 = x_2 = 2$, then

$$T(2x_2\ldots x_kx_{k+1}x_{k+2}x_{k+3}\ldots) = x_2x_3x_4\ldots,$$

if $x_1 = 2$, $x_i = 0,1,3$ $(i = 2,3,\ldots,k)$ and $x_{k+1} = 4$ or $x_1 = 2$, $x_2 = 4$, then

$$T(x_1x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 1, & x_{i+1} = 1 \\ 4, & x_{i+1} = 2 \quad (i \geq 1) \\ 3, & x_{i+1} = 3 \\ 2, & x_{i+1} = 4 \end{cases}$$

if $x_1 = 3$, $x_i = 0,2,4$ $(i = 2,3,\ldots,k)$ and $x_{k+1} = 3$ or $x_1 = x_2 = 3$, then

$$T(3x_2\ldots x_kx_{k+1}x_{k+2}x_{k+3}\ldots) = x_2x_3x_4\ldots$$

if $x_1 = 3$, $x_i = 0,2,4$ $(i = 2,3,\ldots,k)$ and $x_{k+1} = 1$ or $x_1 = 3$, $x_2 = 1$, then

$$T(x_1x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 3, & x_{i+1} = 1 \\ 2, & x_{i+1} = 2 \quad (i \geq 1) \\ 1, & x_{i+1} = 3 \\ 4, & x_{i+1} = 4 \end{cases}$$

if $x_1 = 4$, $x_i = 0,1,3$ $(i = 2,3,\ldots,k)$ and $x_{k+1} = 4$ or $x_1 = x_2 = 4$, then

$$T(4x_2\ldots x_kx_{k+1}x_{k+2}x_{k+3}\ldots) = x_2x_3x_4\ldots$$

if $x_1 = 4$, $x_i = 0,1,3$ $(i = 2,3,\ldots,k)$ and $x_{k+1} = 2$ or $x_1 = 4$, $x_2 = 2$, then

$$T(x_1x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 1, & x_{i+1} = 1 \\ 4, & x_{i+1} = 2 \quad (i \geq 1), \\ 3, & x_{i+1} = 3 \\ 2, & x_{i+1} = 4 \end{cases}$$

which implies $\{B; T\}$ is a dynamical system.

Proof. In order to show that T is well defined on B, we must to check the images of the points, which have two code representations. The form of these points are $\sigma\omega\omega'\omega'\omega'\ldots$ and $\sigma 0\omega\omega\omega\ldots$ where $\sigma = x_1x_2x_3\ldots x_n$. Therefore, we must first examine that the images of the following points indicate the same points on B :

$$\begin{array}{llll} T(0\overline{1}) = \overline{1}, & T(0\overline{2}) = \overline{2}, & T(0\overline{3}) = \overline{3}, & T(0\overline{4}) = \overline{4}, \\ T(1\overline{3}) = \overline{1}, & T(2\overline{4}) = \overline{2}, & T(3\overline{1}) = \overline{3}, & T(4\overline{2}) = \overline{4}, \end{array}$$

$$\begin{array}{llll} T(00\overline{1}) = 0\overline{1}, & T(00\overline{2}) = 0\overline{2}, & T(00\overline{3}) = 0\overline{3}, & T(00\overline{4}) = 0\overline{4}, \\ T(01\overline{3}) = 1\overline{3}, & T(02\overline{4}) = 2\overline{4}, & T(03\overline{1}) = 3\overline{1}, & T(04\overline{2}) = 4\overline{2}, \end{array}$$

$$\begin{array}{llll} T(10\overline{1}) = 0\overline{1}, & T(10\overline{2}) = 0\overline{2}, & T(10\overline{3}) = 0\overline{1}, & T(10\overline{4}) = 0\overline{4}, \\ T(11\overline{3}) = 1\overline{3}, & T(12\overline{4}) = 2\overline{4}, & T(13\overline{1}) = 1\overline{3}, & T(14\overline{2}) = 4\overline{2}, \end{array}$$

$$\begin{array}{llll} T(20\overline{1}) = 0\overline{1}, & T(20\overline{2}) = 0\overline{2}, & T(20\overline{3}) = 0\overline{3}, & T(20\overline{4}) = 0\overline{2}, \\ T(21\overline{3}) = 1\overline{3}, & T(22\overline{4}) = 2\overline{4}, & T(23\overline{1}) = 3\overline{1}, & T(24\overline{2}) = 2\overline{4}, \end{array}$$

$$\begin{array}{llll} T(30\overline{1}) = 0\overline{3}, & T(30\overline{2}) = 0\overline{2}, & T(30\overline{3}) = 0\overline{3}, & T(30\overline{4}) = 0\overline{4}, \\ T(31\overline{3}) = 3\overline{1}, & T(32\overline{4}) = 2\overline{4}, & T(33\overline{1}) = 3\overline{1}, & T(34\overline{2}) = 4\overline{2}, \end{array}$$

$$\begin{array}{llll} T(40\overline{1}) = 0\overline{1}, & T(40\overline{2}) = 0\overline{4}, & T(40\overline{3}) = 0\overline{3}, & T(40\overline{4}) = 0\overline{4}, \\ T(41\overline{3}) = 1\overline{3}, & T(42\overline{4}) = 4\overline{2}, & T(43\overline{1}) = 3\overline{1}, & T(44\overline{2}) = 4\overline{2}. \end{array}$$

Thus, we conclude that both $T(\sigma\omega\omega'\omega'\omega'\ldots)$ and $T(\sigma 0\omega\omega\omega\ldots)$, where $\sigma = x_1x_2x_3\ldots x_n$ are the different code representations of the same points on B respectively. □

3. Investigation of the topological equivalence of some dynamical systems on the Box fractal

In this section, we compute fixed points and some periodic points of F, G and T using the Proposition 2.1, Proposition 2.3 and Proposition 2.4. The number of periodic points can guide in deciding whether the dynamical systems are not topologically conjugate. Moreover, we give an example of topologically equivalent dynamical system with $\{B; G\}$ in Proposition 3.4.

3.1. Computing the periodic points

Thanks to the expression of $\{S; F\}$ in Proposition 2.1, one can find the periodic points of this dynamical system. The points satisfies the equation

$$F(x_1x_2x_3\ldots x_k\ldots) = x_1x_2x_3\ldots x_k\ldots.$$

are found as follows

- $\overline{0} = 00000\ldots$
- $\overline{13} = 131313\ldots$
- $\overline{24} = 242424\ldots$
- $\overline{31} = 313131\ldots$
- $\overline{42} = 424242\ldots.$

We now compute one of 2−periodic points of F. Suppose that $x_1 = 2$ and $x_2 = 1$, if we solve the following equation as

$$F^2(21x_3x_4\ldots x_k\ldots) = 21x_3x_4\ldots x_k\ldots,$$

then it is obtained that $214321432143\ldots$ is a 2−periodic point of F. The other periodic points can be computed in a similar fashion.

Furthermore, we find the periodic points of $\{B; G\}$ by the help of Proposition 2.3 and it is found that there are 6 fixed points;

$\bullet\overline{0} = 00000\ldots$ $\bullet\overline{1} = 11111\ldots$
$\bullet\overline{24} = 242424\ldots$ $\bullet\overline{31} = 313131\ldots$
$\bullet\overline{42} = 424242\ldots$ $\bullet\overline{13} = 131313\ldots.$

Finally, it is seen that the number of fixed points of $\{B; T\}$ is 9, which are found as follows;

$\bullet\overline{0} = 00000\ldots$ $\bullet\overline{1} = 11111\ldots$
$\bullet\overline{13} = 131313\ldots$ $\bullet\overline{2} = 2222\ldots$
$\bullet\overline{24} = 242424\ldots$ $\bullet\overline{3} = 3333\ldots$
$\bullet\overline{31} = 313131\ldots$ $\bullet\overline{4} = 4444\ldots$
$\bullet\overline{42} = 424242\ldots$

As a result, we have 3 dynamical systems, which are constructed by the different number of folding mappings. As can be seen from the above, these dynamical systems have different number of fixed points. By using the statement " If the dynamical systems $\{X_1; f_1\}$ and $\{X_2; f_2\}$ have the different number of $n-$periodic points for at least $n \in \mathbb{N}$, then they are not topologically conjugate (see [13])", we can deduce the following remark:

Remark 3.1. Since $\{B; F\}$, $\{B; G\}$ and $\{B; T\}$ have the different number of fixed points, they are not topologically conjugate.

Theorem 3.2. *$\{B; F\}$ is chaotic in Devaney sense.*

Proof. In order to prove that F satisfy the Devaney chaos conditions, it is sufficient to prove that F is locally eventually onto (l.e.o.) (see [11]) and the periodic points of F are dense.

Let U be open subset of (B, d_{box}). We must obtain $n \in \mathbb{N}$ satisfying $F^n(U) = B$. Since U is an open set, we know that there is $k \in \mathbb{N}$ such that $B(X, \frac{\sqrt{2}}{3^{k-1}}) \subseteq U$, where the adress of $X \in B$ is $x_1x_2\ldots x_{k-1}x_kx_{k+1}\ldots$. Moreover, it may easily be shown that

$$U' = \{x_1x_2x_3\ldots x_kz_{k+1}z_{k+2}z_{k+3}\ldots | x_1, \ldots, x_k \text{ are the first k-terms of X}\}$$

where $z_i \in \{0, 1, 2, 3, 4\}$ for $i = k+1, k+2, k+3, \ldots,$ is the subset of $B(X, \frac{1}{3^{k-1}})$. Since we get $F^k(U') = B$, it obviously means $F^k(U) = B$. Consequently, F satisfies the l.e.o property, that means it is topologically transitive and sensitivity to initial conditions.

To prove that the periodic points of F are dense on B, we must show the existence of a periodic point which is sufficiently close to any points of B. We take the point A with the code representation $a_1a_2a_3\ldots$ and the open set $B(A, \frac{\sqrt{2}}{3^{k-1}})$ and the sets U, U' which are defined above. Since z_i's are arbitrary for all $i = k+1, k+2, k+3, \ldots,$, then we can obtain

$$F^k(a_1a_2\ldots a_kz_{k+1}z_{k+2}z_{k+3}\ldots) = z'_{k+1}z'_{k+2}z'_{k+3}\ldots = a_1a_2\ldots a_kz_{k+1}z_{k+2}z_{k+3}\ldots.$$

This completes the proof. □

Remark 3.3. By following the similar lines of the proof of Theorem 3.2, it can be proven that $\{B; G\}$ and $\{B; T\}$ are chaotic in the sense of Devaney.

3.2. The construction of a topologically equivalent dynamical system with $\{B; G\}$

Now, we also give an example about how to define a topologically conjugate dynamical system on Box fractal by using similar types of folding mappings. By replacing one of the folding mappings g_2 with g'_2 at the same level of the Box fractal, we get a new dynamical system which is topologically equivalent to $\{B, G\}$

and we find a conjugacy map between these dynamical systems. Here, g_2' is a folding mapping that moves the points from upper hand side of the line $y = x + 2$ to the lower hand side. The point is that changing the order of related folding mappings does not affect the function for this structure, we thus construct the topologically conjugate dynamical system $\{B, G'\}$ as follows

$$G' = g_6 \circ g_5 \circ g_3 \circ g_4 \circ g_2' \circ g_1. \tag{4}$$

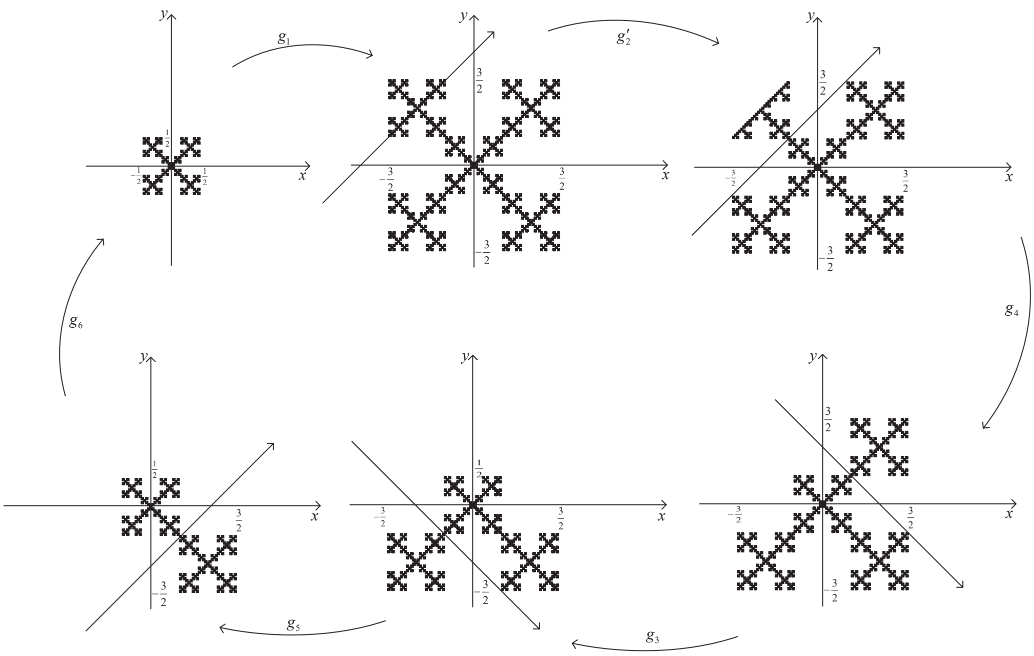

Figure 5: The effect of G' on the Box fractal

We express this dynamical system with the following proposition using the code representations of the points on B.

Proposition 3.4. *Let the points $X, Y \in B$ be $x_1x_2x_3\ldots$ and $y_1y_2y_3\ldots$ respectively, where $x_i, y_i \in \{0, 1, 2, 3, 4\}$ for $i = 1, 2, 3, \ldots$. Then the function $G' : B \to B$ given in (4) is stated as:*
if $x_1 = 0$, then

$$G'(x_1x_2x_3\ldots) = x_2x_3x_4\ldots.$$

if $x_1 = 1$ or $x_1 = 3$, then

$$G'(x_1x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 3, & x_{i+1} = 1 \\ 2, & x_{i+1} = 2 \quad (i \geq 1) \\ 1, & x_{i+1} = 3 \\ 4, & x_{i+1} = 4 \end{cases}$$

if $x_1 = 4$, *then*

$$G'(x_1x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 1, & x_{i+1} = 1 \\ 4, & x_{i+1} = 2 \quad (i \geq 1) \\ 3, & x_{i+1} = 3 \\ 2, & x_{i+1} = 4 \end{cases}$$

if $x_1 = 2$, $x_i = 0, 1, 3$ $(i = 2, 3, \ldots, k)$ *and* $x_{k+1} = 2$ *or* $x_1 = x_2 = 2$, *then*

$$G'(x_1x_2\ldots x_kx_{k+1}x_{k+2}x_{k+3}\ldots) = x_2x_3x_4\ldots,$$

if $x_1 = 2$, $x_2 = 4$ *or* $x_1 = 2$, $x_i = 0, 1, 3$ $(i = 2, 3, \ldots, k)$ *and* $x_{k+1} = 4$, *then*

$$G'(x_1x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 1, & x_{i+1} = 1 \\ 4, & x_{i+1} = 2 \quad (i \geq 1), \\ 3, & x_{i+1} = 3 \\ 2, & x_{i+1} = 4 \end{cases}$$

which means $\{B; G'\}$ *is a dynamical system.*

Lemma 3.5. *Let the code representations of* $X, Y \in B$ *be* $x_1x_2x_3\ldots$ *and* $y_1y_2y_3\ldots$ *respectively, where* $x_i, y_i \in \{0, 1, 2, 3, 4\}$ *for* $i = 1, 2, 3, \ldots$. *Then for all* $X \in B$ *there exists* $H : B \to B$ *such that*

$$H(X) = Y, \quad y_i = \begin{cases} 0, & x_i = 0 \\ 2, & x_i = 1 \\ 1, & x_i = 2 \quad (i \geq 1) \\ 4, & x_i = 3 \\ 3, & x_i = 4 \end{cases} \tag{5}$$

satisfying $H(G(X)) = G'(H(X))$. *Here* H *is called a conjugacy.*

Proof. One can easily check this equality from the definition of G and G'. □

Remark 3.6. It is clear that for all $X, X' \in B$, $d(H(X), H(X')) = d(X, X')$. Moreover, H is surjective. Thus, we conclude that H is a homeomorphism.

Remark 3.7. The dynamical systems $\{B; G\}$ and $\{B; G'\}$ are topologically conjugate because there is a homeomorphism, defined in (5) which satisfies $H(G(X)) = G'(H(X))$ for all $X \in B$. Moreover, $\{B; G'\}$ is chaotic, since $\{B; G\}$ is chaotic, G' is continuous and B is compact (for details, see [8, 13]).

By using the conjugacy H, the periodic points of $\{B; G'\}$ can be easily calculated as long as the periodic points of $\{B; G\}$ are known. Hence, the fixed points of $\{B; G'\}$ are computed as

•$H(\overline{0}) = 00000\ldots$ •$H(\overline{1}) = 2222\ldots$
•$H(\overline{24}) = 131313\ldots$ •$H(\overline{31}) = 424242\ldots$
•$H(\overline{42}) = 313131\ldots$ •$H(\overline{13}) = 242424\ldots$.

Based on the example above, analogously, many dynamical systems which are topologically conjugate to $\{B; G\}$ can be defined by choosing the similar folding mappings instead of g_2.

4. Concluding remarks

We can conclude that using the same number of similar folding mappings at the same levels of Box fractal can lead to the obtaining topologically conjugate dynamical systems. However, any generalization of the derivation of topologically conjugate dynamical systems does not seem possible. The reason for this is that dynamical systems can be constructed with not only through expanding and folding mappings but also through various other ways, such as symmetry groups or rotation and translation mappings etc. The main idea is that in order to have topologically equivalent dynamical systems, there must exist a conjugacy map between these systems. Having the same number of $n-$periodic points is necessary but not sufficient condition for equivalent systems. In other words, if you create any dynamical system with a different number of fixed points, it is certain that these systems are not topologically conjugate. Therefore, to build a dynamical system that may or may not be equivalent to current system, we need to carefully observe which transformations lead to different number of fixed points (or n-periodic points) or a homeomorphism between the systems. In this paper, we exemplify these situations for dynamical systems on the Box fractal by using folding mappings. This study provides a guidance on which folding mappings should be used when obtaining topologically equivalent or non-equivalent dynamical systems on a Box fractal or even on a self-similar sets.

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