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A note on approximation of nonlinear Baskakov operators based on q-integers

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Abstract. In this study, we give a specific family of q-integer-based max-product type approximation operators having the property of pseudo-linearity, a weaker term than classical linearity. We propose a further improvement in max-product type operators which is based on q-integers. Firstly, we construct a new kind of nonlinear Baskakov operators. For construction we use the linear q-Baskakov polynomials and also the max-product algebra. Then we give an error estimation for the q-Baskakov operators of max-product kind. Also, an approximate statistical theorem is presented.

1. Introduction

In this section, it is emphasized some general notations about the max-product kind operators. Over the set of real positive numbers, $\mathbb{R}_+$, we deal with the operations $\bigvee$ (maximum) and $\cdot$ (product). Consequently, $(\mathbb{R}_+, \bigvee, \cdot)$ has a semiring structure and it is called as Max-Product algebra. Take the interval $\mathbb{E} \subset \mathbb{R}$ which is a bounded or unbounded, and describe the space of the function h as follows

$$CB_+(\mathbb{E}) = \{h : \mathbb{E} \to \mathbb{R}_+; h \text{ continuous and bounded on } \mathbb{E}\}.$$

An approximate operator of the discrete max-product type $L_n : CB_+(\mathbb{E}) \to CB_+(\mathbb{E})$, has a general form

$$L_n(h)(x) = \bigvee_{i=0}^{n} K_n(x, x_i) \cdot h(x_i),$$

or

$$L_n(h)(x) = \bigvee_{i=0}^{\infty} K_n(x, x_i) \cdot h(x_i)$$

where $n \in \mathbb{N}$, $h \in CB_+(\mathbb{E})$, $K_n(\cdot, x_i) \in CB_+(\mathbb{E})$ and $x_i \in \mathbb{E}$, for all $i = \{0, 1, 2, \cdots\}$. These are positive nonlinear operators which satisfy a pseudo-linearity condition of the type

$$L_n(\alpha \cdot h \vee \beta \cdot g)(x) = \alpha \cdot L_n(h)(x) \vee \beta \cdot L_n(g)(x), \forall \alpha, \beta \in \mathbb{R}_+, h, g : \mathbb{E} \to \mathbb{R}_+.$$

In order to give some properties of the operators L_n, we present the following auxiliary Lemma.

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Lemma 1.1. ([4]) *Let $\mathbb{E} \subset \mathbb{R}$ be a bounded or unbounded interval,*

$$CB_+(\mathbb{E}) = \{h : \mathbb{E} \to \mathbb{R}_+ : h \text{ continuous and bounded on } \mathbb{E}\}$$

and $L_n : CB_+(\mathbb{E}) \to CB_+(\mathbb{E})$, $n \in \mathbb{N}$ be a sequence of positive homogenous operators satisfying, in addition, the following properties:

i. (Monotoncity) if $h, g \in CB_+(\mathbb{E})$ satisfy $h \leq g$, then $L_n(h) \leq L_n(g)$ for all $n \in \mathbb{N}$;

ii. (Sublinearity) $L_n(h + g) \leq L_n(h) + L_n(g)$ for all $f, g \in CB_+(\mathbb{E})$.

Then for all $h, g \in CB_+(\mathbb{E})$, $n \in \mathbb{N}$ and $x \in \mathbb{E}$ we have

$$|L_n(h)(x) - L_n(g)(x)| \leq L_n(|h - g|)(x).$$

Corollary 1.2. ([4]) *Let $L_n : CB_+(\mathbb{E}) \to CB_+(\mathbb{E})$, $n \in \mathbb{N}$ be a sequence of operators satisfying the requirements (i)-(ii) in Lemma 1.1 and also be a positive homogenous operator. Then we get*

$$|h(x) - L_n(h)(x)| \leq \left[\frac{1}{\delta} L_n(\varphi_x)(x) + L_n(e_0)(x)\right] \omega(h; \delta) + h(x) \cdot |L_n(e_0)(x) - 1|, \ \forall h \in CB_+(\mathbb{E}),\ n \in \mathbb{N},\ x \in \mathbb{E},$$

where $\delta > 0$, $e_0(t) = 1$ for all $t \in \mathbb{E}$, $\varphi_x(t) = |t - x|$ for all $t, x \in \mathbb{E}$. Also,

$$\omega(h; \delta) = \max_{\substack{x,y \in \mathbb{E} \\ |x-y| \leq \delta}} |f(x) - f(y)|$$

is the first modulus of continuity. If $\mathbb{E}$ is unbounded then we assume that there exists $L_n(\varphi_x)(x) \in \mathbb{R}_+ \bigcup \{+\infty\}$, for any $x \in \mathbb{E}, n \in \mathbb{N}$.

Corollary 1.3. ([4]) *Assume, in addition to the qualifications in Corollary 1.2, the sequence $(L_n)_n$ satisfies $L_n(e_0) = e_0$, for all $n \in \mathbb{N}$. Then for all $h \in CB_+(\mathbb{E})$, $n \in \mathbb{N}$ and $x \in \mathbb{E}$ we get*

$$|f(x) - L_n(h)(x)| \leq \left[1 + \frac{1}{\delta} L_n(\varphi_x)(x)\right] \omega(h; \delta).$$

The approximation of a continuous function by a sequence of linear positive operators is fundamental topic in the Korovkin-type approximation theory (see [1], [13]). In the paper [6], nonlinear positive operators in place of linear positive operators has been introduced by Bede et al. They discovered that the nonlinear operators exhibit a similar approximation behavior to the linear operators, despite the fact that the Korovkin theorem fails for these nonlinear operators. In recent years, q-calculus has played an important role in the approximation of functions by a linear positive operator.

Also,the results of convergence are better for q-analogues of approximation operators than for classical ones. Lupas [15] presented q-Bernstein operators and investigated their approximation and shape-preserving properties. In the paper [18], Phillips established the use of q-integers to generalize Bernstein polynomials. Several researchers defined and researched several unique generalizations of linear positive operators based on q and (p, q)-integers ([7], [9], [11], [12], [15]-[18]).

2. Construction of the operators

In [2], Baskakov introduced the positive and linear operators, which are typically associated to functions that are bounded and uniformly continuous to $f \in C[0, +\infty)$ and specified by

$$V_n(f)(x) = (1 + x)^{-n} \sum_{k=0}^{\infty} \binom{n + k - 1}{k} x^k (1 + x)^{-k} f\left(\frac{k}{n}\right), \ \forall n \in \mathbb{N}.$$

It is known that the following pointwise approximation result exists as:

$$| V_n(f)(x) - f(x) | \leq C\omega_2^{\varphi}\left(f; \sqrt{x(1+x)/n}\right), \ x \in [0,\infty), n \in \mathbb{N},$$

where $\varphi(x) = \sqrt{x(1+x)}$ and $I = [0,\infty)$. In this case, $I_h = [h^2/(1-h)^2, +\infty)$, $h \leq \delta < 1$. Moreover, the function f on $[0,+\infty)$ is preserved monotonically and convexly by $V_n(f)$ (see [14]).

The truncated Baskakov operators are identified by

$$U_n(f)(x) = (1+x)^{-n}\sum_{k=0}^{n}\binom{n+k-1}{k}x^k(1+x)^{-k}f\left(\frac{k}{n}\right),$$

for $f \in C[0,1]$.

Truncated Baskakov operator of max product kind $f : [0,1] \to \mathbb{R}$ are described by (see [3])

$$U_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} b_{n,k}(x)}, \ x \in [0,1], n \in \mathbb{N}, \ n \geq 1, \tag{1}$$

where $b_{n,k}(x) = \binom{n+k-1}{k}x^k(1+x)^{-n-k}$, $n \geq 1$, $x \in [0,1]$.

In the paper [5], it was showed that the order of uniform approximation in the whole class $C_+([0,1])$ of positive continuous functions on $[0,1]$ cannot be improved, in the sense that there exists a function $f \in C_+([0,1])$, for which the approximation order by the truncated max-product Baskakov operator is $C\omega_1(f, 1/\sqrt{n})$. The fundamentally better order of approximation $\omega_1(f, 1/n)$ was attained for some functional subclasses, such as the nondecreasing concave functions. Finally, some shape preserving properties were proved.

In this section, we modify the Truncated Baskakov operator of max product kind presented in (1) by using q-analysis. For the construction, we mainly use some properties of q- calculus given below:

Some properties of q- calculus

For the parameter $q > 0$ and $n \in \mathbb{N}$, the q-integers of the number n is defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q} & \text{if} \quad q \neq 1 \\ n & \text{if} \quad q = 1 \end{cases}, \quad [0]_q = 0.$$

For $n \in \mathbb{N}$, the q-factorial $[n]_q!$ is defined as follows

$$[n]_q! = [1]_q[2]_q...[n]_q \text{ and } [0]_q! = 1,$$

and for integers $0 \leq k \leq n$, q-binomial coefficient is introduced as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

Finally, let q-binomial coefficient and $1 \leq j \leq n-1$, one get q-Pascal Rules as follows

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q + q^j\begin{bmatrix} n-1 \\ j \end{bmatrix}_q, \ \begin{bmatrix} n \\ j \end{bmatrix}_q = q^{n-j}\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ j \end{bmatrix}_q.$$

Let us also note the following identity:

$$\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q x^k = \frac{1}{(x;q)_n}, \ | x | < 1,$$

where $(x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x)$

Lemma 2.1. ([5]) *Let an arbitrary function* $f:[0,1]\to\mathbb{R}_+$, $U_n^{(M)}(f)(x)$ *is positive, continuous on* $[0,1]$ *and provides* $U_n^{(M)}(f)(0)=f(0)$ *for all* $n\in\mathbb{N}, n\geq 2$.

Also, $U_n^{(M)}(f)(x)$ satisfies all the conditions given in Lemma 1.1.

Now, we identify our operators as follows:

$$U_n^{(M)}(f;x;q)=\frac{\bigvee_{k=0}^{n} b_{n,k}(x;q) f\left(\frac{[k]_q}{[n]_q}\right)}{\bigvee_{k=0}^{n} b_{n,k}(x;q)}$$

where $n\in\mathbb{N}$, $f\in C_+[0,1]$, $x\in[0,1]$, $q\in(0,1)$ and $b_{n,k}(x;q)$ is given by

$$b_{n,k}(x;q)=\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k.(k-1)}{2}} x^k \prod_{s=1}^{n+k}(1+q^{s-1}x)^{-1}.$$

In this case, we consider the empty product to be one. The operators $U_n^{(M)}(f;x;q)$ reduce to the operators $U_n^{(M)}(f;x)$ given by (1), when $q\to 1^-$. Also, $U_n^{(M)}(f;x;q)$ is well-defined.

According the definition of the operator, we get $f\leq g\Rightarrow U_n^{(M)}(f;x;q)\leq U_n^{(M)}(g;x;q)$, for $f,g\in C_+[0,1]$. With regard to $f\in C_+[0,1]$, $U_n^{(M)}(f;x;q)$ is increasing. In addition, we have $U_n^{(M)}(f+g;x;q)\leq U_n^{(M)}(f;x;q)+U_n^{(M)}(g;x;q)$, for any $f,g\in C_+[0,1]$. So the operators $U_n^{(M)}(f;x;q)$ are not linear over $C_+[0,1]$.

Let $\omega(f,\delta)$, $\delta>0$ indicate the modulus of continuity of $f\in C_+[0,1]$, indicated by $\omega(f,\delta)=\max_{|x-y|\leq\delta}\mid f(x)-f(y)\mid$.

3. An error estimation

Firstly,we need some notations and lemmas to estimate $U_n^{(M)}(\varphi_x;x;q)$ with $\varphi_x(t)=\mid t-x\mid$. We will primarily utilize a method similar to [5] in this part, however we must modify every item to the q-calculus. In determining all estimates, it is sufficient to take into account $x\in(0,1]$ since $U_n^{(M)}(f;0;q)-f(0)=0$ for any $f\in C_+[0,1]$.

For each $n\in\mathbb{N}$, $n\geq 2, k\in\{0,1,2,\cdots,n\}$, $j\in\{0,1,2,\cdots,n-2\}$ and $x\in\left[\frac{[j]_q}{[n-1]_q},\frac{[j+1]_q}{[n-1]_q}\right]$, let identify

$$M_{k,n,j}(x;q)=m_{k,n,j}(x;q)\mid\frac{[k]_q}{[n]_q}-x\mid \tag{2}$$

which

$$m_{k,n,j}(x;q)=\frac{b_{n,k}(x;q)}{b_{n,j}(x;q)} \tag{3}$$

for $x\in(0,1]$, $m_{0,n,0}(0;q)=1$ and $m_{k,n,0}(0;q)=0$ for all $k\in\{0,1,2,\cdots,n\}$.

In this case, (2) and (3) imply respectively that if $k\geq j+2$ then

$$M_{k,n,j}(x;q)=m_{k,n,j}(x;q)\left(\frac{[k]_q}{[n]_q}-x\right) \tag{4}$$

and if $k\leq j$ then

$$M_{k,n,j}(x;q)=m_{k,n,j}(x;q)\left(x-\frac{[k]_q}{[n]_q}\right). \tag{5}$$

Now, for each $n \in \mathbb{N}$, $n \geq 2$, $k \in \{0, 1, 2, \cdots, n\}$, $j \in \{0, 1, 2, \cdots, n-2\}$ with $k \geq j+3$ and $x \in \left[\frac{[j]_q}{[n-1]_q}, \frac{[j+1]_q}{[n-1]_q}\right]$, let identify

$$\overline{M}_{k,n,j}(x;q) = m_{k,n,j}(x;q)\left(\frac{[k]_q}{[n-1]_q} - x\right) \tag{6}$$

and also for each $n \in \mathbb{N}$, $n \geq 2$, $k \in \{0, 1, 2, \cdots, n\}$, $j \in \{0, 1, 2, \cdots, n-2\}$ with $k \leq j-1$ and $x \in \left[\frac{[j]_q}{[n-1]_q}, \frac{[j+1]_q}{[n-1]_q}\right]$, let identify

$$\widehat{M}_{k,n,j}(x;q) = m_{k,n,j}(x;q)\left(x - \frac{[k]_q}{[n-1]_q}\right) \tag{7}$$

Lemma 3.1. *Let $q \in (0,1)$, $n \in \mathbb{N}, n \geq 2$ and $x \in \left[\frac{[j]_q}{[n-1]_q}, \frac{[j+1]_q}{[n-1]_q}\right]$. Then we obtain the following inequalities:*

i. For all $k \in \{0, 1, 2, \cdots, n\}$, $j \in \{0, 1, 2, \cdots, n-2\}$ with $k \geq j+3$, we obtain

$$M_{k,n,j}(x;q) \leq \overline{M}_{k,n,j}(x;q) \leq \left(1 + \frac{1}{q^{n+1} + q^{n+1}}\right) M_{k,n,j}(x;q)$$

ii. For all $k \in \{0, 1, 2, \cdots, n\}$, $j \in \{0, 1, 2, \cdots, n-2\}$ with $k \leq j-1$, we obtain

$$\widehat{M}_{k,n,j}(x;q) \leq M_{k,n,j}(x;q) \leq \left(1 + \frac{1}{q^{n-1}}\right) \widehat{M}_{k,n,j}(x;q)$$

Proof. (i) From the equations (4) and (6), we get the inequality $M_{k,n,j}(x;q) \leq \overline{M}_{k,n,j}(x;q)$. Moreover, using the equality $[n+1]_q = [n]_q + q^n$ we obtain

$$\frac{\overline{M}_{k,n,j}(x;q)}{M_{k,n,j}(x;q)} = \frac{\frac{[k]_q}{[n-1]_q} - x}{\frac{[k]_q}{[n]_q} - x} \leq \frac{\frac{[k]_q}{[n-1]_q} - \frac{[j+1]_q}{[n-1]_q}}{\frac{[k]_q}{[n]_q} - \frac{[j+1]_q}{[n-1]_q}} = \frac{[n]_q\left([k]_q - [j+1]_q\right)}{[k]_q[n-1]_q - [j+1]_q[n]_q} = 1 + \frac{q^{n-1}}{[k]_q - [j+1]_q - q^{n-1}}.$$

By using the facts that $k \geq j+3$ and $j \leq n$ we obtain

$$[k]_q - [j+1]_q - q^{n-1} \geq [j+3]_q - [j+1]_q - q^{n-1} \geq q^{j+2} + q^{j+1} \geq q^{n+2} + q^{n+1}.$$

Hence, we get the proof of (i)

$$\frac{\overline{M}_{k,n,j}(x;q)}{M_{k,n,j}(x;q)} \leq 1 + \frac{1}{q^{n+1} + q^{n+1}}.$$

(ii) We can simply estimate $\widehat{M}_{k,n,j}(x;q) \leq M_{k,n,j}(x;q)$ using (5) and (7). Additionally, using again the equality $[n+1]_q = [n]_q + q^n$, we have

$$\frac{M_{k,n,j}(x;q)}{\widehat{M}_{k,n,j}(x;q)} = \frac{x - \frac{[k]_q}{[n]_q}}{x - \frac{[k]_q}{[n-1]_q}} \leq \frac{\frac{[j]_q}{[n-1]_q} - \frac{[k]_q}{[n]_q}}{\frac{[j]_q}{[n-1]_q} - \frac{[k]_q}{[n-1]_q}} = \frac{[j]_q.[n]_q - [k]_q[n-1]_q}{[n]_q\left([j]_q - [k]_q\right)} \leq \frac{[j]_q - [k]_q + q^{n-1}}{[j]_q - [k]_q}$$
$$= 1 + \frac{q^{n-1}}{[j]_q - [k]_q} \leq 1 + \frac{1}{[j]_q - [k]_q}.$$

Since $k \leq j-1$ and $j \leq n$, we obtain $[j]_q - [k]_q \geq [j]_q - [j-1]_q = q^{j-1} \geq q^{n-1}$. Hence, we obtain

$$\frac{M_{k,n,j}(x;q)}{\widehat{M}_{k,n,j}(x;q)} \leq \left(1 + \frac{1}{q^{n-1}}\right)$$

which completes the proof. □

Lemma 3.2. *Let $q \in (0,1)$, $n \in \mathbb{N}, n \geq 2$. Then for all $k \in \{0,1,2,\cdots,n\}$, $j \in \{0,1,2,\cdots,n-2\}$ and $x \in \left[\frac{[j]_q}{[n-1]_q}, \frac{[j+1]_q}{[n-1]_q}\right]$, we have*

$$m_{k,n,j}(x;q) \leq 1.$$

Proof. Let us notice that for $x = 0$ we necessarily have $j = 0$ which implies $m_{0,n,0}(x;q) = 1$ and $m_{k,n,0}(x;q) = 0$ for all $k \in \{1,2,\cdots,n\}$. Now, assume that $x > 0$ when clearly $m_{k,n,j}(x;q) > 0$. We have two possible cases: 1) $k \geq j$ and 2) $k \leq j$.

Case 1) Let $k \geq j$. Since the function $h(x) = \frac{1+q^{n+k}x}{x}$ is nonincreasing on $\left[\frac{[j]_q}{[n-1]_q}, \frac{[j+1]_q}{[n-1]_q}\right]$, it follows

$$\frac{m_{k,n,j}(x;q)}{m_{k+1,n,j}(x;q)} = \frac{[k+1]_q}{[n+k]_q}.q^{-k}.\frac{1+q^{n+k}x}{x} \geq \frac{[k+1]_q}{[n+k]_q}.q-k.\frac{1+q^{n+k}\frac{[j+1]_q}{[n-1]_q}}{\frac{[j+1]_q}{[n-1]_q}} = \frac{[k+1]_q}{[n+k]_q}.q^{-k}.\frac{[n-1]_q+q^{n+k}[j+1]_q}{[j+1]_q}$$
$$\geq \frac{[k+1]_q}{[j+1]_q}.q^{-k}.\frac{[n-1]_q+q^{n+k}[j+1]_q}{[k+1]_q}$$

By using $[k+1]_q \geq [j+1]_q$, we obtain

$$\frac{m_{k,n,j}(x;q)}{m_{k+1,n,j}(x;q)} \geq q^{-k}\frac{[n-1]_q+q^{n+k}[j+1]_q}{[k+1]_q} = q^{-k}\frac{1-q^{n-1}+q^{n+k}(1-q^{j+1})}{1-q^{k+1}} = 1.$$

Then we have the conclusion that

$$1 = m_{j,n,j}(x;q) \geq m_{j+1,n,j}(x;q) \geq m_{j+2,n,j}(x;q) \geq \cdots \geq m_{n,n,j}(x;q),$$

Therefore, the proof is completed for the case (1).

Case 2) Let $k < j$. Since $h(x) = \frac{x}{1+q^{n+k-1}x}$ is nonincreasing on $\left[\frac{[j]_q}{[n-1]_q}, \frac{[j+1]_q}{[n-1]_q}\right]$, it follows

$$\frac{m_{k,n,j}(x;q)}{m_{k-1,n,j}(x;q)} = \frac{[n+k-1]_q}{[k]_q}q^{k-1}\frac{x}{1+q^{n+k-1}x} \geq \frac{[n+k-1]_q}{[k]_q}q^{k-1}\frac{\frac{[j]_q}{[n-1]_q}}{1+q^{n+k-1}\frac{[j]_q}{[n-1]_q}}$$
$$= \frac{[n+k-1]_q}{[k]_q}q^{k-1}\frac{[j]_q}{[n-1]_q+q^{n+k-1}[j]_q} \geq \frac{1-q^{n+k-1}}{1-q^{n-1}+q^{n+k-1}(1-q^j)} = 1.$$

Then we easily get

$$1 = m_{j,n,j}(x;q) \geq m_{j-1,n,j}(x;q) \geq m_{j-2,n,j}(x;q) \geq \cdots \geq m_{0,n,j}(x;q).$$

Therefore, the proof is completed □

Lemma 3.3. *Let $q \in (0,1)$, $n \in \mathbb{N}, n \geq 2$ and $x \in \left[\frac{[j]_q}{[n-1]_q}, \frac{[j+1]_q}{[n-1]_q}\right]$. Then we have*

i. If $k \in \{j+3, j+4, \cdots, n-1\}$ is such that $[k+1]_q - \sqrt{q^k[k+1]_q} \geq [j+1]_q$, then

$$\overline{M}_{k,n,j}(x;q) \geq \overline{M}_{k+1,n,j}(x;q).$$

ii. If $k \in \{1,2,\cdots,j-2\}$ is such that $[k]_q - \sqrt{q^{k-1}[k]_q} \leq [j]_q$, then $\widehat{M}_{k,n,j}(x;q) \geq \widehat{M}_{k-1,n,j}(x;q)$.

Proof. (i) Let $k \in \{j+3, j+4, \cdots, n-1\}$ and $[k+1]_q - \sqrt{q^k[k+1]_q} \geq [j+1]_q$. We observe that

$$\frac{\overline{M}_{k,n,j}(x;q)}{\overline{M}_{k+1,n,j}(x;q)} = \frac{[k+1]_q}{[n+k]_q} q^{-k} \frac{1+q^{n+k}x}{x} . \frac{[k]_q - x[n-1]_q}{[k+1]_q - x[n-1]_q}.$$

Since $h(x) = \frac{1+q^{n+k}x}{x} . \frac{[k]_q - x[n-1]_q}{[k+1]_q - x[n-1]_q}$ is nonincreasing on the interval $\left[\frac{[j]_q}{[n-1]_q}, \frac{[j+1]_q}{[n-1]_q}\right]$, we get

$$\begin{aligned}
\frac{\overline{M}_{k,n,j}(x;q)}{\overline{M}_{k+1,n,j}(x;q)} &\geq \frac{[k+1]_q}{[n+k]_q} q^{-k} \frac{1+q^{n+k}\frac{[j+1]_q}{[n-1]_q}}{\frac{[j+1]_q}{[n-1]_q}} . \frac{[k]_q - \frac{[j+1]_q}{[n-1]_q}[n-1]_q}{[k+1]_q - \frac{[j+1]_q}{[n-1]_q}[n-1]_q} \\
&\geq q^{-k} \frac{[n-1]_q + q^{n+k}[k+1]_q}{[n+k]_q} \frac{[k+1]_q}{[j+1]_q} \frac{[k]_q - [j+1]_q}{[k+1]_q - [j+1]_q} \\
&= q^{-k} \frac{[k+1]_q}{[j+1]_q} \frac{[k]_q - [j+1]_q}{[k+1]_q - [j+1]_q}.
\end{aligned}$$

By taking into account the fact that $[k+1]_q - \sqrt{q^k[k+1]_q} \geq [j+1]_q$, we have

$$[k+1]_q - \sqrt{[k+1]_q^2 - [k]_q[k+1]_q} \geq [j+1]_q.$$

By simple calculations we get

$$\frac{\overline{M}_{k,n,j}(x;q)}{\overline{M}_{k+1,n,j}(x;q)} \geq 1,$$

which proves (i).

(ii) Let $k \in \{1, 2, \cdots, j-2\}$ is such that $[k]_q - \sqrt{q^{k-1}[k]_q} \leq [j]_q$. Then we have that

$$\frac{\widehat{M}_{k,n,j}(x;q)}{\widehat{M}_{k-1,n,j}(x;q)} = \frac{[n+k-1]_q}{[k]_q} q^{k-1} \frac{x}{1+q^{n+k-1}x} . \frac{x - \frac{[k]_q}{[n-1]_q}}{x - \frac{[k-1]_q}{[n-1]_q}}.$$

Since $f(x) = \frac{x}{1+q^{n+k-1}x} . \frac{[n-1]_q x - [k]_q}{[n-1]_q x - [k-1]_q}$ is nondecreasing on the interval $\left[\frac{[j]_q}{[n-1]_q}, \frac{[j+1]_q}{[n-1]_q}\right]$, we obtain

$$\begin{aligned}
\frac{\widehat{M}_{k,n,j}(x;q)}{\widehat{M}_{k-1,n,j}(x;q)} &\geq \frac{[n+k-1]_q}{[k]_q} q^{k-1} \frac{\frac{[j]_q}{[n-1]_q}}{1+q^{n+k-1}\frac{[j]_q}{[n-1]_q}} \frac{[j]_q - [k]_q}{[j]_q - [k-1]_q} \\
&= \frac{[n+k-1]_q}{[n-1]_q + q^{n+k-1}[j]_q} q^{k-1} \frac{[j]_q}{[k]_q} \frac{[j]_q - [k]_q}{[j]_q - [k-1]_q} \\
&\geq \frac{[n+k-1]_q}{[n-1]_q + q^{n+k-1}[k]_q} q^{k-1} \frac{[j]_q}{[k]_q} \frac{[j]_q - [k]_q}{[j]_q - [k-1]_q} = \frac{[j]_q}{[k]_q} q^{k-1} \frac{[j]_q - [k]_q}{[j]_q - [k-1]_q} \geq 1,
\end{aligned}$$

which proves (ii). □

Lemma 3.4. *Let $q \in (0,1)$, $n \in \mathbb{N}, n \geq 2$ and $j \in \{0, 1, 2, \cdots, n-2\}$. For all $x \in \left[\frac{[j]_q}{[n-1]_q}, \frac{[j+1]_q}{[n-1]_q}\right]$, we have*

$$\bigvee_{k=0}^{n} b_{n,k}(x;q) = b_{n,j}(x;q).$$

Proof. Firstly, we demonstrate that for fixed $n \in \mathbb{N}$, $n \geq 2$ and $0 \leq k < k+1 \leq n$ we get

$$0 \leq b_{n,k+1}(x;q) \leq b_{n,k}(x;q) \Leftrightarrow x \in \left[0, \frac{[k+1]_q}{[n-1]_q}\right]. \tag{8}$$

By the definition, we get

$$\begin{aligned} 0 \leq & b_{n,k+1}(x;q) \leq b_{n,k}(x;q) \\ \Leftrightarrow & 0 \leq \begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q .q^{\frac{(k+1)k}{2}}.x^{k+1}. \prod_{s=1}^{n+k+1} (1+q^{s-1}x)^{-1} \leq \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q .q^{\frac{(k-1)k}{2}}.x^{k}. \prod_{s=1}^{n+k} (1+q^{s-1}x)^{-1} \\ \Leftrightarrow & 0 \leq x.\left(q^k.\begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q - q^{n+k}.\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q\right) \leq \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q. \end{aligned}$$

Then we obtain

$$0 \leq b_{n,k+1}(x;q) \leq b_{n,k}(x;q) \Leftrightarrow 0 \leq x \leq \frac{[k+1]_q}{[n-1]_q}$$

which corrects the claim (8). Therefore, by taking $k = 0, 1, 2, \cdots, n-1$, we obtain that

$$\begin{aligned} b_{n,1}(x;q) \leq b_{n,0}(x;q) \Leftrightarrow & 0 \leq x \leq \frac{1}{[n-1]_q}, \\ b_{n,2}(x;q) \leq b_{n,1}(x;q) \Leftrightarrow & 0 \leq x \leq \frac{[2]_q}{[n-1]_q}, \end{aligned}$$

so on,

$$0 \leq b_{n,k+1}(x;q) \leq b_{n,k}(x;q) \Leftrightarrow 0 \leq x \leq \frac{[k+1]_q}{[n-1]_q}$$

and so on until finally

$$\begin{aligned} 0 \leq b_{n,n-1}(x;q) \leq b_{n,n-2}(x;q) \Leftrightarrow & 0 \leq x \leq 1, \\ 0 \leq b_{n,n}(x;q) \leq b_{n,n-1}(x;q) \Leftrightarrow & 0 \leq x \leq 1. \end{aligned}$$

Using the above inequalities, for all $k = 0, 1, 2, \cdots, n$, we easily get that

$$\begin{aligned} x \in \left[0, \frac{1}{[n-1]_q}\right] & \Rightarrow b_{n,k}(x;q) \leq b_{n,0}(x;q) \\ x \in \left[\frac{1}{[n-1]_q}, \frac{[2]_q}{[n-1]_q}\right] & \Rightarrow b_{n,k}(x;q) \leq b_{n,1}(x;q) \\ x \in \left[\frac{[2]_q}{[n-1]_q}, \frac{[3]_q}{[n-1]_q}\right] & \Rightarrow b_{n,k}(x;q) \leq b_{n,2}(x;q) \\ & \ldots \\ x \in \left[\frac{[j]_q}{[n-1]_q}, \frac{[j+1]_q}{[n-1]_q}\right] & \Rightarrow b_{n,k}(x;q) \leq b_{n,j}(x;q) \end{aligned}$$

Using these last implications with the "if and only if" equivalences mentioned above and writing

$$\bigvee_{k=0}^{n} b_{n,k}(x;q) = \max\left\{\bigvee_{k=0}^{j-1} b_{n,k}(x;q), \bigvee_{k=0}^{n} b_{n,k}(x;q)\right\}$$

the lemma is obvious. □

Theorem 3.5. *Let $f \in C_+[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$, then we obtain*

$$| U_n^{(M)}(f;x;q) - f(x) | \, 24\omega\left(f; \frac{1}{\sqrt{[n+1]_q}}\right). \tag{9}$$

Proof. Firstly we give an estimation for $U_n^{(M)}(\varphi_x;x;q)$ with $\varphi_x(t) =| t-x |$. By the definition we get

$$E_n(x;q) := U_n^{(M)}(\varphi_x;x;q) = \frac{\bigvee_{k=0}^{n} b_{n,k}(x;q) \mid \frac{[k]_q}{[n]_q} - x \mid}{\bigvee_{k=0}^{n} b_{n,k}(x;q)}.$$

Assume that $x \in \left[\frac{[k]_q}{[n-1]_q}, \frac{[k+1]_q}{[n-1]_q}\right]$ where $j \in \{0,1,2,\cdots,n-2\}$ is fixed. By Lemma 3.4, we get

$$E_n(x;q) = \max_{k=0,1\cdots,n}\left\{M_{k,n,j}(x;q)\right\},\ x \in \left[\frac{[k]_q}{[n-1]_q}, \frac{[k+1]_q}{[n-1]_q}\right].$$

Therefore, obtaining an upper estimate for each $M_{k,n,j}(x;q)$ is remained when $j \in \{1,2,\cdots,n-2\}$ is fixed, $x \in \left[\frac{[k]_q}{[n-1]_q}, \frac{[k+1]_q}{[n-1]_q}\right]$ and $k = \{0,1\cdots,n\}$.

For the proof, we take the following cases: 1)$k \in \{j, j+1, j+2\}$, 2)$k \geq j+3$ 3)$k \leq j-1$.

Case 1) If $k=j$ then $M_{j,n,j}(x;q) =| \frac{[j]_q}{[n]_q} - x |= x - \frac{[j]_q}{[n]_q}$. Since $x \in \left[\frac{[k]_q}{[n-1]_q}, \frac{[k+1]_q}{[n-1]_q}\right]$, It is obvious that $M_{j,n,j}(x;q) \leq \frac{1+q^j}{[n]_q} \leq \frac{2}{[n]_q}$. If $k=j+1$ then $M_{j+1,n,j}(x;q) = m_{j+1,n,j}(x;q) \mid \frac{[j+1]_q}{[n]_q} - x \mid$. From Lemma 3.2, we have $m_{j+1,n,j}(x;q) \leq 1$. Since $x \in \left[\frac{[k]_q}{[n-1]_q}, \frac{[k+1]_q}{[n-1]_q}\right]$, it easily follows that $M_{j+1,n,j}(x;q) \leq \frac{q^j}{[n]_q} \leq \frac{1}{[n]_q}$. If $k=j+2$ then $M_{j+2,n,j}(x;q) = m_{j+2,n,j}(x;q)\left(\frac{[j+2]_q}{[n]_q} - x\right)$. For the interval $x \in \left[\frac{[k]_q}{[n-1]_q}, \frac{[k+1]_q}{[n-1]_q}\right]$, we obtain $M_{j+2,n,j}(x;q) \leq \frac{2}{[n]_q}$.

Case2) Subcase a) Suppose first $[k+1]_q - \sqrt{q^k[k+1]_q} < [j+1]_q$. Then we get

$$\overline{M}_k, n, j(x;q) = m_k, n, j(x;q).\left(\frac{[k]_q}{[n-1]_q} - x\right) \leq \frac{[k]_q}{[n-1]_q} - \frac{[j]_q}{[n-1]_q}.$$

By the hypothesis, since $q[j]_q > q[k]_q - \sqrt{q^k[k+1]_q}$, we obtain

$$\overline{M}_k, n, j(x;q) \leq \frac{[k]_q}{[n-1]_q} - \frac{[k]_q - 1/q\sqrt{q^k[k+1]_q}}{[n-1]_q} = \frac{\sqrt{q^{k-2}([k]_q + q^k)}}{[n-1]_q} \leq \frac{3\sqrt{2}}{\sqrt{[n+1]_q}}.$$

Subcase b) Suppose now that $[k+1]_q - \sqrt{q^k[k+1]_q} \geq [j+1]_q$. In this case, the function $g(k) = [k+1]_q - \sqrt{q^k[k+1]_q}$ is nondecreasing, there exists $\bar{k} \in \{0,1,2,\cdots,n\}$, of maximum value, such that

$$[\bar{k}+1]_q - \sqrt{q^{\bar{k}}[\bar{k}+1]_q} < [j+1]_q.$$

Let $k_1 = \bar{k}+1$. Then for all $k \geq k_1$, we obtain $[k+1]_q - \sqrt{q^k[k+1]_q} \geq [j+1]_q$.

$$\begin{aligned}\overline{M}_{\bar{k},n,j}(x;q) =& m_{\bar{k},n,j}(x;q).\left(\frac{[\bar{k}+1]_q}{[n-1]_q} - x\right) \leq \frac{[\bar{k}+1]_q}{[n-1]_q} - x \leq \frac{[\bar{k}+1]_q}{[n-1]_q} - \frac{[j]_q}{[n-1]_q}\\ &\leq \frac{[\bar{k}+1]_q}{[n-1]_q} - \frac{[\bar{k}+1]_q - q^j - \sqrt{q^{\bar{k}}[\bar{k}+1]_q}}{[n-1]_q} = \frac{q^j + \sqrt{q^{\bar{k}}([\bar{k}]_q + q^k)}}{[n-1]_q} \leq \frac{3\sqrt{2} + \sqrt{3}}{\sqrt{[n+1]_q}}.\end{aligned}$$

Also it is easy to check that $k_1 \geq j+3$, since g is nondecreasing function, then we get $g(j+2) < [j]_q$. By Lemma 3.3 (i) it follows that

$$\overline{M}_{\bar{k},n,j}(x;q) \geq \overline{M}_{\bar{k}+2,n,j}(x;q) \geq \cdots \overline{M}_{n,n,j}(x;q).$$

Hence, we get $\overline{M}_{k,n,j}(x;q) \leq \frac{3\sqrt{2}+\sqrt{3}}{\sqrt{[n+1]_q}}$ for any $k \in \left\{\bar{k}+1, \bar{k}+2, \cdots, n\right\}$. By Lemma 3.1 (i) we obtain

$$M_{k,n,j}(x;q) \leq \frac{3\sqrt{2}+\sqrt{3}}{\sqrt{[n+1]_q}}.$$

Case 3) Subcase a) Suppose first that $[k]_q + \sqrt{q^{k-1}[k]_q} \geq [j]_q$. Then we get

$$\widehat{M}_{k,n,j}(x;q) = m_{k,n,j}(x;q)\left(x - \frac{[k]_q}{[n-1]_q}\right) \leq \frac{[j+1]_q}{[n-1]_q} - \frac{[k]_q}{[n-1]_q} \leq \frac{[j]_q + q^j}{[n-1]_q} - \frac{[k]_q}{[n-1]_q}$$
$$\leq \frac{q^j + \sqrt{q^{k-1}([k-1]_q + q^{k-1})}}{[n-1]_q} \leq \frac{\sqrt{2}+1}{\sqrt{[n-1]_q}}.$$

Subcase b) Suppose now that $[k]_q + \sqrt{q^{k-1}[k]_q} < [j]_q$. Let $\tilde{k} \in \{0,1,2,\cdots,n\}$ be the minimum value such that $[\tilde{k}]_q + \sqrt{q^{\tilde{k}-1}[\tilde{k}]_q} \geq [j]_q$. Then $k_2 = \tilde{k}-1$ satisfies $[k_2]_q + \sqrt{q^{k-1}[k_2]_q} < [j]_q$. Then

$$\widehat{M}_{\tilde{k}-1,n,j}(x;q) = m_{\tilde{k}-1,n,j}(x;q)\left(x - \frac{[\tilde{k}-1]_q}{[n-1]_q}\right) \leq \frac{[j+1]_q}{[n-1]_q} - \frac{[\tilde{k}-1]_q}{[n-1]_q} \leq \frac{[j]_q + q^j}{[n-1]_q} - \frac{[\tilde{k}-1]_q}{[n-1]_q}.$$

Since $[\tilde{k}]_q + \sqrt{q^{\tilde{k}-1}[\tilde{k}]_q} \geq [j]_q$ we get

$$\widehat{M}_{\tilde{k}-1,n,j}(x;q) \leq \frac{[\tilde{k}]_q + q^j + \sqrt{q^{\tilde{k}-1}[\tilde{k}]_q}}{[n-1]_q} - \frac{[\tilde{k}-1]_q}{[n-1]_q} = \frac{g^j + q^{\tilde{k}-1} + \sqrt{q^{\tilde{k}-1}[\tilde{k}]_q}}{[n-1]_q} \leq \frac{\sqrt{2}+2}{\sqrt{[n-1]_q}}.$$

Moreover the inequality $k_2 \leq j-1$ is obvious from the case $j \geq 1$. Using Lemma 3.3 (ii), we have

$$\widehat{M}_{\tilde{k}-1,n,j}(x;q) \geq \widehat{M}_{\tilde{k}-2,n,j}(x;q) \geq \cdots \widehat{M}_{0,n,j}(x;q).$$

So, we get $\widehat{M}_{k-1,n,j}(x;q) \leq \frac{\sqrt{2}+2}{\sqrt{[n-1]_q}}$ for any $k_2 \leq j-1$. In both subcases, by Lemma 3.1 (ii) we get $M_{k,n,j}(x;q) \leq \frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{[n+1]_q}}$. Therefore, collect all estimations in the previous cases and subcases we easily complete the proof. □

4. A-statistical approximation

We will find an approximation theorem for the operators $U_n^{(M)}(f;x;q)$. But in order to obtain such an approximation, we must substitute a suitable sequence (q_n) whose terms are in the interval $(0,1)$ for a given $q \in (0,1)$ described in the preceding sections. Phillips [18] used this concept for the q-Bernstein polynomials first.

Now let (q_n) is a real sequence satisfying the following conditions:

$$0 < q_n < 1 \text{ for every } n \in \mathbb{N}, \tag{10}$$

$$st_A - \lim_n q_n = 1, \tag{11}$$

and

$$st_A - \lim q_n^n = 1. \tag{12}$$

The notations given in (11) and (12) denote the A-statistical limit of (q_n), where $A = [a_{jn}]$ $(j, n \in \mathbb{N})$ is an infinite nonnegative regular summability matrix, i.e., $a_{jn} \geq 0$ for every $j, n \in \mathbb{N}$ and $\lim_j \sum_{n=1}^{\infty} a_{jn} x_n = L$ whenever $\lim_n x_n = L$ provided that the series $\sum_{n=1}^{\infty} a_{jn} x_n$ is convergent for each $j \in \mathbb{N}$.

We claim that a particular sequence (x_n) is A-statistically convergent to a number L if, for every $\epsilon > 0, \lim_j \sum_{n:|x_n - L| \geq \epsilon} a_{jn} = 0$ (see [10]). We should note that Fast [8] first presented the idea of statistical convergence, and this method of convergence generalizes both of these ideas.

Lemma 4.1. *Let $A = [a_{jn}]$ nonnegative regular summability matrix. If $\lim_j max_n \left\{a_{jn}\right\} = 0$ then A-statistical convergence stronger than classical convergence.*

Theorem 4.2. *Let (q_n) be a sequence satisfying (10)- (12), and let $A = [a_{jn}]$ be a nonnegative regular summability matrix . Then for every $f \in C_+[0,1]$ we have*

$$st_A - \lim_n \left\{ \sup_{x\in[0,1]} \mid U_n^{(M)}(f; x; q_n) - f(x) \mid \right\} = 0. \tag{13}$$

Proof. Let $f \in C_+[0,1]$. By replacing q with (q_n), taking supremum over $x \in [0,1]$, and also utilizing the monotonicity of the modulus of continuity, we get from Theorem 3.5 that

$$E_n := \sup_{x\in[0,1]} \mid U_n^{(M)}(f; x; q_n) - f(x) \mid \leq 24\omega\left(f; \frac{1}{\sqrt{[n+1]_q}}\right), \ n \in \mathbb{N}. \tag{14}$$

Then it is enough to prove $st_A - \lim_n E_n = 0$. The hypotheses (10)-(12) imply that

$$st_A - \lim_n \frac{1}{\sqrt{[n+1]_q}} = 0.$$

Also, we can write that

$$st_A - \lim_n \omega\left(f; \frac{1}{\sqrt{[n+1]_q}}\right) = 0. \tag{15}$$

So, the proof follows from (10)-(15) immediately. □

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