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A characterization of h-strongly porous subsets of $\mathbb{R}$

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Abstract. In this paper, notion of h-porosity of the subsets of real numbers at zero is investigated. Then, a characterization for h-strongly porous subsets of real numbers is given.

1. Introduction

In real number system, the concept of porosity can be considered as the distribution of numbers within the set. A set is said to be porous if it contains "holes" or "gaps" in itself. More specifically, for a porous set, there exist intervals or neighborhoods that don't contain any elements of the set. First studies about set porosity was given by Denjoy in [5], [6] and Khintchine in [9]. Then, porosity arose a paper about cluster sets [7]. A lot of basic properties of porosity can be found in [10]. By the help of a special function the definition of upper porosity for a subset of real numbers at a point, redefined for the subsets of natural numbers at infinity [1]. Then, porosity convergence of real valued sequences defined by the authors in [2]. Some properties of porosity convergence was defined and studied in [3], [4]. Also, Dovgoshey and Bilet characterized the notion of strongly right upper porosity of a subset of $\mathbb{R}$ at a point [8]. They define a new class of subsets of $\mathbb{R}^+$ which are strongly porous at zero. It has many nontrivial modifications of the notion of porosity.

In this study, we deal with the problem considered in [8] by using h-porosity notion instead of right upper porosity notion.

Let $h : [0, +\infty) \to \mathbb{R}$ be a nonnegative, continuous and increasing function on $[0, +\infty)$ such that

$$h(0) = 0, \quad h(x) > 0 \text{ for all } x > 0$$

holds.

Definition 1.1. The right upper h-porosity of $M \subset \mathbb{R}$ at zero is defined as

$$\overline{p}_h(M) := \limsup_{\delta \to 0^+} \frac{\lambda_h(M, 0, \delta)}{h(\delta)}, \tag{1}$$

where $\lambda(M, 0, \delta)$ denotes the length of the largest open subinterval of $(0, \delta)$ that contains no point of M, and $\lambda_h(M, 0, \delta) := h(\lambda(M, 0, \delta))$.

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The right lower h-porosity for $M \subset \mathbb{R}^+$ at zero can defined similarly, replaced lim inf instead of lim sup.

In this paper, we will take account only the right upper h-porosity of subsets of $\mathbb{R}^+$ and we will use following terminology:

A set $M \subseteq \mathbb{R}^+$ is called:

(i) h-porous at zero if $\overline{p}_h(M) > 0$;

(ii) h-strongly (denoted by h-str) porous at zero if $\overline{p}_h(M) = 1$;

(iii) h-nonporous at zero if $\overline{p}_h(M) = 0$.

Let M_d be the set of all decreasing sequences $\tilde{\mu} = \{\mu_n\}$ with $\mu_n \in M$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \mu_n = 0$ holds.

Remark 1.2. Let's point out that $M_d = \emptyset \Leftrightarrow 0 \notin M'$ (M' denotes the set of all accumulation points of M).

Let I_M be the set of all open interval sequences $\{(k_n, l_n)\} \subseteq \mathbb{R}^+$ such that following conditions hold:

(i) $k_n > 0$ for each $n \in \mathbb{N}$.

(ii) (k_n, l_n) is the interior of a connected component of *extM* (exterior of M) for all $n \in \mathbb{N}$, i.e., $(k_n, l_n) \cap M = \emptyset$ but for every $(k_n, l_n) \subseteq (a, b)$ we have

$$(a, b) \neq (k_n, l_n) \Rightarrow (a, b) \cap M \neq \emptyset.$$

(iii) The limit relations $\lim_{n\to\infty} h(k_n) = 0$ and $\lim_{n\to\infty} \frac{h(l_n)-h(k_n)}{h(l_n)} = 1$ hold.

Let us note that if $0 \notin M'$, then we put $I_M = \emptyset$.

Now, let us define an equivalence relation, "$\overset{h}{\asymp}$" on the set of sequences of $\mathbb{R}^+$ by following way: Let $\tilde{x} = \{x_n\}$ and $\tilde{y} = \{y_n\}$, $n \in \mathbb{N}$. We write $\tilde{x} \overset{h}{\asymp} \tilde{y}$ if there are constants $c_*, c^* > 0$ such that

$$c_* h(x_n) \leq h(y_n) \leq c^* h(x_n) \tag{2}$$

holds, for all $n \in \mathbb{N}$. Equivalently, we can say $\tilde{x} \overset{h}{\asymp} \tilde{y}$ if

$$0 < \liminf_{n\to\infty} \frac{h(x_n)}{h(y_n)} \leq \limsup_{n\to\infty} \frac{h(x_n)}{h(y_n)} < \infty \tag{3}$$

holds.

Definition 1.3. Let $M \subset \mathbb{R}^+$ be a set and $\tilde{\alpha} \in M_d$ be a sequence. If there is an interval sequence $\{(k_n, l_n)\}$ of I_M such that

$$\tilde{\alpha} \overset{h}{\asymp} \tilde{k} \tag{4}$$

where $\tilde{k} = \{k_n\}$, then the set M is called h-$\tilde{\alpha}$-str porous at zero.

The set M is completely h-str porous at zero if M is h-$\tilde{\alpha}$-str porous at zero for every $\tilde{\alpha} \in M_d$.

Remark 1.4. If $0 \notin M'$, then from Remark 1.2 the set M is completely h-str porous at zero.

Let us denote the set of all completely h-str porous at zero subsets of $\mathbb{R}^+$ with $\mathbf{CSP_h(0)}$. Namely,
$\mathbf{CSP_h(0)} := \{M \subseteq \mathbb{R}^+ : M$ is completely h -str porous at zero$\}$

2. A characterization of $\mathbf{CSP_h(0)}$

In this section, we will focus on to characterizing of the sets of $\mathbf{CSP_h(0)}$. At first, we shall start to serve some Lemmas for to achieve our aim.

Lemma 2.1. *Let $M \subset \mathbb{R}^+$ be a set, $\tilde{\alpha} = \{\alpha_n\}_{n\in\mathbb{N}}, \tilde{\beta} = \{\beta_k\}_{k\in\mathbb{N}} \in M_d$ be arbitrary sequences. If M is h-$\tilde{\alpha}$-str porous at zero and there is $n = n(k)$ for every natural number k with*

$$c_* h(\alpha_n) \leq h(\beta_k) \leq c^* h(\alpha_n), \tag{5}$$

hold where $c_*, c^* \in (0, \infty)$ be any constants. Then M is also h-$\tilde{\beta}$-str porous at zero.

Proof. Let $M \subset \mathbb{R}^+$ be h-$\tilde{\alpha}$-str porous at zero set. Then, the inequality (5) and the definition of h-$\tilde{\alpha}$-str porous at zero gives that M is h-$\tilde{\beta}$-str porous at zero. □

Example 2.2. Let us consider a sequence $\tilde{m} = \{m_n\} = \left\{\frac{1}{n!}\right\}_{n\in\mathbb{N}}$ and $h(x) = x^2$ for $x \in \mathbb{R}^+$. Define a set $M \subset \mathbb{R}^+$ as $M := \{0\} \cup \{m_n : n \in \mathbb{N}\}$.

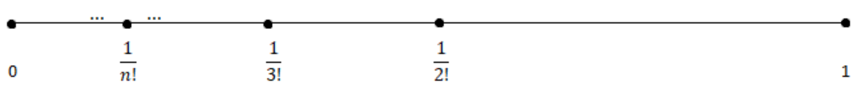

Figure 1: The set M is pointed here

Obviously $\{(m_{n+1}, m_n)\}_{n\in\mathbb{N}} \in I_M$ and M is x^2-$\tilde{m}$-str porous at zero. Also, every sequence $\tilde{\beta} \in M_d$ have the condition of Lemma 2.1 when we take $\tilde{\alpha} = \tilde{m}$, $c_* = c^* = 1$.

So, M is x^2-$\tilde{\beta}$-str porous at zero for every $\tilde{\beta} \in M_d$. Hence, $M \in \mathbf{CSP_h(0)}$ holds.

Example 2.3. Let $\tilde{m}$ be the sequence defined in Example 2.2 and $h(x) = \ln(1 + x)$ for $x \in [0, \infty)$. Define the set N as follows:

$$N = \{0\} \cup \{[m_n, 2m_n] : n \in \mathbb{N}\}.$$

It is clear that $2m_{n+1} < m_n$ for all $n > 1$. Basic Mathematical calculations gives that the sequence $\{(2m_{1+n+1}, m_{1+n})\}$ belongs to I_N. Write $2\tilde{m} = \{2m_n\}$. Then, N is h-$2\tilde{m}$-str porous at zero for $h(x) = \ln(1 + x)$. Let $\tilde{\beta} = \{\beta_k\} \in N_d$. For every $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that

$$\ln(1 + m_n) \leq \ln(1 + \beta_k) \leq \ln(1 + 2m_n). \tag{6}$$

holds. From same reason as in Example 2.2, the inequality (6) gives that $N \in \mathbf{CSP_h(0)}$.

Lemma 2.4. *Let $M \subset \mathbb{R}^+$, $\tilde{\alpha} \in M_d$ and $\{(k_n, l_n)\} \in I_M$. Then, following expressions are equivalent:*

(i) $\tilde{\alpha} \overset{h}{\asymp} \tilde{k}$ *where* $\tilde{k} = \{k_n\}$.

(ii) Following inequalities

$$1 \leq \liminf_{n\to\infty} \frac{h(k_n)}{h(\alpha_n)} \text{ and } \limsup_{n\to\infty} \frac{h(k_n)}{h(\alpha_n)} < \infty$$

hold.

(iii)

$$\limsup_{n\to\infty} \frac{h(k_n)}{h(\alpha_n)} < \infty \quad \text{and} \quad h(\alpha_n) \leq h(k_n)$$

hold for sufficiently large n.

Proof. It is easy to see that (*ii*) implies that (*i*). Let us assume that (*iii*) is true. Since $h(\alpha_n) \leq h(k_n)$, then $1 \leq \frac{h(k_n)}{h(\alpha_n)}$ is true for all $n \in \mathbb{N}$. This implies that $1 \leq \liminf_{n\to\infty} \frac{h(k_n)}{h(\alpha_n)}$ holds. Hence, (*ii*) is proved.

Now, let's prove that (*i*) gives (*iii*). The inequality $\limsup_{n\to\infty} \frac{h(k_n)}{h(\alpha_n)} < \infty$ is obtained by considering the assumption $\tilde{\alpha} \overset{h}{\asymp} \tilde{k}$. Also, we have

$$\lim_{n\to\infty} \frac{h(l_n)}{h(k_n)} = \infty, \tag{7}$$

because of $\{(k_n, l_n)\} \in I_M$. The condition (*i*) implies that the sequence $\left(\frac{h(\alpha_n)}{h(k_n)}\right)$ is bounded from below and upper. So, if we consider (7), then we can say that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{h(\alpha_n)}{h(k_n)} \leq \frac{h(l_n)}{h(k_n)}$$

holds for all $n \geq n_0$. Then, we have

$$h(\alpha_n) \leq h(l_n) \tag{8}$$

for all $n \geq n_0$. From (8) it can be obtained by considering the properties of h that $\alpha_n \leq l_n$ holds for all $n \geq n_0$. In this situation, $k_n \leq \alpha_n$ may be satisfied for all $n \geq n_0$. But this is not possible because $\alpha_n \in M$ and $(k_n, l_n) \cap M = \emptyset$. So, $\alpha_n \leq k_n$ must be hold. Hence, the proof is completed. □

Remark 2.5. Let $M \subset \mathbb{R}^+$ is h-$\tilde{\alpha}$-str porous at zero for $\tilde{\alpha} = \{\alpha_n\} \in M_d$. Then there is an interval sequence $\{(k_n, l_n)\}_{n\in\mathbb{N}}$ in I_M such that the conditions (*ii*) and (*iii*) of Lemma 2.4 are equivalent with the situation of h-$\tilde{\alpha}$-str porous at zero of M.

By the help of Remark 2.5 we can easily establish a set $W \subset \mathbb{R}^+$ such that W is h-str porous at zero but $W \notin \mathbf{CSP_h(0)}$

Example 2.6. Let $\tilde{m} = \left\{\frac{1}{n!}\right\}$ be the sequence in Example 2.2 and $h(x) = x^2$ for $x \in \mathbb{R}^+$. Let us establish the set W as follows:

$$W = \{0\} \cup \left\{\left[\frac{1}{(2n+1)!}, \frac{1}{(2n)!}\right] : n \in \mathbb{N}\right\}.$$

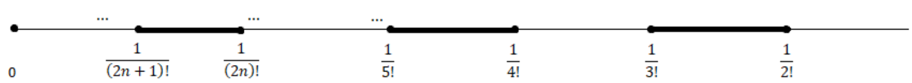

Figure 2: The set W is bold here

The sequence $\left\{\left(\frac{1}{(2n+2)!}, \frac{1}{(2n+1)!}\right)\right\}$ belongs to I_W and $\lim_{n\to\infty} \frac{(\frac{1}{(2n+1)!})^2}{(\frac{1}{(2n+2)!})^2} = \infty$. So, W is x^2-str porous at zero. Now, let us take account the sequence $\tilde{\eta} = \{\eta_n\}$ with $\eta_n = \sqrt{m_{2n+1}m_{2n}} = \sqrt{\frac{1}{(2n+1)!}\frac{1}{(2n)!}}$. It is clear that $\tilde{\eta}$ belongs to W_d. Let $\{(k_n, l_n)\} \in I_W$ be an arbitrary interval sequence and $\eta_n \leq k_n$ for all $n \in \mathbb{N}$. Since, $\eta_n \in \left[\frac{1}{(2n+1)!}, \frac{1}{(2n)!}\right] \subseteq W$, then we have $\eta_n \leq \frac{1}{(2n)!} \leq k_n$. If $W \in \mathbf{CSP_h(0)}$, then W is $\tilde{\eta}$-str porous at zero. Thus, by Remark 2.5, we can take $\{(k_n, l_n)\}$ such that $\eta_n \leq k_n$ for $n \in \mathbb{N}$. So,

$$\limsup_{n\to\infty} \frac{h(k_n)}{h(\eta_n)} \geq \limsup_{n\to\infty} \frac{h(m_{2n})}{h(\eta_n)} = \limsup_{n\to\infty} \frac{[\frac{1}{(2n)!}]^2}{[\frac{1}{(2n+1)!}\frac{1}{(2n)!}]} = \infty.$$

Hence, we obtain from Remark 2.5 that A is not $\tilde{\eta}$-str porous at zero, contrary to the assumption. So, $W \notin \mathbf{CSP_h(0)}$.

If a subset $M \subset \mathbb{R}^+$ is h-$\tilde{\beta}$-str porous at zero for $\tilde{\beta} \in M_d$, then the set M is h-str porous at zero.
Also, we have following proposition.

Proposition 2.7. *Let $M \subset \mathbb{R}^+$ be a set and $0 \in M'$. If M is h-str porous at zero, then there exists $\tilde{\beta} \in M_d$ such that M is h-$\tilde{\beta}$-str porous at zero.*

Remark 2.8. If $0 \notin M'$, then M is h-str porous at zero but from Remark 1.2 there isn't any $\tilde{\beta} \in M_d$.

Definition 2.9. The set $M \subset \mathbb{R}^+$ is uniformly h-str porous at zero if there is a constant $c > 0$, such that for every $\tilde{\beta} \in M_d$ there exists $\{(k_n, l_n)\} \in I_M$ such that

$$1 \leq \liminf_{n\to\infty} \frac{h(k_n)}{h(\beta_n)} \leq \limsup_{n\to\infty} \frac{h(k_n)}{h(\beta_n)} \leq c \tag{9}$$

holds, for all sufficiently large n.

Remark 2.10. If $0 \notin M'$, then from Remark 1.2 the set M is uniformly h-str porous at zero.

Every uniformly h-str porous at zero set belongs to $\mathbf{CSP_h(0)}$. Moreover, the converse of this fact is also true and we will show this at the end of the paper.
A set $I_M(\tilde{\beta}) \subset I_M$ for $\tilde{\beta} \in M_d$ define by the following rule:

$$\{(k_n, l_n)\} \in I_M(\tilde{\beta}) \Leftrightarrow \{(k_n, l_n)\} \in I_M \text{ and } \beta_n \leq k_n \text{ for sufficiently large } n \in \mathbb{N}.$$

Let

$$C(h(\tilde{\beta})) := \inf_{\{(k_n,l_n)\}\in I_M(\tilde{\beta})} \left(\limsup_{n\to\infty} \frac{h(k_n)}{h(\beta_n)} \right) \text{ and } C(h(M)) := \sup_{\tilde{\beta}\in M_d} \left(C(h(\tilde{\beta})) \right). \tag{10}$$

Remark 2.11. Let $M \subset \mathbb{R}^+$ and $0 \notin M'$. M is h-str porous at zero $\Leftrightarrow I_M(\tilde{\beta}) \neq \emptyset$ for every $\tilde{\beta} \in M_d$.
M is completely h-str porous at zero $\Leftrightarrow C(h(\tilde{\beta})) < \infty$ for every $\tilde{\beta} \in M_d$.
M is uniformly h-str porous at zero $\Leftrightarrow C(h(M)) < \infty$.

Lemma 2.12. *Let $M \subset \mathbb{R}^+$ be a set. If $\tilde{\beta} = \{\beta_n\} \in M_d$ and $\{(k_n, l_n)\}_{n\in\mathbb{N}} \in I_M$ is a sequence satisfying $\tilde{k} \overset{h}{\asymp} \tilde{\beta}$, then $\tilde{k} := \{k_n\}$ and $\tilde{l} := \{l_n\}$ are decreasing sequences.*

Proof. It is sufficient to prove that one of the sequence $\tilde{k}$ and $\tilde{l}$ is decreasing. Let us assume that $\tilde{k}$ isn't decreasing. Then, there is a set $E \subseteq \mathbb{N}$, it has infinitely many elements, with

$$k_{n+1} > k_n \tag{11}$$

holds for all $n \in E$. Since $(k_n, l_n) \cap M = \emptyset$, then (11) implies that $h(k_{n+1}) \geq h(l_n) > h(k_n)$. By Lemma 2.4, $h(\beta_n) \leq h(k_n)$ holds for all sufficiently large n. Also, for this n, we can assume that $h(\beta_{n+1}) \leq h(\beta_n)$ because $\tilde{\beta}$ is decreasing. So, we obtain

$$h(k_{n+1}) \geq h(l_n) > h(k_n) \geq h(\beta_n) > h(\beta_{n+1}) \tag{12}$$

for sufficiently large $n \in E$. From (12) we have

$$\frac{h(l_n)}{h(k_n)} \leq \frac{h(k_{n+1})}{h(\beta_{n+1})}.$$

Thus following inequality contradicts to Lemma 2.4

$$\infty = \lim_{\substack{n\in E \\ n\to\infty}} \frac{h(l_n)}{h(k_n)} \leq \limsup_{\substack{n\in E \\ n\to\infty}} \frac{h(k_{n+1})}{h(\beta_{n+1})} \leq \limsup_{n\to\infty} \frac{h(k_{n+1})}{h(\beta_{n+1})}.$$

□

Proposition 2.13. *Let $M \subset \mathbb{R}^+$, $\tilde{\beta} \in M_d$, and $\{(k_n^{(1)}, l_n^{(1)})\}, \{(k_n^{(2)}, l_n^{(2)})\} \in I_M$. If $\tilde{k}^1 \overset{h}{\asymp} \tilde{\beta}$ and $\tilde{k}^2 \overset{h}{\asymp} \tilde{\beta}$, then there exists $N_0 \in \mathbb{N}$ such that*

$$(k_n^{(2)}, l_n^{(2)}) = (k_n^{(1)}, l_n^{(1)}) \tag{13}$$

for every $n \geq N_0$ where $\tilde{k}^i := \{k_n^{(i)}\}$, $i = 1, 2$.

Proof. Let $\tilde{k}^1 \overset{h}{\asymp} \tilde{\beta}$ and $\tilde{k}^2 \overset{h}{\asymp} \tilde{\beta}$ hold. Then, from Lemma 2.12, we obtain $\tilde{k}^i$ are decreasing for $i = 1, 2$. We also have $\tilde{\beta} \overset{h}{\asymp} \tilde{k}^1$ and $\tilde{\beta} \overset{h}{\asymp} \tilde{k}^2$. Also, this implies $\tilde{k}^1 \overset{h}{\asymp} \tilde{k}^2$ holds. From Lemma 2.4 there exists $N_0 \in \mathbb{N}$ such that $\tilde{k}^{(1)} \leq \tilde{k}^{(2)}$ and $\tilde{k}^{(2)} \leq \tilde{k}^{(1)}$ for $n \geq N_0$. Hence, $\tilde{k}^{(1)} = \tilde{k}^{(2)}$ for every $n \geq N_0$ and (13) holds for such n. □

Let's define a set $I_M^d \subseteq I_M$ by the following rule:

$$\{(k_n, l_n)\} \in I_M^d \Leftrightarrow \{(k_n, l_n)\} \text{ and } \{k_n\} \text{ is decreasing.}$$

Remark 2.14. If $\{(k_n, l_n)\} \in I_M^d$ for $M \subseteq \mathbb{R}^+$ then there are $\tilde{\alpha} = \{\alpha_n\}$ and $\tilde{\beta} = \{\beta_n\} \in M_d$ sucht that

$$\lim_{n\to\infty} \frac{h(\alpha_n)}{h(k_n)} = \lim_{n\to\infty} \frac{h(\beta_n)}{h(l_n)} = 1, \tag{14}$$

holds.

Definition 2.15. ([8]) Let $\tilde{K} := \{(k_n, l_n)\}, \tilde{E} := \{(a_n, b_n)\} \in I_M^d$. We say $\tilde{K} \preceq \tilde{E}$ if there are $n_1 = n_1(\tilde{K}, \tilde{E}) \in \mathbb{N}$ and a function $f : \mathbb{N}_{n_1} \to \mathbb{N}$, where $\mathbb{N}_{n_1} := \{n_1, n_1 + 1, ...\}$, such that

$$k_n = a_{f(n)} \tag{15}$$

satisfied for every $n \in \mathbb{N}_{n_1}$.

It is called that $\tilde{E} \in I_M^d$ is universal if $\tilde{K} \preceq \tilde{E}$ for every $\tilde{K} \in I_M^d$.
If $\tilde{K}$ is a subsequence of $\tilde{E}$, then $\tilde{K} \preceq \tilde{E}$ holds.
Let us show that the converse is not true, in general:

Example 2.16. Let $\{m_n\}$ be a strictly decreasing sequence with $\lim_{n\to\infty} \frac{h(m_{n+1})}{h(m_n)} = 0$ and let $M = \{0\} \cup \{m_n : n \in \mathbb{N}\}$. Let us take into account a sequence $\tilde{E} = \{(e_k, f_k)\}$ with $(e_k, f_k) = (m_{n+1}, m_n) \Leftrightarrow n^2 \leq k < (n+1)^2$. From Example 2.2, $\tilde{X} = \{(m_{n+1}, m_n)\} \in I_M$. By Lemma 2.1 we have $\tilde{E} \in I_M$. So, Definition 2.15 implies that $\tilde{E} \preceq \tilde{X}$. It is clear that $\tilde{E} \not\subseteq \tilde{X}$.

Definition 2.15 can be reformulated by the following way:

Proposition 2.17. *Let $\tilde{K} = \{(k_n, l_n)\}, \tilde{E} = \{(a_n, b_n)\} \in I_M^d$. Then, $\tilde{K} \preceq \tilde{E}$ if and only if there are $n_1 = n_1(\tilde{K}, \tilde{E})$ and $f : \mathbb{N}_{n_1} \to \mathbb{N}$ such that*

$$l_n := b_{f(n)}$$

holds for all $n \in \mathbb{N}_{n_1}$.

Proposition 2.18. *Let $M \subset \mathbb{R}$ be a h-str porous set at zero and $0 \notin M'$. The relation " $\preceq$ " is a quasi-ordering (binary relation with reflexive and transitive) on the set I_M^d.*

Proof. It is clear that $\preceq$ is reflexive. So, we must prove that $\preceq$ is transitive. Let $\tilde{K} \preceq \tilde{E}$ and $\tilde{E} \preceq \tilde{T}$ for $\tilde{K} = \{(k_n, l_n)\}, \tilde{E} = \{(a_n, b_n)\}, \tilde{T} = \{(t_n, p_n)\} \in I_M^d$. From Definition 2.15 there exist $f_1 : \mathbb{N}_{n_1} \to \mathbb{N}$ and $f_2 : \mathbb{N}_{n_2} \to \mathbb{N}$ increasing functions such that

$$k_n = a_{f_1(n)} \quad \text{holds for} \quad n \geq n_1 \quad \text{and} \quad a_n = t_{f_2(n)} \quad \text{holds and} \quad n \geq n_2.$$

Take $m_0 := \max\{n \in \mathbb{N} : f_1(n) \leq n_2\}$. Then, $m_0 < \infty$ because f_1 is increasing and unbounded. Let $n_3 := \max\{m_0, n_1\}$. If $\{n \in \mathbb{N} : f_1(n) \leq n_2\} = \emptyset$, then $n_3 = n_1$. So, $n_3 < \infty$ holds. From construction $f_1(n) \geq n_2$ for every $n \in \mathbb{N}_{n_3}$. So, we have

$$k_n = a_{f_1(n)} = t_{f_2(f_1(n))}$$

for such n. Hence, $\tilde{K} \preceq \tilde{E} \wedge \tilde{E} \preceq \tilde{T}$ holds and it implies $\tilde{K} \preceq \tilde{T}$. □

If we use standard procedure we may obtain an equivalence relation " $\equiv$ " on the set I_M^d as follows:

$$\tilde{K} \equiv \tilde{T} \Leftrightarrow \tilde{K} \preceq \tilde{T} \text{ and } \tilde{T} \preceq \tilde{K} \tag{16}$$

Let $\tilde{U} = \{(u_n, v_n)\} \in I_M^d$ be universal for $M \in \mathbb{R}^+$. Define following quantity

$$\mathcal{M}(\tilde{U}) := \limsup_{n\to\infty} \frac{h(u_n)}{h(v_{n+1})}. \tag{17}$$

Let I_M^{sd} be the set of all $\{(k_n, l_n)\} \in I_M^d$ with $\{k_n\}$ is strictly decreasing sequence.

Lemma 2.19. *If $\tilde{T} = \{(t_n, p_n)\} \in I_M^d$ is universal for $M \subset \mathbb{R}^+$, then $\tilde{T}$ have a subsequence $\tilde{T}'$ such that $\tilde{T}' = \{t_{n_k}, p_{n_k}\}$ is also universal and $\tilde{T}' \in I_M^{sd}$.*

Proof. In construction of such subsequence we use mathematical induction. Since $\{t_n\}$ is decreasing, then there is $n_1 \in \mathbb{N}$ such that $t_{n+1} < t_n$ for $n \geq n_1$. Also, from $\lim_{n\to\infty} t_n = 0$ we have, $t_n < t_{n+1}$ for all $n \geq n_1$. Now, let us set

$$n_{k+1} := \min\{n \in \mathbb{N}_{n_k} : t_n < t_{n_k}\}, \quad \text{for } k = 1, 2, ... \tag{18}$$

For every $n \geq n_1$ there is a unique $k \in \mathbb{N}$ such that

$$n_k \leq n < n_{k+1}, \tag{19}$$

holds. Moreover, if n satisfies (19), then the decrease of $\{t_n\}$, $n \in \mathbb{N}_{n_1}$ implies that

$$t_{n_k} = t_n. \tag{20}$$

Now, let us define $f : \mathbb{N}_{n_1} \to \mathbb{N}$ with $f(n) = k$ where k is the unique index satisfying (19). By the above steps we have $\tilde{T} \preceq \tilde{T}'$. Proposition 2.18 gives that " $\preceq$ " is transitive. Also, we have $\tilde{L} \preceq \tilde{T}$ for every $\tilde{L} \in I_M^d$ because $\tilde{T}$ is universal. Then, $\tilde{L} \preceq \tilde{T}'$ for every $\tilde{L} \in I_M^d$. So, $\tilde{T}'$ is also universal. From (18) we have $t_{n_k} > t_{n_{k+1}}$ for every $k \in \mathbb{N}$. So, $\{t_{n_k}\}$ is strictly decreasing, i.e., $\tilde{T}' \in I_M^{sd}$ □

Lemma 2.20. *Let $M \in \mathbf{CSP_h(0)}$. If $\tilde{T} = \{(t_n, p_n)\} \in I_M^{sd}$ is universal, then*

$$\mathcal{M}(\tilde{T}) = C(h(M)) \tag{21}$$

where $C(h(M))$ and $\mathcal{M}(\tilde{T})$ are defined by (10) *and* (17)*, respectively.*

Proof. Let $\tilde{T} \in I_M^{sd}$ be universal. Firstly, we shall prove

$$\mathcal{M}(\tilde{T}) \geq C(h(M)). \tag{22}$$

Equation (22) holds if and only if

$$\mathcal{M}(\tilde{T}) \geq C(h(\tilde{\beta})) \tag{23}$$

for every $\tilde{\beta} \in M_d$ and $C(h(\tilde{\beta}))$ was defined in (10). Now, let $\tilde{\beta} \in M_d$. By the hypothesis, $M \in \mathbf{CSP_h(0)}$. So, there exists $\{(k_n, l_n)\} \in I_M$ so that $\tilde{\beta} \overset{h}{\asymp} \tilde{k}$. By Lemma 2.4 we have

$$\limsup_{n\to\infty} \frac{h(k_n)}{h(\beta_n)} < \infty. \tag{24}$$

Also, for sufficiently large n

$$h(\beta_n) \le h(k_n) \tag{25}$$

holds. Proposition 2.13 and the definition of $C(h(\tilde{\beta}))$ imply

$$C(h(\tilde{\beta})) = \limsup_{n\to\infty} \frac{h(k_n)}{h(\beta_n)}. \tag{26}$$

So, to prove (23) we must show that

$$\mathcal{M}(\tilde{T}) \ge \limsup_{n\to\infty} \frac{h(k_n)}{h(\beta_n)}. \tag{27}$$

By Lemma 2.12 we have

$$\tilde{E} := \{(k_n, l_n)\} \in I_M^d, \tag{28}$$

(28) implies that $\tilde{E} \preceq \tilde{T}$ holds because $\tilde{T}$ is universal. Hence, there are $n_1 \in \mathbb{N}$ and an increasing function $f : \mathbb{N}_{n_1} \to \mathbb{N}$ such that

$$k_n \ge k_{n+1} \text{ and } k_n = t_{f(n)} \tag{29}$$

for every $n \ge n_1$. Since $\tilde{T} = \{(t_n, p_n)\} \in I_M^{sd}$, let us assume that $\tilde{t} = \{t_n\}$ is strictly decreasing. Replacing $\tilde{\beta}$ by a suitable subsequence we may assume that $\tilde{a}$ and $\tilde{\beta}$ are also strictly decreasing, f is strictly increasing, and

$$\beta_1 \le t_1, \ \lim_{n\to\infty} \frac{h(a_n)}{h(\beta_n)} = \limsup_{n\to\infty} \frac{h(a_n)}{h(\beta_n)} \tag{30}$$

hold. The intervals $[p_{n+1}, t_n]$, for $n = 1, 2, \ldots$, together with the interval $[p_1, \infty)$ are a cover of the set $M_0 = M\backslash\{0\}$, i.e.,

$$M_0 \subseteq [p_1, \infty) \cup \left(\bigcup_{n\in\mathbb{N}} [p_{n+1}, t_n] \right).$$

This cover has pairwise disjoint elements and $h(\beta_1) \le h(t_1)$, $n \in \mathbb{N}$. So, there is unique $s(n) \in \mathbb{N}$ with

$$\beta_n \in [p_{s(n)+1}, t_{s(n)}]. \tag{31}$$

We claim that the equality

$$s(n) = f(n) \tag{32}$$

holds for all sufficiently large n. By using (25), (29) and (31) we obtain

$$\beta_n \le t_{f(n)} \text{ and } \beta_n \ge p_{s(n)+1}. \tag{33}$$

Hence, (33) and following inequality

$$h(p_{s(n)+1}) > h(t_{s(n)+1}) > h(t_{s(n)+2}) > h(t_{s(n)+3}) > \ldots$$

imply that

$$f(n) \le s(n) \tag{34}$$

holds. Let us assume that (34) is strict for $n \in E \subseteq \mathbb{N}$, such that E is an infinite set. i.e.,

$$f(n) \le s(n) - 1 \tag{35}$$

for $n \in E$. Since $\tilde{a} \overset{h}{\asymp} \tilde{\beta}$ and $a_n = t_{f(n)}$, there is a constant $c^* \in (0,1)$ such that

$$c^* h(t_{f(n)}) \le h(\beta_n) \le h(t_{f(n)}) \tag{36}$$

holds for all sufficiently large n. From (31), (34) and (36), it follows that

$$c^* h(t_{f(n)}) \le h(\beta_n) \le h(t_{s(n)}) \le h(t_{f(n)}). \tag{37}$$

Since $\tilde{t} = \{t_n\}$ is strictly increasing, then $(t_i, p_i) \cap (t_j, p_j) = \emptyset$ if $i \neq j$ and (35) implies that

$$h(t_{s(n)}) < h(p_{s(n)}) \le h(t_{s(n)-1}) \le h(t_{f(n)}) < h(p_{f(n)}).$$

Together (37) and this inequality

$$c^* h(t_{f(n)}) \le h(\beta_n) \le h(t_{s(n)}) < h(p_{s(n)}) \le h(t_{s(n)-1}) < h(t_{f(n)})$$

for $n \in E$. So, we have

$$\frac{1}{c^*} = \lim_{n\to\infty} \frac{h(t_{f(n)})}{c^* h(t_{f(n)})} \ge \limsup_{\substack{n\to\infty \\ n\in E}} \frac{h(p_{s(n)})}{h(t_{s(n)})}$$

contrary to the limit relation

$$\lim_{n\to\infty} \frac{h(p_n)}{h(t_n)} = \infty.$$

Thus, the set of $n \in \mathbb{N}$ has the condition $f(n) < s(n)$ is finite. So, (32) holds.

Now we can prove (27) easily. By (29) ad (32) we have

$$a_n = t_{f(n)} = t_{s(n)}.$$

Equation (32) implies that $h(\beta_n) \ge h(l_{s(n)+1})$. Consequently

$$\frac{h(a_n)}{h(\beta_n)} \le \frac{h(t_{s(n)})}{h(p_{s(n)+1})}.$$

So,

$$\limsup_{n\to\infty} \frac{h(a_n)}{h(\beta_n)} \le \limsup_{n\to\infty} \frac{h(t_{s(n)})}{h(p_{s(n)+1})} \le \limsup_{n\to\infty} \frac{h(t_n)}{h(p_{n+1})} = \mathcal{M}(\tilde{T}).$$

(27) follows, so (22) is proved.

For to prove

$$\mathcal{M}(\tilde{T}) \le C(h(M)) \tag{38}$$

let us take a sequence $\tilde{\beta} = \{\beta_n\} \in M_d$ such that (31) holds for $s(n) = n$ and

$$\lim_{n\to\infty} \frac{h(p_{n+1})}{h(\beta_n)} = 1. \tag{39}$$

$\tilde{\beta}$ can be constructed as in the proof of Proposition 2.7. The set M is $\tilde{\beta}$-str porous at zero because $M \in \mathbf{CSP_h(0)}$. So, there is $\tilde{a} \overset{h}{\asymp} \tilde{\beta}$ such that $\{(a_n, b_n)\} \in I_M^d$. The sequence $\tilde{a}$ is decreasing from Lemma 2.12.

Since $\beta_n \in [p_{n+1}, t_n]$, then by using (32) we have

$$a_n = p_n$$

for all sufficiently large n. From (26) and (39), we have also

$$\begin{aligned} C(h(\tilde{\beta})) &= \limsup_{n\to\infty} \frac{h(a_n)}{h(\beta_n)} \\ &= \limsup_{n\to\infty} \frac{h(t_n)}{h(p_{n+1})} \frac{h(p_{n+1})}{h(\beta_n)} \\ &= \limsup_{n\to\infty} \frac{h(t_n)}{h(p_{n+1})} \limsup_{n\to\infty} \frac{h(p_{n+1})}{h(\beta_n)} \\ &= \limsup_{n\to\infty} \frac{h(t_n)}{h(p_{n+1})} = \mathcal{M}(\tilde{T}). \end{aligned} \tag{40}$$

Since $C(h(M)) \geq C(h(\tilde{\beta}))$, then (38) follows. □

From (40) we have following result.

Corollary 2.21. *Let $M \in \mathbf{CSP_h(0)}$ for $M \subset \mathbb{R}^+$. If $\tilde{T} = \{(t_n, p_n)\} \in I_M^{sd}$ is universal, then $\mathcal{M}(\tilde{T}) < \infty$.*

Remark 2.22. From Lemma 2.20 $\mathcal{M}(\tilde{T}) = C(h(M))$ holds for every universal $\tilde{T} \in I_M^{sd}$.
Assume that $\tilde{T} \in I_M^d$ is universal but $\tilde{T} \notin I_M^{sd}$. Describe a set $E \subseteq \mathbb{N}$ by the rule

$$n \in E \Leftrightarrow n \in \mathbb{N} \text{ and } (t_{n+1}, p_{n+1}) = (t_n, p_n).$$

Let $\tilde{T}' \in I_M^{sd}$ be the universal element of I_M^d built from $\tilde{T}$ as in Lemma 2.19. If we use the definition of the set E we have

$$\begin{aligned} \mathcal{M}(\tilde{T}) &= \limsup_{n\to\infty} \frac{h(t_{n+1})}{h(p_n)} = \limsup_{\substack{n\to\infty \\ n\in E}} \frac{h(t_{n+1})}{h(p_n)} \vee \limsup_{\substack{n\to\infty \\ n\in \mathbb{N}\setminus E}} \frac{h(t_{n+1})}{h(p_n)} \\ &= \limsup_{\substack{n\to\infty \\ n\in E}} \frac{h(t_n)}{h(p_n)} \vee \mathcal{M}(\tilde{T}') = 0 \vee \mathcal{M}(\tilde{T}') = \mathcal{M}(\tilde{T}'). \end{aligned}$$

So, if $\tilde{T}, \tilde{L} \in I_M^d$ are universal, then $\mathcal{M}(\tilde{T}) = \mathcal{M}(\tilde{L})$.

Now, we are ready to give final theorem.

Theorem 2.23. *Let $M \subseteq \mathbb{R}^+$ be h-str porous set at zero and $0 \in M'$. Then, following conditions are equivalent.*
(i) $M \in \mathbf{CSP_h(0)}$.
(ii) I_M^d has a universal element $\tilde{L} = \{(l_n, m_n)\} \in I_M^{sd}$ with

$$\mathcal{M}(\tilde{L}) < \infty. \tag{41}$$

(iii) M is uniformly h-str porous at zero.

Let us recall that from Remark 2.22 (*ii*) of Theorem 2.23 can be reformulated by the following way: The set of universal elements $\tilde{T} \in I_M^d$ is nonempty and $\mathcal{M}(\tilde{T}) < \infty$ holds for every universal $\tilde{T}$.

Proof. Let $M \in \mathbf{CSP_h(0)}$. Firstly, we shall prove that there is a sequence $\tilde{w} = \{w_n\} \in M_d$ such that for every $\tilde{\beta} = \{\beta_k\} \in M_d$ we can find a function $f : \mathbb{N} \to \mathbb{N}$ which is increasing and satisfy following:

$$\{\beta_k\} \overset{h}{\asymp} \{w_{f(k)}\}. \tag{42}$$

Let us define $\{M_j\}_{j\in\mathbb{N}}$ as follows:

$$\begin{aligned} &M_1 := M \cap [h(1), \infty), \\ &M_2 := M \cap \left[h\left(\frac{1}{2}\right), h(1)\right), \\ &M_3 := M \cap \left[h\left(\frac{1}{4}\right), h\left(\frac{1}{2}\right)\right), \\ &\dots \\ &M_j := M \cap \left[h\left(\frac{1}{2^{j-1}}\right), h\left(\frac{1}{2^{j-2}}\right)\right),\ j \in \mathbb{N}. \end{aligned} \tag{43}$$

There is the unique subsequence $\{M_{j_n}\}$, $n \in \mathbb{N}$ of the sequence $\{M_j\}$, $j \in \mathbb{N}$ such that

$$M\backslash\{0\} = \bigcup_{n\in\mathbb{N}} M_{j_n} \text{ and } M_{j_n} \neq \emptyset$$

for every $n \in \mathbb{N}$. For simplicity, let us take $E_n := M_{j_n}$, $n \in \mathbb{N}$. Let $\{w_n\}$ be a sequence such that $w_n \in E_n$ for every $n \in \mathbb{N}$. Clearly, $\{w_n\} \in M_d$. For every $\tilde{\beta} = \{\beta_k\} \in M_d$, let $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$f(k) = n \Leftrightarrow \beta_k \in E_n.$$

$$M\backslash\{0\} = \bigcup_{n\in\mathbb{N}} E_n \text{ and } E_j \cap E_i = \emptyset \text{ if } i \neq j$$

imply that f is well-defined. Also, (43) gives that

$$f(k) \geq 2 \Rightarrow h\left(\frac{1}{2}\right) h(\beta_k) \leq h(w_{f(k)}) \leq h(2)h(\beta_k)$$

In addition, since $\tilde{\beta}$ and $\tilde{w}$ are decreasing and $\lim_{n\in\mathbb{N}} \beta_n = 0$, the function f is increasing and the set $\{k \in \mathbb{N} : f(k) = 1\}$ is finite. So, we can find constants $c_*, c^* > 0$ such that

$$c^* h(\beta_k) \leq h(w_{f(k)}) \leq c_* h(\beta_k)$$

for all $k \in \mathbb{N}$. So, (42) holds.

Let $\tilde{w} = \{w_n\} \in M_d$ be the sequence constructed above. Since $M \in \mathbf{CSP_h(0)}$, then M is h-$\tilde{w}$-str porous at zero. Thus, there is $\tilde{K} := \{(a_n, b_n)\} \in I_M$ such that

$$\tilde{a} \overset{h}{\asymp} \tilde{w}. \tag{44}$$

holds. Hence, Lemma 2.12 gives that $\tilde{a}$ is decreasing. Namely, $\tilde{K} \in I_M^d$. We claim that $\tilde{K}$ is universal. Indeed, as we shown for every $\tilde{\beta} \in M_d$ there is $f : \mathbb{N} \to \mathbb{N}$ such that (42) holds. The relation $\{w_n\} \overset{h}{\asymp} \{a_n\}$ gives that

$$\{w_{f(k)}\} \overset{h}{\asymp} \{a_{f(k)}\}. \tag{45}$$

$(a_{f(n)}, b_{f(n)})$ is the interior of a connected component of $ExtM$ and $\lim_{n\to\infty} f(n) = \infty$. Then, we have

$$\{(a_{f(k)}, b_{f(k)})\} \in I_M. \tag{46}$$

Also, since f is increasing and $\tilde{K} = \{(a_n, b_n)\} \in I_M^d$, (46) implies

$$\{(a_{f(k)}, b_{f(k)})\} \in I_M^d. \tag{47}$$

(42) and (45) gives that

$$\{\beta_k\} \overset{h}{\asymp} \{a_{f(k)}\}. \tag{48}$$

If we use (47), (48) and Remark 2.14, we can show that $\tilde{T} \preceq \tilde{K}$ for every $\tilde{T} \in I_M^d$, as required.

By Lemma 2.19 we can find a universal element $\tilde{T} \in I_M^{sd}$. According to Corollary 2.21 we have $\mathcal{M}(\tilde{T}) < \infty$. So, we have $(i) \Rightarrow (ii)$ holds.

(iii) $\Rightarrow$ (i) is obvious. Also, if we use Lemma 2.20, we can easily see that (i)$\wedge$(ii)$\Rightarrow$(iii) is true. So, to complete the proof we must show that (ii)$\Rightarrow$(i). Let us assume that $\tilde{\beta} = \{\beta_n\} \in M_d$ and let $\tilde{T} = \{(t_k, p_k)\} \in I_M^{sd}$ be universal. Like in the proof of Lemma 2.20 we can assume that $\{t_n\}$ is strictly decreasing and that $h(\beta_1) \leq h(t_1)$. Then, there is a unique $k(n) \in \mathbb{N}$, $n \in \mathbb{N}$ such that

$$h(p_{k(n)+1}) \leq h(\beta_n) \leq h(t_{k(n)}) \tag{49}$$

(see (31)). (49) implies

$$\limsup_{n\to\infty} \frac{h(t_{k(n)})}{h(\beta_n)} \leq \limsup_{n\to\infty} \frac{h(t_{k(n)})}{h(p_{k(n)+1})} \leq \limsup_{n\to\infty} \frac{h(t_k)}{h(p_{k+1})} = \mathcal{M}(\tilde{T}) < \infty.$$

Since $\{(t_{k(n)}, p_{k(n)})\} \in I_M^d$, then M is h-$\tilde{\beta}$-str porous at zero from Lemma 2.4. So, (i) holds. □

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