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Chaotic behavior for the third-order partial differential equations

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Abstract. In our investigation, our primary focus has been on a third-order partial differential equation, as expressed below:

$$av_{ttt}(y,t) + bv_{tt}(y,t) + cv_t(y,t) - \nu^2 v_{yy}(y,t) - \mu v_{yyt}(y,t) = \eta v(y,t). \tag{0.1}$$

This equation represents the one-dimensional variant of the Moore-Gibson-Thompson equation, which holds significance in the realms of high-intensity ultrasound and the linear vibrations of elastic structures. Notably, our study marks a substantial advancement compared to existing literature. This is particularly evident in our revelation that when the critical parameter $\gamma := b - \frac{a\nu^2}{\mu}$ is negative, the equation **(0.1)** exhibits noteworthy characteristics. Specifically, it manifests a uniformly continuous and chaotic semigroup of bounded linear operators within the Hilbert space $L^2([0,\infty),\mathbb{C})$. This discovery challenges current knowledge and provides fresh insights into the dynamics and behavior of solutions to this equation.

1. Introduction

Chaos theory, a mathematical discipline exploring the behavior of dynamical systems known for their heightened sensitivity to initial conditions, has its roots in the late 19th century. French mathematician Henri Poincaré laid the foundation by delving into the three-body problem in celestial mechanics. However, the contemporary evolution of chaos theory gained substantial traction in the 1960s and 1970s, primarily due to the groundbreaking contributions of American meteorologist Edward Lorenz. His revelations demonstrated that minute alterations in the initial conditions of a weather model could yield significantly divergent outcomes, leading him to coin the term "butterfly effect" to encapsulate this phenomenon [22]. Since then, chaos theory has found applications across diverse fields, including physics, biology, economics, and engineering. James Gleick's influential popular science book, "Chaos: Making a New Science," played a pivotal role in the 1980s by bringing chaos theory into the public eye. The book emphasized the concept that apparently random systems could exhibit order and patterns under specific circumstances. The notion of chaos is commonly linked with nonlinear systems, yet it can also arise in linear dynamical systems within infinite-dimensional spaces. While chaos theory has been firmly grounded in finite-dimensional spaces, notably concerning discrete maps and ordinary differential equations, it historically faced a gap in terms of a coherent theory for partial differential equations. Nevertheless, recent years have witnessed the development of sophisticated mathematical

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tools and techniques dedicated to unraveling chaos in infinite-dimensional spaces. These advancements signify a crucial stride forward, opening avenues for expanded research and applications across a spectrum of disciplines. Various definitions have been put forth by mathematicians to characterize chaos, with Devaney's widely adopted definition incorporating three key components. The initial component encapsulates the core concept of the "butterfly effect," emphasizing that minute alterations in the initial state can result in substantial deviations in the system's trajectory over time. In order to accommodate these perturbations, spaces without isolated points are taken into consideration. The exploration of dynamic behavior in physical and natural systems has been a primary driver of mathematical research. Throughout history, differential equations have served as crucial tools for modeling diverse physical phenomena, including mechanics, optics, electromagnetism, and thermodynamics. Notably, equations such as the wave equation, Schrödinger equation, and heat equation have been subjects of extensive study. Of particular interest is the generalized Moore-Gibson-Thompson equation, which represents a broader form of the heat equation. This equation characterizes the temporal evolution of a physical quantity concerning both its position and time. Its applications extend to modeling phenomena like acoustic wave propagation, mechanical vibrations, and particle diffusion in a medium [23].

Our motivation lies in studying the generalized Moore-Gibson-Thompson equation, an extension of the classical equation given by:

$$\begin{cases} av_{ttt}(y,t) + bv_{tt}(y,t) = \nu^2 v_{yy}(y,t) + \eta v(y,t), \\ v(0,y) = v_1(x), \\ v_t(0,y) = v_2(x), \\ v_{tt}(0,y) = v_3(x), \end{cases} \tag{1.1}$$

where v_1, v_2 and v_3 represent the initial contitions of temperature. We derive inspiration from Conejero's 2014 study, wherein the equation (1.1) underwent investigation. The findings illustrated that it possesses a semigroup on Herzog-type Banach spaces, characterized by uniform continuity, chaos, and topological mixing.

Conejero's research provides valuable insights into the dynamics and behavior of the equation (1.1) in functional spaces. Specifically, the study establishes the existence of a semigroup with chaotic properties, characterized by sensitivity to initial conditions and topological mixing. Moreover, the uniform continuity of the semigroup ensures stability and regularity of solutions over time. These findings deepen our understanding of the generalized Moore-Gibson-Thompson equation and its applications in various fields, including physics, engineering, and mathematical modeling. The knowledge gained from this study can advance the control and manipulation of chaotic systems while laying the groundwork for further research in related areas.

The addition of two terms in the second equation, "$cv_t(y,t)$" and "$\eta v_{yyt}(y,t)$", represents the novelty of the equation (1.1). The term "$cv_t(y,t)$" introduces a linear temporal dependency in the equation, representing the temporal derivative of v with respect to t. On the other hand, the term "$\eta v_{yyt}(y,t)$" accounts for the temporal variation of the curvature of the function v with respect to the y-axis. These added terms allow for modeling more complex physical phenomena, such as temporal variations in the velocity and curvature of the function v. In certain conditions, the solutions of the Moore-Gibson-Thompson equation can become unstable and exhibit chaotic behavior, which intrigues mathematicians and physicists alike.

In conclusion, the Moore-Gibson-Thompson equation, a generalization of the heat equation, finds applications in diverse physical phenomena and exhibits intriguing chaotic behavior that has been the focus of extensive mathematical and physical studies.

This paper is structured as follows: Section 2 provides a review of the essential definitions and tools needed to present the main result. Specifically, we introduce a valuable spectral criterion for identifying Devaney Chaos in C_0-semigroups. Section 3 presents our primary finding, Theorem 3.1, which demonstrates that the Moore-Gibson-Thompson equation exhibits chaos.

2. Preliminareis

Let Y be an infinite-dimensional separable Banach space. A family $\{S_t\}_{t\geq 0}$ of linear and continuous operators on Y is said to be a C_0-semigroup if $S_0 = Id, S_tS_s = S_{t+s}$ for all $t, s \geq 0$, and $\lim_{t\to s} S_t y = S_s y$ for all $y \in Y$ and $s \geq 0$. Given a C_0-semigroup $\{S_t\}_{t\geq 0}$ on Y, it can be shown that an operator defined by

$$Gy := \lim_{t\to 0^+} \frac{S_t y - y}{t},$$

exists on a dense subspace of X; denoted by $D(G)$. Then $(G, D(G))$ is called the (infinitesimal) generator of the C_0-semigroup $\{S_t\}_{t\geq 0}$. If $D(G) = Y$, then the C_0-semigroup can be rewritten as $\left\{e^{tG}\right\}_{t\geq 0}$. Such a semigroup is the corresponding solution C_0-semigroup of the abstract Cauchy problem

$$\begin{cases} v'(t, y) &= Gv(t, y) \\ v(0, y) &= \varphi(y), \end{cases}$$

The solutions to this problem can be expressed as $v(t, y) = e^{tG}\varphi(y)$, where $\varphi(y) \in Y$. Further information on C_0-semigroups can be found in [1].

Given a family of operators $\{S_t\}_{t\geq 0}$ we say that it is transitive if for every pair of non-void open sets $V, W \subset Y$ there exists some $t > 0$ such that $S_t(V) \cap W \neq \emptyset$. Furthermore, if there is some $t_0 > 0$ such that the condition $S_t(V) \cap W \neq \emptyset$ holds for every $t \geq t_0$ we say that it is topologically mixing. A C_0-semigroup is hypercyclic if there exists some $y \in Y$ such that the set $\{S_t y : t \geq 0\}$ is dense in Y. In this setting, transitivity coincides with hypercyclicity, but it is strictly weaker than topologically mixing [3–6].

We recall that an element $y \in Y$ is said to be a periodic point of $\{S_t\}_{t\geq 0}$ if there exists some $t_0 > 0$ such that $S_{t_0} y = y$. A C_0-semigroup $\{S_t\}_{t\geq 0}$ is said to be chaotic in the sense of Devaney if it is hypercyclic and there exists a dense set of periodic points in Y. The following criterion let us prove the Devaney chaos for a C_0-semigroup. This result can be compared with the Desch-Schappacher-Webb Criterion [[7] Th 3.1], or any of its extensions [8, 9]. In order to prove this chaotic behavior of solutions, we will apply the following criterion, first stated for operators by Godefroy and Shapiro [11], see also [[12], Proposition 3.3]. This criterion is the counterpart for semigroups. More information on sufficient conditions for hypercyclicity and chaos for semigroups and C_0 operators can be found in [5, 7, 13–21] .

Theorem 2.1. ***Desch-Schappacher-Webb criterion.****[7, Proposition 1.2] Let Y be a complex separable Banach space and $\{S_t\}_{t\geq 0}$ a C_0-semigroup on Y with intinitesimal generator $(G, D(G))$, where $D(G)$ denotes its domain. Assume that there exists a nonempty open connected subset V of $\mathbb{C}$ and weakly holomorphic functions $f_j : V \to Y$, $j \in J$, such that*

1. $V \cap i\mathbb{R} \neq \emptyset$,
2. $f_j(\lambda) \in \ker(\lambda I - G)$ *for every* $\lambda \in V, j \in J$,
3. *for any* $y^* \in Y^*$, *if* $\left\langle f_j(\lambda), y^* \right\rangle = 0$ *for all* $\lambda \in V$ *and* $j \in J$ *then* $y^* = 0$.

Then $\{S_t\}_{t\geq 0}$ is topologically mixing and Devaney chaotic.

Theorem 2.2. ***Eigenvalue Criterion for Chaos.*** *Let Y be an infinite-dimensional separable Banach space. Suppose that the sets*

$$Y_0 := \operatorname{span}\left\{y \in Y : \exists\lambda > 0, S_t y = e^{\lambda t} y, \forall t \geq 0\right\}$$

$$Y_1 := \operatorname{span}\left\{y \in Y : \exists\lambda < 0, S_t y = e^{\lambda t} y, \forall t \geq 0\right\}$$

$$Y_p := \operatorname{span}\left\{y \in Y : \exists\lambda \in \mathbb{Q}, S_t y = e^{\pi\lambda i t} y, \forall t \geq 0\right\}$$

are dense in Y, then $\{S_t\}_{t\geq 0}$ is chaotic.

The third condition in this result is utilized to establish the density of the space generated by specific sets of eigenvectors associated with eigenvalues of G having a real part greater than, equal to, or less than 0. A criterion phrased in these terms was initially proposed for operators by Godefroy and Shapiro in their article [11].

corollary 2.1. *Let Y be a subspace of X. Then, Y is dense if and only if every linear form $L \in X'$ that vanishes on Y also vanishes on the entire space X.*

3. Chaotic behavior for third order partial differential equations

In this section, we will consider the following equation in $Y = \mathbf{L}^2([0,\infty), \mathbb{C})$ given by the formula (0.1).

$$av_{ttt}(y,t) + bv_{tt}(y,t) + cv_t(y,t) - \nu^2 v_{yy}(y,t) - \mu v_{yyt}(y,t) = \eta v(y,t)$$

Using the notation $u_1 := v$, $u_2 := v_t$ and $u_3 := v_{tt}$ the second order in time Cauchy problem in (0.1) can be rewritten as a first-order differential equation.

$$\frac{\partial}{\partial t}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ \frac{\eta}{a}I + \frac{\nu^2}{a}\partial_{yy} & -\frac{c}{a}I + \frac{\mu}{a}\partial_{yy} & \frac{-b}{a}I \end{pmatrix}}_{G}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \tag{3.1}$$

where

$$\begin{pmatrix} u_1(0,y) \\ u_2(0,y) \\ u_3(0,y) \end{pmatrix} = \begin{pmatrix} v_1(y) \\ v_2(y) \\ v_3(y) \end{pmatrix}$$

where We recall the definition of the space of analytic functions of Herzog type [2]. Let

$$Y_\theta = \Big\{ f(y) = \sum_{n \geq 0} \frac{\alpha_n}{n!}(\theta y)^n : \quad (\alpha_n)_n \in c_0(\mathbb{N}_0) \Big\}, \tag{3.2}$$

with $\theta > 0$ and being $c_0(\mathbb{N}_0)$ the Banach space of all complex-valued sequences tending to 0. These are Banach spaces when endowed with the norm

$$\|f\| := \sup_{n \in \mathbb{N}_0} \sup_{y \in \mathbb{R}} \theta^{-n} e^{-\theta |y|} \left| f^{(n)}(y) \right|. \tag{3.3}$$

In other words, the spaces Y_θ, $\theta > 0$, are Banach spaces of analytic functions with certain increasing control at infinity. The spaces Y_θ were introduced by Herzog [2] in connection with the study of dynamical behaviour of the heat equation. Observe that for any θ fixed, the space Y_θ is naturally isomorphic to $c_0(\mathbb{N}_0)$. In particular, its dual Y_θ^* is isomorphic to the Banach space $l_1(\mathbb{N}_0)$ which consists of all complex-valued sequences (a_n) such that $\sum_{n=0}^{\infty} |a_n| < \infty$.

Exemple 3.1. *Considiring $b \in \mathbb{C}$, the function $f(x) = \sinh(\sqrt{b}x)$ is a member of Y_θ if and only if $\theta^2 = |b|$. Similarly, the function $g(x) = e^{bx}$ also belongs to Y_θ if and only if $\theta > |b|$.*

Since for each $\theta > 0$ the operator $D : Y_\theta \to Y_\theta$ defined by $Df(y) = \partial_{yy} f(y)$ is clearly bounded, it follows that the operator-valued matrix G in (3.1) is a bounded linear operator on any space $Y_\theta \oplus Y_\theta \oplus Y_\theta$, $\theta > 0$.

Theorem 3.1. *Let a, b c and η are real numbers and $\mu\, \nu > 0$ be given. Assume that $b - \frac{a\nu^2}{\mu} < 0$. Then, the solution semigroup $\left\{e^{tG}\right\}_{t \geq 0}$ of (3.1) is chaotic on $Y_\theta \oplus Y_\theta \oplus Y_\theta$ for each $\theta > \sqrt{\frac{1}{2}[\frac{c}{\mu} + \frac{b\nu^2}{\mu^2} + \frac{a\nu^6}{\mu^3} + \frac{\eta}{\nu^2}]}$.*

Proof. Letting $GV = \lambda V$ with $V = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$. We have $\begin{cases} \varphi_2 = \lambda \varphi_1 \\ \varphi_3 = \lambda \varphi_2 = \lambda^2 \varphi_1 \\ \frac{\eta}{a}\varphi_1 + \frac{\nu^2}{a}\varphi_1'' - \frac{c\lambda}{a}\varphi_1 + \frac{\mu\lambda}{a}\varphi_1'' - \frac{b\lambda^2}{a}\varphi_1 = \lambda^3 \varphi_1 \end{cases}$, which implies that

$$\left(\frac{\nu^2}{a} + \lambda \frac{\mu}{a}\right)\varphi_1'' - \left(-\frac{\eta}{a} + \frac{c\lambda}{a} + \frac{\lambda^2 b}{a} + \lambda^3\right)\varphi_1 = 0.$$

So,

$$\left(\nu^2 + \lambda\mu\right)\varphi_1'' - \left(-\eta + c\lambda + \lambda^2 b + a\lambda^3\right)\varphi_1 = 0.$$

Let $a_0,\ a_1 \in \mathbb{R}$, we define the following function

$$\varphi_{1,\lambda,a_0,a_1}(x) = a_0 \sum_{n\leq 0} \frac{(R_\lambda)^n x^{2n}}{(2n)!} + a_1 \sum_{n\leq 0} \frac{(R_\lambda)^n x^{2n+1}}{(2n+1)!},$$

where $\mathcal{R}_\lambda = \frac{c\lambda+b\lambda^2+\lambda^3 a-\eta}{\nu^2+\mu\lambda}$. It is clear that the function $\varphi_{1,\lambda,a_0,a_1}$ is in Y_θ for all λ in, say, certain open disc D_c of radius $r < \frac{\nu^2}{\mu}$ centered at zero. Indeed, we have

$$\begin{aligned} |\frac{\mathcal{R}_\lambda}{\theta^2}| &< \frac{c|\lambda| + b|\lambda|^2 + a|\lambda|^3 - \eta}{\nu^2 + \mu\lambda} \\ &< \frac{c\frac{nu^2}{\mu} + b\frac{nu^4}{\mu^2} + a\frac{nu^8}{\mu^3} + \eta}{\nu^2 + \mu\lambda}. \end{aligned}$$

Since $\theta > \sqrt{\frac{1}{2}[\frac{c}{\mu} + \frac{b\nu^2}{\mu^2} + \frac{a\nu^6}{\mu^3} + \frac{\eta}{\nu^2}]}$. Then, $|\frac{\mathcal{R}_\lambda}{\theta^2}| < 1$. It fllows that the function $\varphi_{1,\lambda,a_0,a_1}$ is in Y_θ.

If we set

$$f_{\lambda,a_0,a_1} = \left(\varphi_{1,\lambda,a_0,a_1}, \lambda\varphi_{1,\lambda,a_0,a_1}, \lambda^2\varphi_{1,\lambda,a_0,a_1}\right)^\intercal.$$

Then, we have

$$e^{tG}\varphi_1 = e^{\lambda t}\varphi_1,$$

for all $t \geq 0$. So that, if we prove that the sets

$$\begin{aligned} Y_0 &:= \operatorname{span}\left\{f_{\lambda,a_0,a_1} :\ \ 0 < \lambda < r,\ a_0, a_1 \in \mathbb{R}\right\} \\ Y_1 &:= \operatorname{span}\left\{f_{\lambda,a_0,a_1} :\ -r < \lambda < 0,\ a_0, a_1 \in \mathbb{R}\right\} \\ Y_p &:= \operatorname{span}\left\{f_{\lambda,a_0,a_1} :\ \ \lambda \in \pi i\mathbb{Q},\ |\lambda| < r,\ a_0, a_1 \in \mathbb{R}\right\} \end{aligned}$$

are dense in $Y_\theta \oplus Y_\theta \oplus Y_\theta$, then the eigenvalue criterion asserts that the C_0-semigroup generated by G is chaotic on $Y_\theta \oplus Y_\theta \oplus Y_\theta$.

Since Y_0, Y_1, Y_p are linear subspaces of Y_θ, it suffices to prove that they are weakly dense, that is: given $f \in Y^*_\theta \oplus Y^*_\theta \oplus Y^*_\theta$, if $\langle y, f\rangle = 0$ for all $y \in Y$ (where Y is either Y_0, Y_1 or Y_p) then necessarily $f = 0$. In other words, given $f = \left((\rho_n)_n, (\zeta_n)_n, (\sigma_n)_n\right) \in \ell^1 \oplus \ell^1 \oplus \ell^1$, if

$$\begin{aligned} a_0 \sum_{n\geq 0} \mathcal{R}^n_\lambda \rho_{2n} + a_1 \sum_{n\geq 0} \mathcal{R}^n_\lambda \rho_{2n+1} + \lambda a_0 \sum_{n\geq 0} \mathcal{R}^n_\lambda \zeta_{2n} + \lambda a_1 \sum_{n\geq 0} \mathcal{R}^n_\lambda \zeta_{2n+1} \\ + \lambda^2 a_0 \sum_{n\geq 0} \mathcal{R}^n_\lambda \sigma_{2n} + \lambda^2 a_1 \sum_{n\geq 0} \mathcal{R}^n_\lambda \sigma_{2n+1} = 0 \end{aligned} \tag{3.4}$$

for all $a_0, a_1 \in \mathbb{R}$ and for all $0 < \lambda < r$ (respectively $-r < \lambda < 0, \mu = \pi i q$ with $q \in \mathbb{Q}$ and $|\lambda| < r$), then $\rho_n = \zeta_n = \sigma_n = 0$ for all $n \geq 0$. Indeed, set $f(\mu)$ as the left part of (3.4). Then $f(\mu)$ is a meromorphic function with a unique pole equals to $\frac{-\nu^2}{\mu}$. Since $r < \frac{\nu^2}{\mu}$, then this function is a holomorphic function that vanishes on open disc D_c, the subset of $\mathbb{C}$ with an accumulation point. Therefore, all coefficients of its power series should be 0 . The independent coefficient is $a_0\rho_0 + a_1\rho_1$, and this should be zero for any choice of $a_0, a_1 \in \mathbb{R}$, therefore $\rho_0 = \rho_1 = 0$. Now, since $P(\lambda) = \lambda^3 + b\lambda^2 + ac\lambda - \eta$ is a polynomial of order 3, then the equation $P(\lambda) = 0$ has three solutions λ_1, λ_2 and λ_3 in $\mathbb{C}$. Then, if $\lambda = \lambda_1$ or $\lambda = \lambda_2$, then $\mathcal{R}_\lambda = 0$ and we have

$$\begin{cases} a_0\zeta_0 + a_1\zeta_1 + \lambda_1(a_0\sigma_0 + a_1\sigma_1) &= 0, \\ a_0\zeta_0 + a_1\zeta_1 + \lambda_2(a_0\sigma_0 + a_1\sigma_1) &= 0, \end{cases}$$

for all $a_0, a_1 \in \mathbb{R}$. This yields $\zeta_0 = \zeta_1 = \sigma_0 = \sigma_1 = 0$.

Suppose that all $\rho_0 = \cdots = \rho_{2n-1} = 0$, $\zeta_0 = \cdots = \zeta_{2n-1} = 0$ and all $\sigma_0 = \cdots = \sigma_{2n-1} = 0$. If we divide $f(\lambda)$ by R_λ^n then we obtain an entire function that vanishes on a set with an accumulation point. Therefore, all its coefficients should be 0 . The independent coefficient is $a_0\rho_{2n} + a_1\rho_{2n+1}$ with $a_0, a_1 \in \mathbb{R}$. A similar argument as before yields $\rho_{2n} = \rho_{2n+1} = 0$. Finally, taking $\lambda = \lambda_1$ and $\lambda = \lambda_2$ we get

$$\begin{cases} a_0\zeta_{2n} + a_1\zeta_{2n+1} + \lambda_1(a_0\sigma_{2n} + a_1\sigma_{2n+1}) = 0, \\ a_0\zeta_{2n} + a_1\zeta_{2n+1} + \lambda_2(a_0\sigma_{2n} + a_1\sigma_{2n+1}) = 0, \end{cases}$$

for any choice of a_0, a_1, and then $\zeta_{2n} = \zeta_{2n+1} = \sigma_{2n} = \sigma_{2n+1} = 0$.

If we consider Y_θ just as the corresponding space containing only the real sequences, the hypercyclicity of $\left\{e^{tG}\right\}_{t\geq 0}$ can be deduced on $Y_\theta \oplus Y_\theta \oplus Y_\theta$. with a similar proof, avoiding the part of proving the density of Yp. Accordin to Thoerem 2.2, then $\left\{e^{tG}\right\}_{t\geq 0}$ is chaotic. □

Remark 3.1. *In the context of problem **(0.1)**, setting $c = \eta = 0$, we have that the associated uniformly continuous semigroup of*

$$av_{ttt}(y,t) + bv_{tt}(y,t) - \nu^2 v_{yy}(y,t) - \mu v_{yyt}(y,t) = 0. \tag{3.5}$$

is chaotic on $Y_\theta \oplus Y_\theta \oplus Y_\theta$ for all $\theta > 0$, see [24].

4. Conclusion

The Moore-Gibson-Thompson equation, a generalization of the heat equation, has proven to be a versatile model for various physical phenomena. However, it was also revealed in the article that the solutions to this equation can display instability and chaotic behavior under specific circumstances. This intriguing chaotic nature has captured the attention of numerous mathematicians and physicists, leading to extensive research in the field. The article provided a comprehensive overview of the essential definitions and tools necessary to comprehend this phenomenon. Ultimately, the article successfully demonstrated that the Moore-Gibson-Thompson equation does indeed exhibit chaotic behavior. This significant conclusion adds valuable insights to the realm of chaos theory, and it further solidifies the equation's importance as a dynamic and complex model in the study of physical systems [1–5].

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References

[1] K. J. Engel and R. Nagel, One-parameter semigroups for linear evolution equations, volume 194 of Graduate Texts in Mathematics. Springer-Verlag, New York,.With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt, (2000).
[2] G. Herzog, On a universality of the heat equation. Math. Nachr., 188, 169–171 (1997).
[3] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey, On Devaney's definition of chaos. Amer. Math. Monthly, 99(4):332–334 (1992).
[4] J. Banasiak and M. Lachowicz, Chaotic linear dynamical systems with applications. In Semigroups of operators: theory and applications (Rio de Janeiro, 2001), pages 32–44. Optimization Software, New York, (2002).
[5] T. Bermúdez, A. Bonilla, J. A. Conejero and A. Peris, Hypercyclic, topologically mixing and chaotic semigroups on Banach spaces. Studia Math., 170(1):57–75 (2005).
[6] K. G. Grosse-Erdmann and A. Peris. Linear chaos. Universitext. Springer, London, (2011).
[7] W. Desch, W. Schappacher and G. F. Webb, Hypercyclic and chaotic semigroups of linear operators. Ergodic Theory Dynam. Systems, 17(4):793–819, (1997).
[8] J. Banasiak and M. Moszynski, A generalization of DeschSchappacher-Webb criteria for chaos. Discrete Contin. Dyn. Syst., 12(5):959–972, (2005).
[9] J. A. Conejero and E. M. Mangino, Hypercyclic semigroups generated by Ornstein-Uhlenbeck operators. Mediterr. J. Math., 7(1):101–109, (2010).
[10] Barrachina and J. A. Conejero, Devaney chaos and distributional chaos in the solution of certain partial differential equations. Abstr. Appl. Anal., pages Art. ID 457019, 11, (2012).
[11] G. Godefroy and J. H.Shapiro, Operators with dense, invariant, cyclic vector manifolds, J. Funct.Anal. 98, 229–269 (1991).
[12] J. Bonet, F. Martınez-Gimenez and A. Peris, Linear chaos on Fréchet spaces," Int. J. Bifurcation and Chaos 13, 1649–1655; Dynamical Systems and Functional Equations (Murcia, 2000) (2003).
[13] J. P. Bes and A. Peris, Hereditarily hypercyclic operators, J. Funct. Anal. 167, 94–112 ,1999.
[14] L. Bernal-Gonzalez and K. G. Grosse-Erdmann, The hypercyclicity criterion for sequences of operators, Studia Math. 157, 17–32 (2003).
[15] T. Bermudez, A. Bonilla and A. Peris, On hypercyclicity and supercyclicity criteria,Bull. Austral. Math. Soc. 70, 45–54 (2004).
[16] J. Banasiak and M. Moszynski, A generalization of Desch–Schappacher–Webb criteria for chaos, Discr. Contin. Dyn. Syst. 12, 959–972, (2005).
[17] J. A. Conejero and A. Peris, Linear transitivity criteria, Topol. Appl. 153, 767–773 (2005).
[18] S. El Mourchid, The imaginary point spectrum and hypercyclicity, Semigroup Forum 73, 313–316 (2006).
[19] T. Kalmes, On chaotic C_0-semigroups and infinitely regular hypercyclic vectors, Proc. Amer. Math. Soc. 134, 2997–3002 (electronic), (2006).
[20] J. Bes and A. Peris, Disjointness in hypercyclicity, J. Math. Anal. Appl. 336, 297–315 (2007).
[21] J. A. Conejero and A. Peris, Hypercyclic translation C0-semigroups on complex sectors, Discr. Contin. Dyn. Syst. 25, 1195–1208 (2009).
[22] E. Lorenz, The butterfly effect. World Scientific Series on Nonlinear Science Series A, 39, 91-94 (2000).
[23] A.E. Abouelregal, H. Ersoy and Ö. Civalek, Solution of Moore–Gibson–Thompson equation of an unbounded medium with a cylindrical hole. Mathematics, 9(13), 15-36 (2021).
[24] J.A. Conejero, C. Lizama and F. D. A. Ródenas Escribá, Chaotic behaviour of the solutions of the Moore-Gibson-Thompson equation. Applied Mathematics and Information Sciences, 9(5), 2233-2238 (2015).
[25] F.J. Solis, L. Jódar and B. Chen, Chaos in the one-dimensional wave equation, Appl. Math. Lett. 18 (1), 85–90 (2005).
[26] J. A. Conejero, A. Peris and M.Trujillo, Chaotic asymptotic behavior of the hyperbolic heat transfer equation solutions. International Journal of Bifurcation and Chaos, 20(09), 2943-2947 (2010).
[27] Baraket S., Jaidane R., Weighted Schrodinger-Kirchhoff type problem in dimension 2 with non-linear double exponential growth, Filomat, 37 (16), 5373–5398, (2023).
[28] Eddine N.C., Nguyen P.D., Ragusa M.A., Existence and multiplicity of solutions for a class of critical anisotropic elliptic equations of Schrodinger-Kirchhoff-type, Mathematical Methods in the Applied Sciences, doi: 10.1002/mma.9474, (2023).
[29] Sun, D.D.,Ground state solutions of Schrodinger-Kirchhoff equations with potentials vanishing at infnity, Journal of Function Spaces, vol.2023, art.n.8829268, doi: 10.1155/2023/8829268, (2023).