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Some properties of $\mathcal{I}$-convergence in cone metric spaces

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Abstract. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$, $\mathcal{I}$-sequential compactness, $\mathcal{I}$-sequential countable compactness and $\mathcal{I}$-completeness in cone metric spaces are discussed. We also construct a bounded sequence in an infinite discrete metric space which is not $\mathcal{I}$-convergent, which gives a negative answer to an open problem posed by P. Das [12, Open problem 2.3].

1. Introduction

The notion of statistical convergence, which is an extension of the idea of usual convergence was formerly given under the name "almost convergence" by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [43]. The concept of statistical convergence was formally introduced by Fast [16] and Steinhaus [35] independently, and later was reintroduced by Schoenberg [34], and also independently by Buck [5]. Although statistical convergence was introduced over nearly the last ninety years, it has become an active area of research for forty years with the contributions by several authors, Šalát [33], Fridy [18, 19], Di Maio and Kočinac [13], Çakallı and Khan [8]. Statistical convergence has many applications in different fields of mathematics, see [6, 10, 13, 21, 28, 36] etc.

The concept of ideal convergence (or $\mathcal{I}$-convergence) of real sequences was introduced by Nuray and Ruckle in [31] who called it generalized statistical convergence as a generalization of statistical convergence, and also independently by Kostyrko, Šalát, and Wilczynski in [22]. Over the last 20 years a lot of work has been done on $\mathcal{I}$-convergence and associated topics, for more details see [11, 12, 24, 30, 39–42] etc.

A choice of a suitable definition of distance between images naturally leads to an environment in which many possible metrics can be considered simultaneously and cone metric spaces lend themselves to this requirement. One specific instance of this is in the analysis of the structural similarity (SSIM) index of images. SSIM is used to improve the measuring of visual distortion between images. In both of these contexts the difference between two images is calculated using multiple criteria, which leads in a natural way to consider vector-valued distances. In 1934, Kurepa [23] introduced an abstract metric space, in which the metric takes values in an ordered vector space. The metric spaces with vector valued are studied under various names [29, 38]. Huang and Zhang in [25] called such spaces as cone metric spaces. Beg,

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Abbas, and Nazir [4], Beg, Azam, and Arshad [3] replaced the set of an ordered Banach space by a locally convex Hausdorff topological vector space in the definition of a cone metric and a generalized cone metric space. The connection between topological vector space valued cone metric spaces and standard metric spaces and the respective fixed point results were considered by several authors [9, 14, 17, 20]. A lot of work has been done on the theory of cone metric spaces, see [1, 7, 25, 26, 37] etc. S.K. Pal et al. studied $\mathcal{I}$ and $\mathcal{I}^*$-Cauchy sequences in cone metric spaces [32]. K. Li et al. investigated statistical convergence in cone metric spaces, they discussed statistically-sequentially compact cone metric spaces and characterized statistical completeness of cone metric spaces [28].

In this paper, we consider some properties of $\mathcal{I}$-convergence in cone metric spaces which are not considered in [32]. We prove that: (a) If $\mathcal{I}$ is a P-ideal, then the following are equivalent for a cone metric space (X, d): (1) X is $\mathcal{I}$-sequentially compact; (2) X is $\mathcal{I}$-sequentially countably compact; (3) X is compact; (4) X is countably compact; (b) A cone metric space (X, d) is $\mathcal{I}$-complete if and only if for each decreasing sequence $\{F_n\}$ of non-empty $\mathcal{I}$-closed sets of X, if there is a sequence $\{b_n\}$ converging to 0 in E^+ such that b_n is an upper bound of the set $\{F_n\}$ for each $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point. Let ℓ_∞ and $C(\mathcal{I})$ be the set of all bounded sequences and the set of all bounded $\mathcal{I}$-convergent sequences of (X, d), respectively. We show that: If (X, d) is a totally bounded complete cone metric space and $\mathcal{I}$ is an admissible ideal on $\mathbb{N}$, then $C(\mathcal{I}) = \ell_\infty$ if and only if $\mathcal{I}$ is maximal. We also construct a bounded sequence in an infinite discrete metric space which is not $\mathcal{I}$-convergent, which gives a negative answer to [12, Open problem 2.3].

2. Preliminaries

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers. Readers may consult [15] for notation and terminology not given here.

Definition 2.1. ([2]) Let E be a real Banach space and P a subset of E. We call P a *cone* and (E, P) a *cone space* if

(C1) P is non-empty, closed, and $P \neq \{0\}$;
(C2) $0 \leq a, b \in \mathbb{R}$ and $x, y \in P \Rightarrow ax + by \in P$;
(C3) $x \in P$ and $-x \in P \Rightarrow x = 0$.

A partial ordering $\leq$ with respect to P is defined by $x \leq y \Leftrightarrow y - x \in P$, and $x < y \Leftrightarrow x \leq y$ and $x \neq y$. $x \ll y$ indicates that $y - x \in \text{int}P$, where int$P$ denotes the interior of P (with the topology of the Banach space E). The relation $\ll$ is transitive and antisymmetric but not in general reflective. In this paper, we always assume that int$P \neq \emptyset$, and denote $E^+ = \{c \in E : 0 \ll c\}$, i.e., $E^+ = \text{int}P$.

Let $c \in E^+$ and $e \in E$. If $\{a_n\}$ is a non-negative sequence in $\mathbb{R}$ such that it converges to 0, it is clear that the sequence $\{c - a_n e\}$ in E converges to c. So there is $n \in \mathbb{N}$ such that $c - a_n e \in E^+$, i.e., $0 \ll c - a_n e$. It follows that $a_n e \ll c$ for some $n \in \mathbb{N}$.

Definition 2.2. ([25]) Let (E, P) be a cone space, X a non-empty set and $d : X \times X \to E$ a mapping that satisfies the following conditions:

($CM1$) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;
($CM2$) $d(x, y) = d(y, x)$ for all $x, y \in X$;
($CM3$) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X and (X, E, P, d) (or shortly, (X, d)) a *cone metric space*.

It is obvious that every metric space is a cone metric space. Every cone metric space (X, d) is a topological space [37]. In fact, for any $c \in E^+$, let $B(x, c) = \{y \in X : d(x, y) \ll c\}$ (a c-ball in a cone metric space). Then

$$\mathcal{B} = \{B(x, c) : x \in X,\ c \in E^+\}$$

is a base of a topology $\tau_d = \{U \subseteq X : \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U\}$ on X. It can be shown that the topology τ_d is Hausdorff and first-countable [37]. A subset A of X is said to be *upper bounded* [37] if there exists $c \in E^+$ such that $d(x, y) \leq c$ for all $x, y \in A$; the c is called an *upper bound* of A. It is clear that a subset A of X is upper bounded if and only if there are $c \in E^+$ and $x_0 \in X$ such that $A \subseteq B(x_0, c)$.

Definition 2.3. ([32]) Let (X, d) be a cone space and $\{x_n\}$ a sequence in X.

(1) If $x \in X$ and for each $c \in E^+$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$, then $\{x_n\}$ is said to be *convergent* and $\{x_n\}$ converges to x.

(2) If for each $c \in E^+$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m > n_0$, then $\{x_n\}$ is called a *Cauchy sequence* in X.

(3) (X, d) is said to be *complete* if every Cauchy sequence in X is convergent in X.

Let $\mathcal{I}$ be a family of non-empty subsets on $\mathbb{N}$, $\mathcal{I}$ is said to be an *ideal* if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, (ii) $A \in \mathcal{I}$, $B \subseteq A$ implies $B \in \mathcal{I}$. An ideal $\mathcal{I}$ is said to be *non-trivial* if $\mathbb{N} \notin \mathcal{I}$ and $\mathcal{I} \neq \{\emptyset\}$. The family of sets $\mathcal{F}(\mathcal{I}) = \{\mathbb{N} - A : A \in \mathcal{I}\}$ is a filter called the *associated filter* of $\mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called *admissible* if $\mathcal{I} \supseteq \{\{x\} : x \in \mathbb{N}\}$. An admissible ideal $\mathcal{I}$ is said to satisfy the *condition (AP)* (or is called a *P-ideal* or sometimes *AP-ideal*) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ from $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j$ is finite for each $j \in \mathbb{N}$ and $\bigcup_{k=1}^{\infty} B_k \in \mathcal{I}$ [12]. It is clear that $B_j \in \mathcal{I}$ for each $j \in \mathbb{N}$. In the following, if no otherwise specified, we always consider $\mathcal{I}$ is an admissible ideal on the set $\mathbb{N}$.

Let X be a topological space. A sequence $\{x_n\}$ in X is said to be *$\mathcal{I}$-convergent* to a point $x \in X$ if for every neighborhood U of x, we have the set $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$, which is denoted by $x_n \xrightarrow{\mathcal{I}} x$ or $x = \mathcal{I}\text{-}\lim x_n$ [22]. Especially, if $\mathcal{I}$ is the class $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$, then $\mathcal{I}_f$ is an admissible ideal and $\mathcal{I}_f$-convergence coincides with the usual convergence of sequences; if $\mathcal{I}_d$ is the class of all $A \subseteq \mathbb{N}$ with $d(A) = 0$, where $d(A)$ denotes the asymptotic density of a set A, then $\mathcal{I}_d$ is an admissible ideal and $\mathcal{I}_d$-convergence coincides with the statistical convergence. A set $P \subseteq X$ is said to be an *$\mathcal{I}$-closed set* of X if whenever a sequence $\{x_n\}$ in P with $x_n \xrightarrow{\mathcal{I}} x$ in X, the $\mathcal{I}$-limit point $x \in P$ [30]. By means of $\mathcal{I}$-convergence, for each $F \subseteq X$, put $[F]_{\mathcal{I}_s} = \{x \in X :$ there is a sequence $\{x_n\}$ in F such that $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = x\}$, which is called the *$\mathcal{I}_s$-hull* of the set F in X. Thus a set F is an $\mathcal{I}$-closed subset in X if and only if $F = [F]_{\mathcal{I}_s}$.

Definition 2.4. ([25]) Let (X, d) be a cone space, $\{x_n\}$ be a sequence in X and $x \in X$.

(1) If for every $c \in E^+$ the set $\{n \in \mathbb{N} : d(x_n, x) \ll c\} \in \mathcal{F}(\mathcal{I})$, then $\{x_n\}$ is said to be *$\mathcal{I}$-convergent* to x and we write $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = x$.

(2) The sequence $\{x_n\}$ is said to be *$\mathcal{I}$-Cauchy* if for every $c \in E^+$ there exists $n_0 \in \mathbb{N}$ such that $\{n \in \mathbb{N} : d(x_n, x_{n_0}) \ll c\} \in \mathcal{F}(\mathcal{I})$.

Lemma 2.5. ([27, Theorem 8 (i)]) *If $\mathcal{I}$ is a P-ideal and (X, τ) a first-countable space, then for an arbitrary sequence $\{x_n\}$ in X, $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = x$ implies $\mathcal{I}^*\text{-}\lim_{n\to\infty} x_n = x$, i.e., there is $K \in \mathcal{F}(\mathcal{I})$ such that $\{x_n\}_{n\in K}$ converges to x.*

3. Main results

Definition 3.1. Let (X, d) be a cone metric space, and $F \subseteq X$. Put

$$F^{\mathcal{I}-d} = \{x \in X : x \in [F \setminus \{x\}]_{\mathcal{I}_s}\}.$$

The set $F^{\mathcal{I}-d}$ is called the *$\mathcal{I}$-sequential derived* set of F in X. Every point in $F^{\mathcal{I}-d}$ is called an *$\mathcal{I}$-sequential accumulation point* of F.

A point $x \in X$ is called an *$\mathcal{I}$-sequential accumulation point* of a sequence $\{x_n\}$ in a cone metric space (X, d), if there is a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $\mathcal{I}\text{-}\lim_{k\to\infty} x_{n_k} = x$.

Definition 3.2. Let (X, d) be a cone metric space.

(1) A subset F of X is said to be *$\mathcal{I}$-sequentially countably compact* if any infinite subset of F has at least one $\mathcal{I}$-sequential accumulation point in F.

(2) A subset F of X is said to be *$\mathcal{I}$-sequentially compact* if for any sequence $\{x_n\}$ in F there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ is $\mathcal{I}$-convergent to $x \in F$.

Definition 3.3. ([37]) Let $\mathcal{A}$ be an open cover of a cone metric space (X, d). An element $c \in E^+$ is called a *Lebesgue element* for the cover $\mathcal{A}$ if a subset B of X has an upper bound c, then $B \subseteq A$ for some $A \in \mathcal{A}$.

Lemma 3.4. *Let $\mathcal{I}$ be a P-ideal, then every open cover of an $\mathcal{I}$-sequentially compact cone metric space has a Lebesgue element.*

Proof. Let (X, d) be an $\mathcal{I}$-sequentially compact cone metric space, and $\mathcal{A} = \{A_\alpha\}_{\alpha\in\Lambda}$ an open cover of (X, d). Without loss of generality, we may assume that $X \notin \mathcal{A}$. Suppose that $\mathcal{A}$ does not have a Lebesgue element. Fix $c \in E^+$. Then, for each $n \in \mathbb{N}$, there is a non-empty subset B_n of X such that $\frac{c}{n}$ is an upper bound of B_n and $B_n \nsubseteq A_\alpha$ for each $\alpha \in \Lambda$. Choosing $x_n \in B_n$ for each $n \in \mathbb{N}$. Since X is $\mathcal{I}$-sequentially compact, there is a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $\{x_{n_k}\}$ $\mathcal{I}$-converges to $x \in X$. Since $\mathcal{I}$ is a P-ideal, by Lemma 2.5, there exists a subsequence $x_{n_{k_m}}$ of the sequence x_{n_k} such that $x_{n_{k_m}}$ converges to x. Note that $\mathcal{A}$ is an open cover of X, there exists $\alpha_0 \in \Lambda$ such that $x \in A_{\alpha_0}$. Find $c_1 \in E^+$ such that $B(x, c_1) \subseteq A_{\alpha_0}$. Hence there is $m_0 \in \mathbb{N}$ such that $d(x, x_{n_{k_{m_0}}}) \ll \frac{c_1}{2}$ and $\frac{c_1}{n_{k_{m_0}}} \ll \frac{c_1}{2}$. If $y \in B_{n_{k_{m_0}}}$, then

$$d(x, y) \le d(x, x_{n_{k_{m_0}}}) + d(x_{n_{k_{m_0}}}, y) \ll \frac{c_1}{2} + \frac{c_1}{n_{k_{m_0}}} \ll \frac{c_1}{2} + \frac{c_1}{2} = c_1.$$

It follows that $B_{n_{k_{m_0}}} \subseteq B(x, c_1) \subseteq A_{\alpha_0}$, which is a contradiction. Thus $\mathcal{A}$ has a Lebesgue element. □

Now we can prove our first main result.

Theorem 3.5. *Let $\mathcal{I}$ be a P-ideal, then the following are equivalent for a cone metric space (X, d):*
(1) *X is $\mathcal{I}$-sequentially compact;*
(2) *X is $\mathcal{I}$-sequentially countably compact;*
(3) *X is compact;*
(4) *X is countably compact.*

Proof. (3) ⇒ (4) is clear. Since every cone metric space is first-countable, (4) ⇒ (2) holds. We will show that (2) ⇒ (1) and (1) ⇒ (3).

(2) ⇒ (1). Let $\{x_n\}$ be a sequence in X. Put $A = \{x_n : n \in \mathbb{N}\}$. We may assume that A is an infinite set. Since X is $\mathcal{I}$-sequentially countably compact, there exists $x \in A^{\mathcal{I}-d}$. Thus there is a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $\mathcal{I}\text{-}\lim_{k\to\infty} x_{n_k} = x$, which shows that X is $\mathcal{I}$-sequentially compact.

(1) ⇒ (3). Assume that (X, d) is an $\mathcal{I}$-sequentially compact cone metric space.

Claim: For each $c \in E^+$, the open covering $\{B(x, c)\}_{x\in X}$ of X has a finite covering.

If this fails to be true, there exists $c \in E^+$ such that X cannot be covered by finitely many c-balls. Therefore, we can construct a sequence $\{x_n\}$ in X as follows: First, fix a point $x_1 \in X$, and take a point $x_2 \in X \setminus B(x_1, c)$ by $X \neq B(x_1, c)$. In general, given $\{x_i\}_{i\le n}$ in X, choose a point $x_{n+1} \in X \setminus \bigcup_{i\le n} B(x_i, c)$ because $X \neq \bigcup_{i\le n} B(x_i, c)$. Then $d(x_{n+1}, x_i) \not\ll c$ for each $i \le n$, thus $\{x_n : n \in \mathbb{N}\}$ is a closed discrete subspace of X. It follows that the sequence $\{x_n : n \in \mathbb{N}\}$ does not contain any $\mathcal{I}$-convergent subsequence, which is a contradiction.

We will prove that X is compact. Assume that $\mathcal{U}$ is an open cover of X. By Lemma 3.4, there is $\delta \in E^+$ such that δ is a Lebesgue element for the open cover $\mathcal{U}$. Put $c = \frac{\delta}{3}$. There exists a finite subset F of X such that $X = \bigcup_{x\in F} B(x, c)$. For each $x \in F$, since $2c$ is an upper bound of the set $B(x, c)$, there is $U_x \in \mathcal{U}$ such that $B(x, c) \subseteq U_x$. Hence, $\{U_x\}_{x\in F}$ is a finite subcover of $\mathcal{U}$. Therefore, X is compact. □

Proposition 3.6. *Let (X, d) be an $\mathcal{I}$-sequentially compact cone metric space and $F \subseteq X$. If F is $\mathcal{I}$-closed, then F is $\mathcal{I}$-sequentially compact.*

Proof. Let $\{x_n\}$ be an arbitrary sequence in F. Since X is $\mathcal{I}$-sequentially compact, there is a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $\mathcal{I}\text{-}\lim_{k\to\infty} x_{n_k} = x \in X$. It follows from F is $\mathcal{I}$-closed that $x \in F$, i.e., $\mathcal{I}\text{-}\lim_{k\to\infty} x_{n_k} = x \in F$. Thus F is $\mathcal{I}$-sequentially compact in X. □

A cone metric space (X, d) is said to be $\mathcal{I}$*-complete* if every $\mathcal{I}$-Cauchy sequence in (X, d) is $\mathcal{I}$-convergent. Using $\mathcal{I}$-closed sets, we have a useful criterion for $\mathcal{I}$-completeness of cone metric spaces.

Theorem 3.7. *A cone metric space (X, d) is $\mathcal{I}$-complete if and only if for each decreasing sequence $\{F_n\}$ of non-empty $\mathcal{I}$-closed sets of X, if there is a sequence $\{b_n\}$ converging to 0 in E^+ such that b_n is an upper bound of the set $\{F_n\}$ for each $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.*

Proof. Assume that the cone metric space (X, d) is $\mathcal{I}$-complete. Let $\{F_n\}$ be a decreasing sequence of $\mathcal{I}$-closed non-empty sets such that there is a sequence $\{b_n\}$ converging to 0 in E^+ and b_n is an upper bound of the set $\{F_n\}$ for each $n \in \mathbb{N}$. Choosing $x_n \in F_n$ for each $n \in \mathbb{N}$, then the sequence $\{x_n\}$ is Cauchy, and hence $\{x_n\}$ is $\mathcal{I}$-Cauchy. Since the space (X, d) is $\mathcal{I}$-complete, it follows that the sequence $\{x_n\}$ is $\mathcal{I}$-convergent to some $x \in X$. Noting that $\{x_{n+k} : k \in \mathbb{N}\} \subseteq F_n$ for each $n \in \mathbb{N}$, then the sequence $\{x_{n+k}\}$ is $\mathcal{I}$-convergent to x as $k \to \infty$. Since $\{F_n\}$ is $\mathcal{I}$-closed, it follows that $x \in F_n$. Therefore, $x \in \bigcap_{n \in \mathbb{N}} F_n$. If $y \in \bigcap_{n \in \mathbb{N}} F_n$, then $x, y \in F_n$ for each $n \in \mathbb{N}$. Thus $0 \leq d(x, y) \ll b_n$, and therefore $d(x, y) \ll c$ for each $c \in E^+$. Hence, we have $d(x, y) = 0$, which means that $x = y$, i.e., $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Conversely, suppose that $\{x_n\}$ is an $\mathcal{I}$-Cauchy sequence in (X, d). Fix $e \in E^+$. For each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $\{n \in \mathbb{N} : d(x_n, x_{n_k}) \ll \frac{e}{2^{k+4}}\} \in \mathcal{F}(\mathcal{I})$. Since the ideal $\mathcal{I}$ is non-trivial, every element in $\mathcal{F}(\mathcal{I})$ is infinite. We can assume that $n_k < n_{k+1}$ and $d(x_{n_{k+1}}, x_{n_k}) \leq \frac{e}{2^{k+4}}$ for each $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, let $b_k = \frac{e}{2^k}$ and

$$F_k = [B(x_{n_k}, \frac{e}{2^{k+2}})]_{\mathcal{I}_s} = [\{y \in X : d(x_k, y) \ll \frac{e}{2^{k+2}}\}]_{\mathcal{I}_s}.$$

Then $\lim_{k \to \infty} b_k = 0$ and b_k is an upper bound of F_k. If $y \in F_{k+1}$, then $d(y, x_{n_{k+1}}) \leq \frac{e}{2^{k+3}}$ and $d(x_{n_{k+1}}, x_{n_k}) \leq \frac{e}{2^{k+4}}$, thus

$$d(y, x_{n_k}) \leq d(y, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k}) \leq \frac{e}{2^{k+2}}.$$

Hence $y \in F_k$. It follows that $F_{k+1} \subseteq F_k$. By hypothesis, there is $x \in \bigcap_{n=1}^{\infty} F_n$.

We will show that $\mathcal{I}\text{-}\lim_{n \to \infty} x_n = x$. For any $c \in E^+$, since $\lim_{n \to \infty} \frac{e}{2^n} = 0$ in E^+, there exists $k \in \mathbb{N}$ such that $\frac{e}{2^n} \ll c$ for each $n \geq k$. Since $x \in F_k$, we have $d(x, x_{n_k}) \leq \frac{e}{2^{k+2}}$. Therefore, for each $n > n_k$, if $d(x_n, x_{n_k}) \ll \frac{e}{2^{k+4}}$,

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \leq \frac{e}{2^{k+4}} + \frac{e}{2^{k+2}} \ll c.$$

It follows that

$$\{n \in \mathbb{N} : d(x_n, x_{n_k}) \ll \frac{e}{2^{k+4}}\} \subseteq \{n \in \mathbb{N} : d(x_n, x) \ll c\}.$$

Since $\{n \in \mathbb{N} : d(x_n, x_{n_k}) \ll \frac{e}{2^{k+4}}\} \in \mathcal{F}(\mathcal{I})$, we can conclude that $\{n \in \mathbb{N} : d(x_n, x) \ll c\} \in \mathcal{F}(\mathcal{I})$. So the sequence $\{x_n\}$ $\mathcal{I}$-converges to x. Hence (X, d) is $\mathcal{I}$-complete. □

Corollary 3.8. *Let $\mathcal{I}$ be a P-ideal, then every compact cone metric space is $\mathcal{I}$-complete.*

Proof. Let (X, d) be a compact cone metric space. Assume that $\{F_n\}$ is a decreasing sequence of $\mathcal{I}$-closed non-empty subsets of X, and there is a sequence $\{b_n\}$ converging to 0 in E^+ such that b_n is an upper bound of the set $\{F_n\}$ for each $n \in \mathbb{N}$. By Proposition 3.6 and Theorem 3.5, $\{F_n\}$ is compact for each $n \in \mathbb{N}$. Therefore, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Since b_n is an upper bound of the set $\{F_n\}$ for each $n \in \mathbb{N}$ and $\{b_n\}$ converges to 0 in E^+, we can deduce that $\bigcap_{n=1}^{\infty} F_n$ is a single-point set. According to Theorem 3.7, (X, d) is $\mathcal{I}$-complete. □

Using Zorn's lemma, we can show that in the family of all admissible ideals of $\mathbb{N}$, there exists a maximal ideal (with respect to inclusion).

Lemma 3.9. ([12]) *Let $\mathcal{I}_0$ be an admissible ideal on $\mathbb{N}$. Then $\mathcal{I}_0$ is maximal if and only if*

$$(A \in \mathcal{I}_0) \vee (\mathbb{N} \setminus A \in \mathcal{I}_0)$$

holds for each $A \subseteq \mathbb{N}$.

Let ℓ_∞ be the set of all bounded sequences of real numbers, and $C(\mathcal{I})$ the set of all bounded $\mathcal{I}$-convergent real sequences. P. Das proved the following result:

Theorem 3.10. ([12]) *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. Then $C(\mathcal{I}) = \ell_\infty$ if and only if $\mathcal{I}$ is a maximal ideal in $\mathbb{N}$.*

Thus P. Das posed the following open problem:

Question 3.11. ([12, Open problem 2.3]) *Can the above result be extended to an arbitrary metric space or uniform space?*

Let (X, d) be a cone metric space. A subset A of X is said to be a *c-net* in X if $X = \bigcup_{z\in A} B(z, c)$, where $B(z, c) = \{y \in X : d(z, y) \ll c\}$ for a fixed element c of E^+ [37]. (X, d) is called *totally bounded* if it has a finite c-net in X for each $c \in E^+$ [37]. A subspace (A, d_A) of (X, d) is said to be *totally bounded* if it is totally bounded as a cone metric space in its own right. A sequence $\{x_n\} \subseteq X$ is said to be *bounded* if the set $\{x_n : n \in \mathbb{N}\}$ is upper bounded. Also, let ℓ_∞ and $C(\mathcal{I})$ be the set of all bounded sequences and the set of all bounded $\mathcal{I}$-convergent sequences of (X, d), respectively. The proof of the following lemma is similar to [15, Theorem 4.3.2], so we omit it.

Lemma 3.12. *If (X, d) is a totally bounded cone metric space, then for every subset M of X the space (M, d) is totally bounded.*

We will prove the third main result in the following.

Theorem 3.13. *Let (X, d) be a totally bounded complete cone metric space. Then $C(\mathcal{I}) = \ell_\infty$ if and only if $\mathcal{I}$ is a maximal ideal.*

Proof. Let $\{x_n\} \in \ell_\infty$. We will show that $\{x_n\}$ is $\mathcal{I}$-convergent. Since $\{x_n\} \in \ell_\infty$, there are $x_0 \in X$ and $\epsilon_0 > 0$ such that $\{x_n : n \in \mathbb{N}\} \subseteq \overline{B(x_0, \epsilon_0)}$. Since X is totally bounded, there exist $y^1_1, y^1_2, \ldots, y^1_{k_1}$ such that $\overline{B(x_0, \epsilon_0)} \subseteq X = \bigcup_{i=1}^{i=k_1} B(y^1_i, \epsilon_0/2)$. Put $A^1_i = \{n : x_n \in B(y^1_i, \epsilon_0/2)\}$ for every $i \in \{1, \ldots, k_1\}$. Then $\bigcup_{i=1}^{i=k_1} A^1_i = \mathbb{N}$. Since $\mathcal{I}$ is a nontrivial ideal, there is $A^1_{i_1} \notin \mathcal{I}$ for some $i_1 \in \{1, \ldots, k_1\}$. Set $J_1 = \overline{B(y_0, \epsilon_0)} \cap \overline{B(y^1_{i_1}, \epsilon_0/2)}$ and $D_1 = \{n : x_n \in J_1\} \supseteq A^1_{i_1}$. Since $A^1_{i_1} \notin \mathcal{I}$, $D_1 \notin \mathcal{I}$.

Proceeding as above we can construct by induction a sequence of closed sets $J_1 \supseteq J_2 \supseteq \ldots, J_n = J_{n-1} \cap \overline{B(y^n_{i_n}, \epsilon_0/2^n)}$ with $D_k = \{n : x_n \in J_k\} \notin \mathcal{I}$ and $\epsilon_0/2^{n-1}$ is an upper bound of J_n for each $n \in \mathbb{N}$. Since the cone metric space (X, d) is complete, by [32, Lemma 2], $\bigcap_{n=1}^\infty J_n \neq \emptyset$. Let $\xi \in \bigcap_{n=1}^\infty J_n$. For every $\epsilon > 0$, put $M(\epsilon) = \{n : d(x_n, \xi) \ll \epsilon\}$. For sufficiently large n we have $J_n \subseteq B(\xi, \epsilon)$. Since $D_n \notin \mathcal{I}$, we have $M(\epsilon) \notin \mathcal{I}$. Note that $\mathcal{I}$ is maximal, by Lemma 3.9 we have $\mathbb{N} \setminus M(\epsilon) \in \mathcal{I}$. Hence $M(\epsilon) = \{n : d(x_n, \xi) \ll \epsilon\} \in \mathcal{F}(\mathcal{I})$. Therefore, $\mathcal{I}$-$\lim x_n = \xi$.

Conversely, if $C(\mathcal{I}) = \ell_\infty$, then the ideal $\mathcal{I}$ is maximal. In fact, assume that $\mathcal{I}$ is not maximal, by Lemma 3.9, there is a infinite set $M = \{m_1 < m_2 < \ldots\}$ such that $M \notin \mathcal{I}$ and $\mathbb{N} \setminus M \notin \mathcal{I}$. Take two different elements $a, b \in X$. Define the sequence $\{x_n\}$ by

$$x_n = \begin{cases} a, & if\ n \in M; \\ b, & if\ n \in \mathbb{N} \setminus M. \end{cases}$$

Then $\{x_n\} \in \ell_\infty$. However, for each $c \in X$, since the cone metric space (X, d) is Hausdorff, there is $\epsilon_c \in E^+$ such that $\{a, b\} \not\subseteq B(c, \epsilon_c)$. Then the set $\{n : d(x_n, c) \ll \epsilon_c\}$ is equal to $\emptyset$ or M or $\mathbb{N} \setminus M$ and neither of these sets belong to $\mathcal{F}(\mathcal{I})$. Hence $\mathcal{I}$-$\lim x_n$ does not exist. Therefore, the ideal $\mathcal{I}$ is maximal. □

However, the following example shows that the condition totally bounded cannot be omitted in Theorem 3.13.

Example 3.14. Let (X, d) be an infinite discrete metric space. Then there is a bounded sequence which is not $\mathcal{I}$-convergent.

Proof. Since (X, d) is a discrete metric space, every sequence in X is bounded. Choose a sequence $\{x_n\}$ in X with $x_n \neq x_m$ for each $n \neq m$. For every $x \in X$, there is at most one $x_i \in \{x_n : n \in \mathbb{N}\}$ such that $x = x_i$. Therefore, $\{n : x_n \neq x\} = \mathbb{N}$ or $\mathbb{N} \setminus \{i\}$. Note that the ideal $\mathcal{I}$ is non-trivial, then $\{n : x_n \neq x\} \notin \mathcal{I}$. This implies that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is not $\mathcal{I}$-convergent to x. □

Example 3.14 also shows that Theorem 3.10 cannot be extended to an arbitrary metric space nor uniform space, which gives a negative answer to Question 3.11.

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