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Almost Schouten solitons and perfect fluid spacetimes

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Abstract. In this study, we assume that a perfect fluid is the source of the gravitational field while analyzing the solutions to the Einstein field equations. With this new and innovative approach, we study almost Schouten and gradient Schouten solitons on perfect fluid spacetimes. It is demonstrated that in a perfect fluid spacetime obeying almost Schouten solitons the spacetime represents generalized Robertson-Walker spacetime. Also, we study a perfect fluid spacetime satisfying almost Schouten solitons and gradient Schouten solitons whose potential vector field is a V-Ric vector field. Finally, we study perfect fluid spacetimes obeying almost Schouten solitons and gradient Schouten solitons whose potential vector field is a torse-forming vector field.

1. Introduction

Schouten solitons (briefly, **SS**) are the solutions of an inherent flow familiar as a Schouten flow ([3], [4]) and described as

$$\pounds_V g + 2S_t = -2\boldsymbol{\beta} g, \tag{1}$$

in which V indicates the potential vector field and the Schouten tensor S_t is described by

$$S_t = \frac{1}{n-2}(S - \frac{r}{2(n-1)}g), \tag{2}$$

$\boldsymbol{\beta} \in \mathbb{R}$, S is the Ricci tensor and r indicates the scalar curvature. This soliton is referred to be shrinking, stable or expanding for $\boldsymbol{\beta} < 0$, $\boldsymbol{\beta} = 0$ or $\boldsymbol{\beta} > 0$, respectively. The Einstein manifold is the most straightforward illustration of **SS**. In a Riemannian manifold, the author provided an example of a **SS** in [3].

We broaden the aforementioned concept, which we refer to as almost **SS**, by supposing that $\boldsymbol{\beta}$ is a smooth function.

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The gradient **SS** notion was first introduced by Catino and Mazzieri [4] and described as

$$S + \nabla^2 f = \{\frac{r}{2(n-1)} + \boldsymbol{\beta}\}g, \tag{3}$$

in which f is a smooth function and $\boldsymbol{\beta}$ indicates a real constant. In [4] both the gradient **SS** and the compact gradient **SS** are examined. They demonstrated that each compact gradient **SS** is trivial. Additionally, they showed that if a gradient type steady **SS** is complete, then it is Ricci flat and trivial. Additionally, they proved that every complete gradient shrinking **SS** in dimension 3 is isometric to a finite quotient of either $\mathbb{R}^3$ or S^3 or $\mathbb{R} \times S^2$. Pina and Menezes have characterised gradient **SS** which is complete in [19]. Recently, Borges presented a research paper [3] that used the gradient **SS**. If $r = 0$, (3) entails that gradient **SS** becomes gradient Ricci soliton.

An n-dimension Lorentzian manifold M with the Lorentzian metric g of signature $(-, \underbrace{+, +, \cdots, +}_{(n-1)\,times})$, which permits a globaly time oriented vector is named a spacetime.

The n-dimensional ($n \geq 3$) Lorentzian manifold M is called a generalized Robertson-Walker (shortly, $\mathcal{GRW}$) spacetime [1] if it is constructed as a warped product $M = -I \times \varrho^2 M^*$, where I is an open interval, M^* is an $(n-1)$-dimensional Riemannian manifold and $\varrho > 0$ is the scalar function. If M^* is of constant curvature and dim. 3, then the spacetime reduces to Robertson-Walker (briefly, $\mathcal{RW}$) spacetime. The properties of $\mathcal{GRW}$ spacetimes have been studied in ([11], [17]). Mantica and Molinari [17] have proven the subsequent theorem.

Theorem A.([17]) A Lorentzian manifold of dimension n ($n \geq 3$) is a $\mathcal{GRW}$ spacetime if and only if it admits a torse-forming vector field which is unit time-like : $\nabla_{E_1} u = \phi[E_1 + A(E_1)u]$, A is a one-form given by $g(E_1, u) = A(E_1)$ for all E_1 which is also an eigen vector of the Ricci tensor.

M is described as a perfect fluid spacetime (shortly, PF-spacetime) if its non-zero Ricci tensor S fulfills

$$S = b\eta \otimes \eta + ag, \tag{4}$$

in which a, b are scalars and a unit time-like vector ζ (velocity vector) is described as $g(E_1, \zeta) = \eta(E_1)$, η is a 1-form and $g(\zeta, \zeta) = -1$. Each and every $\mathcal{RW}$ spacetime is a PF-spacetime [18]. In dim. 4, the $\mathcal{GRW}$ spacetime turns into a PF-spacetime iff it is a $\mathcal{RW}$ spacetime. Many geometers have studied several properties of PF-spacetime ([2], [20]) and many others. Geometers called a manifold to be a quasi-Einstein manifold if S is of the shape (4).

For a gravitational constant $\boldsymbol{\kappa}$, the Einstein's field equations with vanishing cosmological constant are of the shape

$$S - \frac{\boldsymbol{r}}{2}g = \boldsymbol{\kappa T}, \tag{5}$$

where $\boldsymbol{T}$ stands for the energy momentum tensor. In PF-spacetime, $\boldsymbol{T}$ is described by

$$\boldsymbol{T} = (\sigma + p)\eta \otimes \eta + pg, \tag{6}$$

p and $\boldsymbol{\sigma}$ denote the isotropic pressure and energy density of the PF-spacetime[12]. Using equations (5) and (6) in equation (4), we acquire

$$a = \frac{\boldsymbol{\kappa}(\sigma - p)}{n-2}, \quad b = \boldsymbol{\kappa}(\sigma + p). \tag{7}$$

The equation (4) can be obtained from (5) and (6).

Additionally, p and σ are linked by an equation of state (shortly, **EoS**) of the type $p = p(\sigma)$, and the *PF*-spacetime is said to as isentropic. Further, if $\sigma = p$, the *PF*-spacetime is known as stiff matter fluid. If $p = 0$, $\sigma + p = 0$, and $p = \frac{\sigma}{3}$, respectively, the *PF*-spacetime is considered to be the dust matter fluid, the dark matter era, and the radiation era.

On the otherhand Weyl tensor plays a significant role in both geometry and relativity theory. Several researchers have characterized spacetimes with Weyl tensor. The Weyl tensor C is defined by

$$\begin{aligned} C(E_1, H_1)G_1 &= R(E_1, H_1)G_1 \\ &\quad -\frac{1}{n-2}[g(QH_1, G_1)E_1 - g(QE_1, G_1)H_1 + g(H_1, G_1)QE_1 - g(E_1, G_1)QH_1] \\ &\quad +\frac{r}{(n-1)(n-2)}[g(H_1, G_1)E_1 - g(E_1, G_1)H_1], \end{aligned}$$

E_1, H_1, G_1 any vector fields, R being the Riemann curvature tensor and Q being the Ricci operator expressed by $g(QE_1, H_1) = S(E_1, H_1)$. Moreover, we known that

$$\begin{aligned} (div\, C)(E_1, H_1)G_1 &= \frac{n-3}{n-2}[\{(\nabla_{E_1} S)(H_1, G_1) - (\nabla_{H_1} S)(E_1, G_1)\} \\ &\quad -\frac{1}{2(n-1)}\{(E_1 \boldsymbol{r})g(H_1, G_1) - (H_1 \boldsymbol{r})g(E_1, G_1)\}], \end{aligned} \tag{8}$$

'*div*' indicates the divergence. The Weyl tensor is called harmonic if $div\, C = 0$. The harmonicity of the tensor appears in conservation laws of physics.

Theorem B. ([16]) On every $\mathcal{GRW}$ spacetime $(div\, C)(E_1, H_1)G_1 = 0$ if and only if $C(E_1, H_1)\zeta = 0$ holds for dimension 4.

Many researchers have recently researched various types of solitons in spacetimes, including Ricci solitons [6], Yamabe solitons [10], gradient Ricci solitons ([9], [10]), gradient Yamabe solitons [10], *k*-almost yamabe solitons [7], gradient *m*-quasi Einstein solitons[8], respectively.

The research mentioned above inspire us to characterize almost **SS** and gradient **SS** on *PF*-spacetimes. To be more precise, we obtain the following results:

Theorem 1.1. *If a PF-spacetime permits an almost* **SS**, *then the spacetime represents a* $\mathcal{GRW}$ *spacetime and the integral curves generated by* ζ *are geodesics.*

Theorem 1.2. *If a PF-spacetime admits a gradient* **SS**, *then either*
(i) the **EoS** *of M is presented by the* $p = \frac{3-n}{n-1}\sigma$ *or,*
(ii) it presents stiff matter era or it is trivial.

Theorem 1.3. *If a PF-spacetime M permits an almost* **SS** *whose soliton vector is V-Ric, then r is constant, provided* $\psi \neq -\frac{1}{2(n-1)}$.

Theorem 1.4. *If a PF-spacetime M with V-Ric vector field admits a gradient* **SS**, *then r is constant and the scalar curvature vanishes if and only if the soliton is steady, provided* $\psi \neq -\frac{n-2}{2(n-1)}$.

Theorem 1.5. *Let M be a PF-spacetime with torse-forming vector field* ζ. *If M admits an almost* **SS**, *then*
(i) the soliton becomes **SS**
(ii) the scalar curvature is constant
(iii) $\sigma = 0$.

Theorem 1.6. *Let M be a PF-spacetime with a torse-forming vector field* ζ. *If M permits a gradient* **SS**, *then either the spacetime presents stiff matter era or the soliton is trivial.*

2. Perfect Fluid Spacetimes

Equation (4) provides the following

$$S(E_1, H_1) = ag(E_1, H_1) + b\eta(E_1)\eta(H_1). \tag{9}$$

Equation (9) implies

$$r = na - b \Rightarrow \ r \text{ is constant } \ \Leftrightarrow n(E_1 a) = (E_1 b). \tag{10}$$

Lemma 2.1. *Every PF-spacetime obeys*

$$(i)\ (\nabla_{E_1}\eta)\zeta = 0 \text{ and } (ii)\ \eta(\nabla_{E_1}\zeta) = 0. \tag{11}$$

Definition 2.2. *([14]) A vector field V is named a V-Ric vector field if*

$$\nabla_{E_1} V = \psi Q E_1 \tag{12}$$

for any E_1 and ψ = constant.

We will look at the scenario when ζ is a torse-forming vector field ([2], [24]) of the shape:

$$\nabla_{E_1}\zeta = E_1 + \eta(E_1)\zeta. \tag{13}$$

Lemma 2.3. *([23]) A PF-spacetime satisfies the following:*

$$\eta(\nabla_\zeta \zeta) = 0, \ \nabla_\zeta \zeta = 0, \tag{14}$$

$$(\nabla_{E_1}\eta)H_1 = g(E_1, H_1) + \eta(E_1)\eta(H_1), \tag{15}$$

$$R(E_1, H_1)\zeta = \eta(H_1)E_1 - \eta(E_1)H_1, \tag{16}$$

$$S(E_1, \zeta) = -(1 - n)\eta(E_1), \tag{17}$$

$$(\pounds_\zeta g)(E_1, H_1) = 2[g(E_1, H_1) + \eta(E_1)\eta(H_1)]. \tag{18}$$

3. Proof of the Main Theorems

Proof of Theorem 1.1. Equations (1) and (2) together imply

$$(\pounds_V g)(E_1, H_1) + \frac{2}{n-2}S(E_1, H_1) + [2\boldsymbol{\beta} - \frac{r}{(1-n)(2-n)}]g(E_1, H_1) = 0, \tag{19}$$

which implies

$$g(\nabla_{E_1}V, H_1) + g(E_1, \nabla_{H_1}V) + \frac{2}{n-2}S(E_1, H_1) + [2\boldsymbol{\beta} - \frac{r}{(n-2)(n-1)}]g(E_1, H_1) = 0. \tag{20}$$

Putting $V = \zeta$ in (20), we obtain

$$g(\nabla_{E_1}\zeta, H_1) + g(E_1, \nabla_{H_1}\zeta) + \frac{2}{n-2}S(E_1, H_1) + [2\boldsymbol{\beta} - \frac{r}{(n-1)(n-2)}]g(E_1, H_1) = 0. \tag{21}$$

Again, putting $E_1 = H_1 = \zeta$ in (21) entails that

$$\boldsymbol{\beta} = \frac{1}{n-2}[b - a + \frac{r}{2(n-1)}]. \tag{22}$$

Using (9) and (22) in (21), we get

$$g(\nabla_{E_1}\zeta, H_1) + g(E_1, \nabla_{H_1}\zeta) + \frac{2b}{n-2}[g(E_1, H_1) + \eta(E_1)\eta(H_1)] = 0. \tag{23}$$

Putting $E_1 = \zeta$ in (23), we infer

$$\nabla_\zeta \zeta = 0. \tag{24}$$

Contracting (23), we obtain

$$div\, \zeta = -b\frac{n-1}{n-2} = \Theta\, (say). \tag{25}$$

Using (25) in (23) gives

$$g(\nabla_{E_1}\zeta, H_1) + g(E_1, \nabla_{H_1}\zeta) - \frac{2\Theta}{n-1}[g(E_1, H_1) + \eta(E_1)\eta(H_1)] = 0. \tag{26}$$

This shows that the velocity vector field is shear-free ($\sigma(E_1, H_1) = 0$), where

$$\begin{aligned} \sigma(E_1, H_1) \quad = \quad & \frac{1}{2}[g(\nabla_{E_1}\zeta, H_1) + g(E_1, \nabla_{H_1}\zeta) - \frac{2\Theta}{n-1}\{g(E_1, H_1) + \eta(E_1)\eta(H_1)\}] \\ & + \frac{1}{2}[g(\nabla_\zeta \zeta, H_1)\eta(E_1) + g(\nabla_\zeta \zeta, E_1)\eta(H_1)]. \end{aligned} \tag{27}$$

If an **EoS** holds for the perfect fluid in the Einstein equation and it satisfies the shear-free conjucture by Eills [21], then $\Theta\omega = 0$, where ω^2 is the square of the vorticity tensor. With $\omega(E_1, H_1) = 0$ the velocity of the perfect fluid gives

$$\nabla_{E_1}\zeta = \frac{\Theta}{n-1}[E_1 + \eta(E_1)\zeta]$$

hence the spacetime presents a $\mathcal{GRW}$ spacetime.
Therefore, the proof is finished.

Proof of Theorem 1.2. Equation (3) provides

$$\nabla_{E_1}Df + QE_1 = (\frac{r}{2(n-1)} + \boldsymbol{\beta})E_1. \tag{28}$$

Equation (28) implies that

$$\nabla_{H_1}\nabla_{E_1}Df + \nabla_{H_1}QE_1 = \frac{H_1 r}{2(n-1)}E_1 + (\frac{r}{2(n-1)} + \boldsymbol{\beta})\nabla_{H_1}E_1. \tag{29}$$

Interchanging E_1 and H_1 from the above equation, we provide

$$\nabla_{E_1}\nabla_{H_1}Df + \nabla_{E_1}QH_1 = \frac{E_1 r}{2(n-1)}H_1 + (\frac{r}{2(n-1)} + \boldsymbol{\beta})\nabla_{E_1}H_1. \tag{30}$$

Again, equation (28) implies

$$\nabla_{[E_1,H_1]}Df + Q([E_1,H_1]) = (\frac{r}{2(n-1)} + \beta)([E_1,H_1]). \tag{31}$$

Above equations together imply

$$\begin{aligned} R(E_1,H_1)Df &= -(\nabla_{E_1}Q)H_1 + (\nabla_{H_1}Q)E_1 \\ &\quad + \frac{E_1 r}{2(n-1)}H_1 - \frac{H_1 r}{2(n-1)}E_1. \end{aligned} \tag{32}$$

Contracting E_1 from (32), we infer

$$S(H_1, Df) = 0. \tag{33}$$

Substituting E_1 by Df in (9) yields

$$S(H_1, Df) = a(H_1 f) + b(\zeta f)\eta(H_1). \tag{34}$$

In view of (33) and (34), we obtain

$$a(H_1 f) + b(\zeta f)\eta(H_1) = 0. \tag{35}$$

Putting $H_1 = \zeta$ in (35) infers

$$(b - a)\zeta f = 0, \tag{36}$$

which implies either $b = a$ or, $b \neq a$.

Case i: If $b = a$, then (7) yields

$$p = \frac{3-n}{n-1}\sigma. \tag{37}$$

Hence the **EoS** of the spacetime is presented by $p = \frac{3-n}{n-1}\sigma$.

Case ii: If $b \neq a$, then (36) implies $\zeta f = 0$. Hence (35) implies

$$a(H_1 f) = 0, \tag{38}$$

which implies either $a = 0$ or, $H_1 f = 0$.

Case 1: If $a = 0$, then from (7), we find $p = \sigma$. Hence it represents stiff matter era [22].

Case 2: If $H_1 f = 0$, then f =constant. Hence the soliton is trivial.
This finishes the proof.

Consequently, from the foregoing theorem for $r = 0$, we obtain:

Corollary 3.1. *If a PF-spacetime M admits a gradient Ricci soliton, then either*
(i) the **EoS** *of M is presented by the $p = \frac{3-n}{n-1}\sigma$ or,*
(ii) it presents stiff matter era or it is trivial.

In particular, for dimension 4, we get form (37) $3p + \sigma = 0$. Therefore, the spacetime represents Phantom era [5]. As a result, we argue:

Corollary 3.2. *If a 4-dimensional PF-spacetime permits a gradient* **SS***, then the spacetime represents Phantom era, provided $\zeta f \neq 0$.*

Proof of Theorem 1.3. From equations (1) and (2), we provide

$$g(\nabla_{E_1}V,H_1)+g(E_1,\nabla_{H_1}V)+\frac{2}{n-2}S(E_1,H_1)+[2\boldsymbol{\beta}-\frac{r}{(n-2)(n-1)}]g(E_1,H_1)=0. \tag{39}$$

Using (12) in (39) gives

$$[2\psi+\frac{2}{n-2}]S(E_1,H_1)+[2\boldsymbol{\beta}-\frac{r}{(n-2)(n-1)}]g(E_1,H_1)=0. \tag{40}$$

Contracting the above equation, we get

$$(2\psi+\frac{1}{n-1})r=-2\boldsymbol{\beta}n, \tag{41}$$

which implies r is constant for $\psi\neq-\frac{1}{2(n-1)}$.
Hence the proof is completed.

Again for dimension 4, from Theorem B, we obtain that in a $\mathcal{GRW}$ spacetime, $C(E_1,H_1)\zeta=0$ iff $(div\, C)(E_1,H_1)G_1=0$. Also, $C(E_1,H_1)\zeta=0$ implies C is purely electric [13] and therefore the spacetime is of Petrov classification I, D or O ([22], p. 73). Thus, we have:

Corollary 3.3. *If a 4-dimensional PF-spacetime permits an almost* **SS** *whose soliton vector is V-Ric, then the spacetime is of Petrov type I, D or O.*

In 4-dimension, $C(E_1,H_1)\zeta=0$ is equivalent to ([15], p. 128)

$$\begin{aligned}&\eta(Z_5)C(E_1,H_1,G_1,Z_4)+\eta(E_1)C(H_1,Z_5,G_1,Z_4)\\&\quad+\eta(H_1)C(Z_5,E_1,G_1,Z_4)=0,\end{aligned} \tag{42}$$

where $\eta(E_1)=g(E_1,\zeta)$ and $C(E_1,H_1,G_1,Z_4)=g(C(E_1,H_1)G_1,Z_4)$ for all E_1, H_1, G_1, Z_4, Z_5.
Replacing Z_5 by ζ in the last equation yields

$$C(E_1,H_1,G_1,Z_4)=0, \tag{43}$$

which tells that the spacetime is conformally flat. Hence, we have:

Corollary 3.4. *If a PF-spacetime of dim. 4 permits an almost* **SS** *whose soliton vector is V-Ric, then the spacetime is conformally flat.*

Again, for dimension 4 and we take $\boldsymbol{\beta}=0$, equation (40) implies

$$(2\psi+1)S(E_1,H_1)-\frac{r}{6}g(E_1,H_1)=0. \tag{44}$$

Contracting the above equation, we get either $r=0$ or $\psi=-\frac{1}{6}$. Again, using $r=0$ in (10) gives $b-4a=0$ and therefore from (7), we acquire $\boldsymbol{\sigma}-3p=0$. Therefore, the spacetime presents radiation era. Hence we have:

Corollary 3.5. *If a 4-dimensional PF-spacetime admits an almost* **SS** *whose soliton vector is V-Ric, then the spacetime represents radiation era, provided* $\psi\neq-\frac{1}{6}$*.*

Proof of Theorem 1.4. Equation (3) yields

$$\nabla_{E_1} Df + QE_1 = (\frac{r}{2(n-1)} + \boldsymbol{\beta})E_1. \tag{45}$$

Again, replacing V by Df in (12) gives

$$\nabla_{E_1} Df = \psi QE_1. \tag{46}$$

In view of (45) and (46), we provide

$$(\psi + 1)S(E_1, H_1) = (\frac{r}{2(n-1)} + \boldsymbol{\beta})g(E_1, H_1). \tag{47}$$

Contracting E_1 and H_1 in the previous equation infers

$$r(\psi + \frac{n-2}{2(n-1)}) = \boldsymbol{\beta} n, \tag{48}$$

which means that r is constant for $\psi \neq -\frac{n-2}{2(n-1)}$.
Hence the proof is completed.

In particular, for $n = 4$ and $\boldsymbol{\beta} = 0$, equation (48) implies $r = 0$. Hence, we get $4a - b = 0$. Therefore, from (7), we get $3p - \boldsymbol{\sigma} = 0$. Hence the spacetime represents radiation era [5]. Hence we have:

Corollary 3.6. *If a 4-dimensional PF-spacetime admits a steady gradient* **SS** *whose potential vector field is V-Ric, then the spacetime represents radiation era, provided* $\psi \neq -\frac{1}{3}$.

Again, for $r = 0$, equation (48) implies $\boldsymbol{\beta} = 0$. Hence the soliton is steady. Hence we write:

Corollary 3.7. *If a PF-spacetime permits a gradient Ricci soliton whose potential vector field is V-Ric, then the soliton is steady.*

Proof of Theorem 1.5. Equations (1) and (2) together provide

$$(\pounds_V g)(E_1, H_1) + \frac{2}{n-2}S(E_1, H_1) + [2\boldsymbol{\beta} - \frac{r}{(1-n)(2-n)}]g(E_1, H_1) = 0. \tag{49}$$

Putting $V = \zeta$ in (49) and using (18) yields

$$2[g(E_1, H_1) + \eta(E_1)\eta(H_1)] + \frac{2}{n-2}S(E_1, H_1) + [2\boldsymbol{\beta} - \frac{r}{(n-2)(n-1)}]g(E_1, H_1) = 0. \tag{50}$$

Contracting E_1 and H_1 in (50) gives

$$2\boldsymbol{\beta} n + 2(n+1) + \frac{r}{n-1} = 0. \tag{51}$$

Setting $H_1 = \zeta$ in (50) entails that

$$2\boldsymbol{\beta} = \frac{r}{(1-n)(2-n)} - \frac{2(1-n)}{2-n}. \tag{52}$$

In view of (51) and (52), we provide

$$r = 2(n-1) = \textit{constnat and hence } \boldsymbol{\beta} = \textit{constant}. \tag{53}$$

Hence from (10), we find

$$na - b = 2(n-1). \tag{54}$$

Again, putting $H_1 = \zeta$ in (9) and comparing with (17), we obtain

$$a - b = n - 1. \tag{55}$$

From (54) and (55), we infer

$$a = 1 \textit{ and } b = 2 - n. \tag{56}$$

Equations (7) and (56) together imply $\boldsymbol{\sigma} = 0$.
This finishes the proof.

Now, we state a theorem:
Theorem C. ([22], p. 601) All conformally flat perfect fluid spacetime with non-zero energy density are of embedding class one (that is, can be embedded as a hypersurface of a Minkowski spacetime), and are hence all contained either in the generalized Schwarzschild metrics or generalized Friedmann metrics. All conformally flat solutions with zero energy density are flat (Minkowski spacetime).
Hence, from the above result, we have:

Corollary 3.8. *Let M be a conformally flat PF-spacetime with torse-forming vector field ζ. If M admits an almost* **SS**, *then*
(i) the soliton becomes **SS**
(ii) the scalar curvature is constant
(iii) the spacetime becomes flat.

Proof of Theorem 1.6. Equation (3) implies

$$\nabla_{E_1} Df + QE_1 = \left(\frac{r}{2(n-1)} + \boldsymbol{\beta}\right)E_1. \tag{57}$$

The above equation implies

$$\nabla_{H_1}\nabla_{E_1} Df + \nabla_{H_1} QE_1 = \frac{1}{2}\left[\frac{H_1 r}{(n-1)}E_1 + \left(\frac{r}{(n-1)} + 2\boldsymbol{\beta}\right)\nabla_{H_1} E_1\right]. \tag{58}$$

Interchanging E_1 and H_1 from the foregoing equation, we acquire

$$\nabla_{E_1}\nabla_{H_1} Df + \nabla_{E_1} QH_1 = \frac{1}{2}\left[\frac{E_1 r}{(n-1)}H_1 + \left(\frac{r}{(n-1)} + 2\boldsymbol{\beta}\right)\nabla_{E_1} H_1\right]. \tag{59}$$

Equation (57) implies

$$\nabla_{[E_1,H_1]} Df + Q([E_1,H_1]) = \left(\frac{r}{2(n-1)} + \boldsymbol{\beta}\right)([E_1,H_1]). \tag{60}$$

Above equations together imply

$$\begin{aligned} R(E_1,H_1)Df &= -(\nabla_{E_1}Q)H_1 + (\nabla_{H_1}Q)E_1 \\ &\quad + \frac{1}{2}\left[\frac{E_1 r}{(n-1)}H_1 - \frac{H_1 r}{(n-1)}E_1\right]. \end{aligned} \tag{61}$$

Contracting E_1 from (61), we obtain

$$S(H_1, Df) = 0. \tag{62}$$

From (17) and (62), we get

$$\zeta f = 0. \tag{63}$$

Replacing E_1 by Df in (9) and comparing with (62), we get

$$a(H_1 f) + b(\zeta f)\eta(H_1) = 0. \tag{64}$$

Using (63) in (64), we get

$$a(H_1 f) = 0. \tag{65}$$

which implies either $a = 0$ or, $H_1 f = 0$.

Case 1: If $a = 0$, then from (7), we find $p = \boldsymbol{\sigma}$. Hence it represents stiff matter era [22].

Case 2: If $H_1 f = 0$, then f =constant. Hence the soliton is trivial.
Thus, the proof is completed.

Conclusions

In their purest form, solitons are nothing more than waves. After colliding with another wave of the same kind, waves physically propagate with the least amount of energy loss and maintain their speed and shape. Solitons play a key role in the resolution of initial-value problems for wave propagation-related nonlinear PDEs.

Several researchers have studied different types of solitons in *PF*-spacetime. In this study, we investigate the almost **SS** and gradient **SS** in *PF*-spacetime.

In this article it is proved that if a *PF*-spacetime admits an almost **SS** then the spacetime presents a $\mathcal{GRW}$ spacetime and the integral curves generated by ζ are geodesics. Also, we show that if a *PF*-spacetime admits an almost **SS** whose soliton vector is V-Ric, then the scalar curvature is constant. Next, we show that if M be a *PF*-spacetime equipped with a torse-forming vector field ζ and M admits a gradient **SS**, then either M represents a stiff matter era or the soliton is trivial.

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