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Solvability of nonlinear multivalued elliptic problems involving degenerate coercivity with variable exponents and L^1−data.

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Abstract. This paper is interested in studying existence results for a general class of nonlinear elliptic problems with variable exponents associated with the differential inclusion and degenerate coercivity. We prove an existence of entropy solutions for this non-coercive differential inclusion. Moreover, we will give some regularity results for these solutions.

1. Introduction

In this paper, we deal with the following nonlinear multivalued elliptic problem with Dirichlet boundary conditions:

$$(S,f)\quad \begin{cases} \gamma(u) - div\left(\dfrac{a(x,\nabla u)}{(1+|u|)^{\theta(p(x)-1)}}\right) \ni f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, with sufficiently smooth boundary denoted by $\partial\Omega$, γ is a maximal monotone graph such that $0 \in \gamma(0)$ and $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function which verifies the natural extensions of Leray-Lions assumptions to the variable exponents case such that $p(\cdot)$ satisfies the log-Hölder continuity condition (see (1) below) such that $1 < p^- \leq p^+ < N$. As well as the right-hand side $f \in L^1(\Omega)$.

We point out that, in the case of constant exponents, the solvability of the problem (S,f) is very well understood (see [2]). In this previous paper, if $f \in L^\infty(\Omega)$, we proved the existence and uniqueness of a bounded weak energy solution to the problem (S,f). While if $f \in L^1(\Omega)$, we established the existence and some regularity results for the so-called entropy solution to the problem (S,f) only for $0 \leq \theta(p-1) < \frac{N(p-1)}{N-1} - 1$.

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More precisely, that is a pair of functions (u, w) satisfies the two conditions:
(1) $u : \Omega \to \mathbb{R}$ is measurable and $w \in L^1(\Omega)$ such that $u(x) \in D(\gamma(x))$ and $w(x) \in \gamma(u(x))$ for almost every x in Ω.
(2) For every $k > 0$, $T_k(u) \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega wT_k(u-v)dx + \int_\Omega \frac{a(x,\nabla u)}{(1+|u|)^{\theta(p(x)-1)}} \cdot \nabla T_k(u-v)dx \leq \int_\Omega fT_k(u-v)dx$$

holds for all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Moreover, we showed that $u \in W_0^{1,q}(\Omega)$, with $q \in \left(1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}\right)$.

The natural framework to solve the problem (S, f) is that of Sobolev spaces with variable exponents. In recent years, the study of variational problems and partial differential equations with non-standard growth conditions has received considerable attention from many models coming from various fields, including biology, engineering, finance and many branches of mathematical physics, such as elastic mechanics, image processing and electro-rheological fluid dynamics, etc. There is a lengthy list of references regarding open problems and recent developments can be found in Diening et al. [10].

Later, in the absence of the maximal monotone graph ($\gamma \equiv 0$) and for $\theta = 0$, Sanchón and Urbano in [18] obtained the existence and uniqueness of an entropy solution to the $p(x)$-Laplacian problem.

In another important work [20], Wittbold and Zimmermann adapted the notion of renormalized solution to a new and interesting elliptic problem type diffusion-convection in the framework of variable exponent Sobolev spaces

$$\begin{cases} \gamma(u) - diva(x, \nabla u) - divF(u) \ni f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

They proved for F locally Lipschitz continuous, γ maximal monotone mapping with $0 \in \gamma(0)$, $f \in L^1(\Omega)$ and a continuous exponent $p(\cdot)$, the existence and uniqueness of a renormalized solution.

Recently, Zhang and Fu [21] established a partial generalization of results from the pioneering paper [3] of Alvino et al. to the variable exponents case. More precisely, they considered the nonlinear elliptic equation (S, f), in the case where $\gamma \equiv 0$, with principal part having degenerate coercivity, $p(\cdot) \in C(\overline{\Omega})$ such that $1 < p^- \leq p^+ < N$ and $f \in L^{m(x)}(\Omega)$. If $m^- > \frac{N}{p^-}$, they proved the existence of a bounded weak energy solution to (S, f) for $0 \leq \theta \leq \frac{p^--1}{p^+-1}$. While if $m^- \leq \frac{N}{p^-}$, the existence of weak or entropic solutions is achieved only for $0 \leq \theta < \frac{p^--1}{p^+-1}$.

To our knowledge, there is no literature has considered the non-coercive multivalued problem (S, f) in the setting of Sobolev spaces with variable exponents $W_0^{1,p(x)}(\Omega)$. Motivated by [21] and [18], it is our purpose in this paper to extend our results in [2] to the variable exponent case. We will study the existence and some $q(x)$-regularity results of entropy solution to the nonlinear multivalued elliptic problem (S, f) with L^1-data and for any maximal monotone graph γ where the functional setting involves Sobolev spaces with variable exponents.

We want to mention that there are some difficulties associated with these kinds of problems:

Firstly, $Au := -div\left(\frac{a(x,\nabla u)}{(1+|u|)^{\theta(p(x)-1)}}\right)$ is a monotone operator from $W_0^{1,p(x)}(\Omega)$ into its dual. However, when $0 < \theta < 1$, the term $A(u)$ becomes degenerate, which can result in a slow diffusion effect when the solution u becomes large. As a result of this degeneracy, classical methods for elliptic equations are not applicable, even when the datum f is regular.

The second difficulty arises when a variable exponential growth condition is given for $a(\cdot, \cdot)$ in the equation. In this case, the operator A exhibits a more complex nonlinearity, making some techniques used in the constant exponent case inapplicable.

Considering the fact that the problem (S, f) involves data $f \in L^1(\Omega)$, it is pertinent to explore the concept of entropy solutions, which require less regularity than the usual weak solutions. The concept of entropy solution was first introduced by Bénilan et al. in [5] for nonlinear elliptic problems with constant p. Not long ago, in [18], Sanchón and Urbano investigated the Dirichlet problem associated with the $p(x)$-Laplace equation and established the existence and uniqueness of entropy solutions for L^1-data, as well as integrability results for the solution and its gradient.

In this article, we utilize the approximation procedure to obtain the results. Specifically, we focus on a sequence of nondegenerate Dirichlet problems, which ensures the existence of solutions. By obtaining a priori estimates on the approximate solutions and subsequently taking the limit, we are able to find a solution for (S, f).

The present paper is structured as follows: Section 2 provides definitions and results related to variable exponent Sobolev spaces. In section 3, we outline the assumptions and introduce the concepts of weak and entropy solutions for the problem (S, f), we also establish the existence of our main result. Section 4 introduces and solves approximating problems for any L^∞−data f, which is necessary to prove the main result. Finally, section 5 focuses on proving the existence of entropy solutions and some regularity results when $f \in L^1(\Omega)$.

2. Preliminaries and notations

2.1. Lebesgue and Sobolev spaces with variable exponents

In the following, we will briefly review some definitions and fundamental properties of Lebesgue and Sobolev spaces with variable exponents for the reader's convenience. We suggest the reader to [8–14, 22–24] and the references therein for more background about these spaces.

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, we say that a real-valued continuous function $p(\cdot)$ is log-Hölder continuous on $\overline{\Omega}$ if

$$|p(x) - p(y)| \leq \frac{C}{-log|x-y|} \text{ for all } x, y \in \overline{\Omega} \text{ such that } 0 < |x-y| \leq \frac{1}{2}, \tag{1}$$

where C is a constant.

We denote

$$C_+(\overline{\Omega}) = \Big\{ \text{log-Hölder continuous function } p(\cdot) : \overline{\Omega} \to \mathbb{R} \text{ such that } 1 < p^- \leq p(x) \leq p^+ < N \Big\},$$

where $p^- = \min_{x\in\overline{\Omega}} p(x)$ and $p^+ = \max_{x\in\overline{\Omega}} p(x)$.

We define the variable exponent Lebesgue space for $p(\cdot) \in C_+(\overline{\Omega})$ by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

The space $L^{p(x)}(\Omega)$ under the norm

$$||u||_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}$$

is a uniformly convex Banach space and therefore reflexive. The conjugate space of $L^{p(x)}(\Omega)$ will be denoted by $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Proposition 2.1 (See [13], [22]).

i) For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the Hölder type inequality

$$\left| \int_\Omega uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) ||u||_{p(x)} ||v||_{p'(x)}.$$

ii) For all $p_1(\cdot), p_2(\cdot) \in C_+(\overline{\Omega})$ such that $p_1(x) \le p_2(x)$, we have

$$L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$$

and the embedding is continuous.

Proposition 2.2 (See [13], [22]). *If we denote*

$$\rho(u) = \int_\Omega |u(x)|^{p(x)} dx \ \ \text{for all } u \in L^{p(x)}(\Omega),$$

then, the following assertions hold

i) $\|u\|_{p(x)} < 1$ *(resp.* $= 1, > 1$*)* $\Leftrightarrow \rho(u) < 1$ *(resp.* $= 1, > 1$*).*

ii) $\|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p^-} \le \rho(u) \le \|u\|_{p(x)}^{p^+}$ *and* $\|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p^+} \le \rho(u) \le \|u\|_{p(x)}^{p^-}$.

iii) $\|u\|_{p(x)} \to 0 \Leftrightarrow \rho(u) \to 0$ *and* $\|u\|_{p(x)} \to \infty \Leftrightarrow \rho(u) \to \infty$.

Now, we define the Sobolev space with variable exponent by

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}$$

and it can be equipped with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}. \tag{2}$$

By $W_0^{1,p(x)}(\Omega)$, we denote the subspace of $W^{1,p(x)}(\Omega)$, which is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2).

The critical Sobolev exponent is defined as $p^*(x) = \frac{Np(x)}{N-p(x)}$ for $p(x) < N$, and the dual space of $W_0^{1,p(x)}(\Omega)$ will be denoted by $W^{-1,p'(x)}(\Omega)$.

Proposition 2.3 (See [13], [14]).

i) The spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.

ii) If $q(\cdot) \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for evey $x \in \Omega$, then the embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$$

is continuous and compact.

iii) Poincaré's inequality: there exists a constant $C > 0$, such that

$$\|u\|_{p(x)} \le C\|\nabla u\|_{p(x)} \ \ \text{for every } u \in W_0^{1,p(x)}(\Omega). \tag{3}$$

iv) Sobolev inequality: there exists another constant $C > 0$, such that

$$\|u\|_{p^*(x)} \le C\|\nabla u\|_{p(x)} \ \ \text{for every } u \in W_0^{1,p(x)}(\Omega).$$

Remark 2.4. *The log-Hölder continuous condition* (1) *is used to obtain several regularity results for Sobolev spaces with variable exponents; specially, $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega) = W^{1,p(x)}(\Omega) \cap W_0^{1,1}(\Omega)$. Furthermore, the Sobolev embedding also holds (see [8]) for $q(x) = p^*(x)$, i.e., $W^{1,p(x)}(\Omega) \subset L^{p^*(x)}(\Omega)$.*

Remark 2.5. *By* (3)*, we deduce that* $\|u\|_{1,p(x)}$ *and* $\|\nabla u\|_{p(x)}$ *are equivalent norms on* $W_0^{1,p(x)}(\Omega)$*. Note that the following inequality*

$$\int_\Omega |u(x)|^{p(x)}dx \leq C \int_\Omega |\nabla u(x)|^{p(x)}dx,$$

in general does not hold (see [12]). But by Proposition 2.2 and (3)*, we have*

$$\int_\Omega |u(x)|^{p(x)}dx \leq C \max\left\{\|\nabla u\|_{p(x)}^{p^+}, \|\nabla u\|_{p(x)}^{p^-}\right\}.$$

2.2. Notations

Throughout the paper, we will use the following notations. Giving $E \subset \Omega$ a Lebesgue measurable set, we will denote its characteristic function by χ_E and its Lebesgue measure by $|E|$.

For any $j \geq 0$ and $v : \Omega \to \mathbb{R}$, the set $\{x \in \Omega : |v(x)| \geq (>, \leq, <, =) j\}$ will be written by $\{|v| \geq (>, \leq, <, =) j\}$.

For any given $\eta > 0$, let $h_\eta : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$h_\eta(s) = \begin{cases} 0 & \text{if } |s| \geq \eta + 1, \\ \eta + 1 - |s| & \text{if } \eta < |s| < \eta + 1, \\ 1 & \text{if } |s| \leq \eta. \end{cases}$$

For $k > 0$, the standard truncation function $T_k : \mathbb{R} \to \mathbb{R}$ is defined as

$$T_k(s) = \begin{cases} k & \text{if } s \geq k, \\ s & \text{if } |s| < k, \\ -k & \text{if } s \leq -k, \end{cases}$$

and we define $G_k : \mathbb{R} \to \mathbb{R}$ by

$$G_k(s) = \begin{cases} s - k & \text{if } s \geq k, \\ 0 & \text{if } |s| < k, \\ s + k & \text{if } s \leq -k. \end{cases}$$

Let $\delta > 0$ and $S_\delta^+ : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$S_\delta^+(s) = \begin{cases} 1 & \text{if } s \geq \delta, \\ \dfrac{s}{\delta} & \text{if } 0 < s < \delta, \\ 0 & \text{if } s \leq 0. \end{cases}$$

Obviously, S_δ^+ is an approximation of the function $sign_0^+ : \mathbb{R} \to \mathbb{R}$ defined as

$$sign_0^+(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \leq 0. \end{cases}$$

3. Essential assumptions and statements of result

3.1. Basic assumptions

The basic assumptions of the problem (S, f) are presented in the following:
Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with sufficiently smooth boundary $\partial\Omega$, $p \in C_+(\overline{\Omega})$ and let us consider $a(x, \xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ to be a Carathéodory function (i.e. $a(x, \xi)$ is continuous in ξ for a.e. $x \in \Omega$

and measurable in x for every $\xi \in \mathbb{R}^N$) such that
Assumption (H_1)

$$a(x,\xi)\cdot\xi \geq \alpha|\xi|^{p(x)}, \tag{4}$$

$$|a(x,\xi)| \leq C_0\left(a_0(x) + |\xi|^{p(x)-1}\right), \tag{5}$$

$$(a(x,\xi) - a(x,\xi'))\cdot(\xi - \xi') > 0, \tag{6}$$

for almost every $x \in \Omega$ and all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$, where $\alpha > 0$, $C_0 > 0$ and a_0 is a nonnegative function in $L^{p'(x)}(\Omega)$.
Assumption (H_2)
The nonlinearity $\gamma : \mathbb{R} \to 2^{\mathbb{R}}$ is a set valued, maximal monotone graph such that $0 \in \gamma(0)$.
Assumption (H_3)
Throughout this paper, if not otherwise specified, we always assume that

$$2 - \frac{1}{N} < p^- \leq p(x) \leq p^+ < N \text{ and } 0 \leq \theta(p^+ - 1) < \frac{N(p^- - 1)}{N-1} - 1. \tag{7}$$

3.2. Statement of the main result

Let us now turn to our main result. In the same spirit of [2], we introduce the following classes of solutions for the problem (S, f).

Definition 3.1. *Let $f \in L^\infty(\Omega)$. A weak solution of the problem (S, f) is a couple of functions $(u, w) \in W_0^{1,p(x)}(\Omega) \times L^\infty(\Omega)$ such that $w \in \gamma(u)$ a.e. in Ω and*

$$\int_\Omega wv\,dx + \int_\Omega \frac{a(x,\nabla u)}{(1+|u|)^{\theta(p(x)-1)}}\cdot\nabla v\,dx = \int_\Omega fv\,dx \text{ holds for all } v \in W_0^{1,p(x)}(\Omega). \tag{8}$$

In this paper, our assumption on the data $f \in L^1(\Omega)$ does not assure the existence of a weak solution. This forces us to use the notion of an entropy solution, which will be presented in the definition that follows:

Definition 3.2. *Let $f \in L^1(\Omega)$. A couple of measurable functions (u, w) is called entropy solution of (S, f) if this couple satisfies the two conditions:*
(C1) *$u : \Omega \to \mathbb{R}$ is measurable and $w \in L^1(\Omega)$ such that $u(x) \in D(\gamma(x))$ and $w(x) \in \gamma(u(x))$ for a.e. x in Ω.*
(C2) *For each $k > 0$, $T_k(u) \in W_0^{1,p(x)}(\Omega)$ and*

$$\int_\Omega wT_k(u-v)\,dx + \int_\Omega \frac{a(x,\nabla u)}{(1+|u|)^{\theta(p(x)-1)}}\cdot\nabla T_k(u-v)\,dx \leq \int_\Omega fT_k(u-v)\,dx \tag{9}$$

holds for all $v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$.

Remark 3.3. *We recall that the gradient of u which appears in (9) is defined as in Lemma 2.1 in [5] to be the unique measurable function $\phi : \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_k(u) = \phi\chi_{\{|u|<k\}}$$

for a.e. $x \in \Omega$ and for any $k > 0$.
Furthermore, if $u \in W_0^{1,1}(\Omega)$, then $\phi \equiv \nabla u$ in the usual weak sense.

Remark 3.4. *Observe that the use of $T_k(u-v)$ as a test function gives meaning to all the terms in* **(C2)**. *Indeed, the second integral in the left-hand side is only on the set $\{|u-v|\le k\}$, then, the integral is well defined, and on this set, we have $|u|\le k+\|v\|_\infty = M$, as a consequence, we get*

$$\int_\Omega \frac{a(x,\nabla u)}{(1+|u|)^{\theta(p(x)-1)}}\cdot \nabla T_k(u-v)dx = \int_\Omega \frac{a(x,\nabla T_M(u))}{(1+|T_M(u)|)^{\theta(p(x)-1)}}\cdot \nabla T_k(u-v)dx,$$

which is finite by the growth condition (5) *on a.*

The aim of this paper is to prove the following result stated bellow:

Theorem 3.5. *Assume that (H_1)-(H_3) hold true and $f\in L^1(\Omega)$, there exists at least one entropy solution (u,w) of the problem (S,f) in the sense of Definition 3.2. Furthermore, $u\in W_0^{1,q(x)}(\Omega)$, with $q(x)\in\left(1,\frac{(p(x)-1-\theta(p^+-1))N}{N-1-\theta(p^+-1)}\right)$.*

4. Existence of weak solutions for $f\in L^\infty(\Omega)$

We shall introduce and resolve the approximate problems listed in the subsection 4.1 before establishing the aforementioned Theorem 3.5.

For any $f\in L^1(\Omega)$ and $m,n\in\mathbb{N}$, let $f_{m,n}:\Omega\to\mathbb{R}$ be the function defined by $f_{m,n}(x)=max(-n,min(f(x),m))$ for almost every $x\in\Omega$. Evidently, one has $f_{m,n}\in L^\infty(\Omega)$ for every $m,n\in\mathbb{N}$ and $|f_{m,n}(x)|\le|f(x)|$ a.e. in Ω, as a result $\lim_{m\to\infty}\lim_{n\to\infty} f_{m,n}=f$ in $L^1(\Omega)$ and a.e. in Ω.

The following theorem will give us an existence of weak solutions $(u_{m,n},w_{m,n})$ of $(S,f_{m,n})$ for every fixed $m,n\in\mathbb{N}$:

Theorem 4.1. *Assume that (H_1)-(H_3) hold true and $f\in L^\infty(\Omega)$, there exists at least one weak solution (u,w) of the problem (S,f).*

Remark 4.2. *Observe that the assumption on f given in the above theorem yields $L^\infty(\Omega)$ solutions for nonlinear multivalued elliptic problems involving degenerate coercivity with variable exponents. The result (which is not depending on θ) is not surprising, since if one looks for bounded solutions then the lack of coercivity of the operator $A(u):=-div\left(\frac{a(x,\nabla u)}{(1+|u|)^{\theta(p(x)-1)}}\right)$ (which is created by unbounded functions) "disappears".*

We will divide our proof of the above Theorem 4.1 in several steps and we will use standard techniques.

4.1. Approximate problems

Let $f\in L^\infty(\Omega)$, we consider the approximate problem of (S,f), for $0<\varepsilon\le 1$, by

$$(S_\varepsilon,f)\quad\begin{cases}\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))+\varepsilon arctan(u_\varepsilon)+A_\varepsilon(u_\varepsilon)=f & \text{in }\Omega,\\[2ex] u_\varepsilon=0 & \text{on }\partial\Omega,\end{cases}$$

where $A_\varepsilon(u)=-div\left(\frac{a(x,\nabla u)}{(1+|T_{\frac{1}{\varepsilon}}(u)|)^{\theta(p(x)-1)}}\right)$ and $\gamma_\varepsilon(\cdot):\mathbb{R}\to\mathbb{R}$ is the Yosida regularization of $\gamma(\cdot)$ given by

$$\gamma_\varepsilon=\frac{1}{\varepsilon}\left(I-(I+\varepsilon\gamma)^{-1}\right).$$

The following property of a maximal monotone graph in $\mathbb{R}^2$ can be found, e.g. in [6, 7, 17].

Proposition 4.3. *Let $\gamma:\mathbb{R}\to 2^{\mathbb{R}}$ be any maximal monotone graph. Then, there exists a convex, lower semi-continuous and proper function $j:\mathbb{R}\to(-\infty,+\infty]$ such that γ is the subdifferential of j, i.e. $\gamma=\partial j$.*

The function j is uniquely determined up to an additive constant and it is superpositionally measurable.

Note that the proof of Proposition 4.3 provides a method for the determination of j, when γ is given.

To regularize γ, we consider

$$j_\varepsilon(s) = \min_{r\in\mathbb{R}} \left\{ \frac{1}{2\varepsilon}|s-r|^2 + j(r) \right\}, \text{ for all } s \in \mathbb{R} \text{ and any } \varepsilon > 0.$$

Thanks to [6, Proposition 2.11], we have the next results:

Proposition 4.4.

i) $D(\gamma) \subset D(j) \subset \overline{D(j)} \subset \overline{D(\gamma)}$.

ii) $j_\varepsilon(s) = \frac{\varepsilon}{2}|\gamma_\varepsilon(s)|^2 + j(J_\varepsilon(s))$ *where* $J_\varepsilon := (I + \varepsilon\gamma)^{-1}$.

iii) j_ε *is a convex, Frechet-differentiable function and* $\gamma_\varepsilon = \partial j_\varepsilon$.

iv) $j_\varepsilon(s) \uparrow j(s)$ *as* $\varepsilon \downarrow 0$ *for all* $s \in \mathbb{R}$.

Moreover, for any $\varepsilon > 0$, γ_ε is a nondecreasing and Lipschitz-continuous function.

We introduce the operator $B_\varepsilon : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$, defined by

$$\langle B_\varepsilon u, v\rangle = \int_\Omega \Big(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u)) + \varepsilon arctan(u) + A_\varepsilon(u)\Big) v dx, \text{ for any } u, v \in W_0^{1,p(x)}(\Omega).$$

Lemma 4.5. *The operator* $B_\varepsilon : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ *is bounded, coercive and pseudo-monotone.*

Proof. Due to (5) and (6), B_ε is monotone and well-defined. As *arctan* and $\gamma_\varepsilon \circ T_{\frac{1}{\varepsilon}}$ are continuous and bounded, again by the growth assumption (5), we conclude that B_ε is pseudo-monotone (see [16], p.157).

Next, for all $u \in W_0^{1,p(x)}(\Omega)$, one has

$$\langle B_\varepsilon u, u\rangle = \int_\Omega \frac{a(x,\nabla u)\cdot\nabla u}{(1+|T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta(p(x)-1)}} dx + \int_\Omega \Big(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u)) + \varepsilon arctan(u)\Big) u dx,$$

since the second term on the right hand side is nonnegative, from (4), we get

$$\begin{aligned}\langle B_\varepsilon u, u\rangle &\geq \frac{\alpha}{(1+\frac{1}{\varepsilon})^{\theta(p^+-1)}} \int_\Omega |\nabla u|^{p(x)} dx \\ &\geq \alpha' \|\nabla u\|_{p(x)}^{\rho},\end{aligned}$$

with

$$\rho = \begin{cases} p^- & \text{if } \|\nabla u\|_{p(x)} \geq 1, \\ p^+ & \text{if } \|\nabla u\|_{p(x)} < 1. \end{cases}$$

Hence, B_ε is coercive on $W_0^{1,p(x)}(\Omega)$. □

In light of Lemma 4.5, for evey fixed $\varepsilon > 0$, the problem (S_ε, f) has at least one solution u_ε (see [15], [16]). In other word, for any $0 < \varepsilon \leq 1$ and $f \in W^{-1,p'(x)}(\Omega)$ there exists at least one solution $u_\varepsilon \in W_0^{1,p(x)}(\Omega)$ such that

$$\int_\Omega \Big(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\Big) v dx + \int_\Omega \frac{a(x,\nabla u_\varepsilon)\cdot\nabla v}{(1+|T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta(p(x)-1)}} dx = \langle f, v\rangle \tag{10}$$

holds for all $v \in W_0^{1,p(x)}(\Omega)$, where $\langle .,. \rangle$ denotes the duality pairing between $W_0^{1,p(x)}(\Omega)$ and $W^{-1,p'(x)}(\Omega)$.

The following proposition gives the uniqueness of solutions u_ε of (S_ε, f) with the right-hand side $f \in L^\infty(\Omega)$ through a comparison principle that will play an important role in the approximation of solutions to (S, f) with $f \in L^1(\Omega)$.

Proposition 4.6. *For $0 < \varepsilon \leq 1$ fixed and $f, \tilde{f} \in L^\infty(\Omega)$, let $u_\varepsilon, \tilde{u}_\varepsilon \in W_0^{1,p(x)}(\Omega)$ be solutions of (S_ε, f) and $(S_\varepsilon, \tilde{f})$, respectively. Then, the following comparison principle holds*

$$\varepsilon \int_\Omega (arctan(u_\varepsilon) - arctan(\tilde{u}_\varepsilon))^+ dx \leq \int_\Omega (f - \tilde{f}) sign_0^+(u_\varepsilon - \tilde{u}_\varepsilon) dx. \tag{11}$$

Proof. By taking $S_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon)$ as a test function in (10) for u_ε and $\tilde{u}_\varepsilon$. Subtracting the resulting inequalities, we get

$$I^1_{\varepsilon,\delta} + I^2_{\varepsilon,\delta} + I^3_{\varepsilon,\delta} = I^4_{\varepsilon,\delta}$$

where

$$I^1_{\varepsilon,\delta} = \int_\Omega \left(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon))\right) S_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon) dx,$$

$$I^2_{\varepsilon,\delta} = \varepsilon \int_\Omega (arctan(u_\varepsilon) - arctan(\tilde{u}_\varepsilon)) S_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon) dx,$$

$$I^3_{\varepsilon,\delta} = \int_\Omega \left(\frac{a(x, \nabla u_\varepsilon)}{(1 + |T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta(p(x)-1)}} - \frac{a(x, \nabla \tilde{u}_\varepsilon)}{(1 + |T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon)|)^{\theta(p(x)-1)}} \right) \cdot \nabla S_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon) dx,$$

$$I^4_{\varepsilon,\delta} = \int_\Omega (f - \tilde{f}) S_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon) dx.$$

It is easy to see that, for $\tau \geq 0$ and $b \geq a \geq 0$, the following inequality holds

$$\left| \frac{1}{(1+a)^\tau} - \frac{1}{(1+b)^\tau} \right| = \left| \frac{\tau(a-b)}{(1+c)^{\tau+1}} \right| \leq \tau |a - b| \text{ for some } c \in [a, b]. \tag{12}$$

By using (5) and (12), we have

$$\begin{aligned} I^3_{\varepsilon,\delta} &\geq \frac{1}{\delta} \int_{\{(u_\varepsilon - \tilde{u}_\varepsilon)^+ < \delta\}} \left(\frac{1}{(1 + |T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta(p(x)-1)}} - \frac{1}{(1 + |T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon)|)^{\theta(p(x)-1)}} \right) a(x, \nabla \tilde{u}_\varepsilon) \cdot \nabla (u_\varepsilon - \tilde{u}_\varepsilon)^+ dx \\ &\geq -C \int_\Omega \left(a_0(x) + |\nabla \tilde{u}_\varepsilon|^{p^+-1} \right) \cdot |\nabla (u_\varepsilon - \tilde{u}_\varepsilon)^+| \chi_{\{(u_\varepsilon - \tilde{u}_\varepsilon)^+ < \delta\}} dx, \end{aligned}$$

then $\liminf_{\delta \to 0} I^3_{\varepsilon,\delta} \geq 0$.

By letting $\delta \downarrow 0$, we deduce (11). □

Remark 4.7. *Let $u_\varepsilon, \tilde{u}_\varepsilon \in W_0^{1,p(x)}(\Omega)$ be solutions of (S_ε, f) and $(S_\varepsilon, \tilde{f})$, respectively, where $f, \tilde{f} \in L^\infty(\Omega)$ be such that $f \leq \tilde{f}$ a.e. in Ω and $\varepsilon > 0$. So, it is a direct result of Proposition 4.6 that $u_\varepsilon \leq \tilde{u}_\varepsilon$ a.e. in Ω. Moreover, by the monotonicity of $\gamma_\varepsilon \circ T_{\frac{1}{\varepsilon}}$ it follows that also $\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \leq \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon))$ a.e. in Ω.*

Next, we need a priori estimates on the weak solution u_ε of approximate problems (S_ε, f) and show that the solution has the regularity needed to give sense to the weak formulation and that it is possible to pass to the limit in the part with the differential operator.

4.2. A priori estimates

If we choose the test function $G_k(H(u_\varepsilon))$ in the weak formulation (10), where

$$H(s) = \int_0^s \frac{dt}{(1+t)^\theta},$$

we obtain

$$\int_{X_k} \Big(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\Big) G_k(H(u_\varepsilon))dx + \int_{X_k} \frac{a(x,\nabla u_\varepsilon)\cdot \nabla u_\varepsilon}{(1+|T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta p(x)}}dx = \int_{X_k} f G_k(H(u_\varepsilon))dx,$$

such that $X_k = \{|H(u_\varepsilon)| > k\}$.

By using (4) and the fact that γ_ε is nondecreasing with $\gamma_\varepsilon(0) = 0$, it follows that

$$\alpha \int_\Omega |\nabla G_k(H(u_\varepsilon))|^{p(x)} dx \leq \int_\Omega |f||G_k(H(u_\varepsilon))|dx. \tag{13}$$

In view of the Sobolev inequality, we have

$$\int_\Omega |\nabla G_k(H(u_\varepsilon))|^{p(x)} dx \geq C_1 \left(\int_\Omega |G_k(H(u_\varepsilon))|^{p^*(x)} dx \right)^{\frac{\rho_2}{\rho_1}}, \tag{14}$$

where

$$\rho_1 = \begin{cases} p^{*+} & \text{if } \ \|G_k(H(u_\varepsilon))\|_{p^*(x)} \geq 1, \\ p^{*-} & \text{if } \ \|G_k(H(u_\varepsilon))\|_{p^*(x)} < 1. \end{cases} ; \rho_2 = \begin{cases} p^{+} & \text{if } \ \|\nabla G_k(H(u_\varepsilon))\|_{p(x)} < 1, \\ p^{-} & \text{if } \ \|\nabla G_k(H(u_\varepsilon))\|_{p(x)} \geq 1. \end{cases}$$

By (13) and (14), we obtain

$$\begin{aligned} \left(\int_\Omega |G_k(H(u_\varepsilon))|^{p^*(x)} dx \right)^{\frac{\rho_2}{\rho_1}} &\leq C_2 \|\chi_{X_k}\|_{p^{*\prime}(x)} \|G_k(H(u_\varepsilon))\|_{p^*((x)} \\ &\leq C \|\chi_{X_k}\|_{p^{*\prime}(x)} \left(\int_\Omega |G_k(H(u_\varepsilon))|^{p^*(x)} dx \right)^{\frac{1}{\rho_1}}. \end{aligned}$$

Thus

$$\left(\int_\Omega |G_k(H(u_\varepsilon))|^{p^*(x)} dx \right)^{\frac{\rho_2 - 1}{\rho_1}} \leq C_3 |X_k|^{\frac{1}{\rho_3}}$$

where

$$\rho_3 = \begin{cases} p^{*\prime +} & \text{if } \ \|\chi_{X_k}\|_{p^{*\prime}(x)} < 1, \\ p^{*\prime -} & \text{if } \ \|\chi_{X_k}\|_{p^{*\prime}(x)} \geq 1. \end{cases}$$

Let $k' > k$, it is clear that $X_{k'} \subset X_k$ and

$$|G_k(H(u_\varepsilon))| \geq k' - k$$

on the set $X_{k'}$. Hence

$$|X_{k'}| \leq \frac{C}{(k'-k)^{p^{*-}}} |X_k|^{\frac{\rho_1}{(\rho_2 - 1)\rho_3}}. \tag{15}$$

We have $\frac{\rho_1}{(\rho_2-1)\rho_3} > 1$, as a result, by Stampacchia's Lemma, $H(u_\varepsilon)$ is uniformly bounded (see [19, Lemma 4.1]).

The properties of the function H (especially the fact that $\lim_{s\to\infty} H(s) = \infty$) give a bound for u_ε in $L^\infty(\Omega)$, that is

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leq C_\infty, \tag{16}$$

with C_∞ is a constant that don't depend on ε.

Next, by choosing u_ε as a test function in (10), we get

$$\int_\Omega \Big(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\Big) u_\varepsilon dx + \int_\Omega \frac{a(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon}{(1+|T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta(p(x)-1)}} dx = \int_\Omega f u_\varepsilon dx,$$

since the first term on the left hand side is nonnegative, from (4), we obtain

$$\alpha \int_\Omega \frac{|\nabla u_\varepsilon|^{p(x)}}{(1+|u_\varepsilon|)^{\theta(p^+-1)}} dx \leq \int_\Omega |f||u_\varepsilon| dx.$$

As the norm of u_ε in $L^\infty(\Omega)$ is bounded, we conclude that

$$\|\nabla u\|_{p(x)}^{\rho_4} \leq \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx \leq C \tag{17}$$

where

$$\rho_4 = \begin{cases} p^- & \text{if } \|\nabla u\|_{p(x)} \geq 1, \\ p^+ & \text{if } \|\nabla u\|_{p(x)} < 1. \end{cases}$$

On the other hand, by using the same arguments as in [2], i.e. taking $\frac{1}{\delta}(T_{k+\delta}(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) - T_k(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))))$ as a test function in (10), passing to the limit with $\delta \to 0$ and choosing $k > \|f\|_\infty$, we obtain

$$\|\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\|_\infty \leq \|f\|_\infty. \tag{18}$$

4.3. Basic convergence results

This step's goal is to present the convergence results that are required for the proof that solutions exist. Taking into account (18), there exists $w \in L^\infty(\Omega)$ such that

$$\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \overset{*}{\rightharpoonup} w \text{ in } L^\infty(\Omega). \tag{19}$$

According to (17), u_ε remains bounded in $W_0^{1,p(x)}(\Omega)$, we deduce that, there exist a subsequence (not relabeled) and a function u such that

$$u_\varepsilon \rightharpoonup u \text{ weakly in } W_0^{1,p(x)}(\Omega) \text{ and a.e. in } \Omega. \tag{20}$$

We will demonstrate that

$$u_\varepsilon \to u \text{ strongly in } W_0^{1,p(x)}(\Omega). \tag{21}$$

By choosing $u_\varepsilon - u$ as test function in (10), we get that, for $\dfrac{1}{\varepsilon} > C_\infty$,

$$\int_\Omega \Big(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\Big)(u_\varepsilon - u) dx + \int_\Omega \frac{a(x,\nabla u_\varepsilon)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}} \cdot (\nabla u_\varepsilon - \nabla u) dx = \int_\Omega f(u_\varepsilon - u) dx. \tag{22}$$

Concerning the first term on the left-hand side of (22). In virtue of (20), we get $u_\varepsilon \to u$ in $L^1(\Omega)$ and by (19), we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega \Big(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\Big)(u_\varepsilon - u)dx = 0. \tag{23}$$

Likewise, since f belongs to $L^\infty(\Omega)$, we have

$$\lim_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon - u)dx = 0. \tag{24}$$

On the other hand, by (23) and (24) one has

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{a(x,\nabla u_\varepsilon)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}}\cdot(\nabla u_\varepsilon - \nabla u)dx = 0.$$

By using (5) and Vitali's Theorem, we get that

$$\frac{a(x,\nabla u)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}} \to \frac{a(x,\nabla u)}{(1+|u|)^{\theta(p(x)-1)}} \text{ strongly in } (L^{p'(x)}(\Omega))^N.$$

We deduce, by (20), that

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{a(x,\nabla u)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}}\cdot(\nabla u_\varepsilon - \nabla u)dx = 0.$$

Then, we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega \left(\frac{a(x,\nabla u_\varepsilon)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}} - \frac{a(x,\nabla u)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}}\right)\cdot(\nabla u_\varepsilon - \nabla u)dx = 0. \tag{25}$$

According to (6), the integrand function in the left-hand side in (25) is nonnegative, so,

$$\left(\frac{a(x,\nabla u_\varepsilon)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}} - \frac{a(x,\nabla u)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}}\right)\cdot(\nabla u_\varepsilon - \nabla u) \to 0 \text{ in } L^1(\Omega).$$

So, up to a subsequence still indexed by ε, we have

$$\left(\frac{a(x,\nabla u_\varepsilon(x))}{(1+|u_\varepsilon(x)|)^{\theta(p(x)-1)}} - \frac{a(x,\nabla u(x))}{(1+|u_\varepsilon(x)|)^{\theta(p(x)-1)}}\right)\cdot(\nabla u_\varepsilon(x) - \nabla u(x)) \to 0$$

for almost every x in Ω.

Consequently, there exists a subset Γ of Ω of zero measure, such that for every x in $\Omega\backslash\Gamma$, one has

$$D_\varepsilon(x) = \left(\frac{a(x,\nabla u_\varepsilon(x))}{(1+|u_\varepsilon(x)|)^{\theta(p(x)-1)}} - \frac{a(x,\nabla u(x))}{(1+|u_\varepsilon(x)|)^{\theta(p(x)-1)}}\right)\cdot(\nabla u_\varepsilon(x) - \nabla u(x)) \to 0,$$

$|u(x)| < \infty$, $|\nabla u(x)| < \infty$, $|a_0(x)| < \infty$ and $u_\varepsilon(x) \to u(x)$.

As a result, from (5), (4) and (16) we can write

$$D_\varepsilon(x) \geq \frac{\alpha}{(1+|C_\infty|)^{\theta(p^+-1)}}|\nabla u_\varepsilon(x)|^{p(x)} - c(x)(1+|\nabla u_\varepsilon(x)| + |\nabla u_\varepsilon(x)|^{p(x)-1}), \tag{26}$$

where $c(x)$ is a constant which depends on x but does not depend on ε, which shows, in view of (26), that the sequence $|\nabla u_\varepsilon(x)|$ is uniformly bounded in $\mathbb{R}^N$ with respect to ε. Now, arguing as in the same way as in Lemma 3.4. in [4], we get

$$u_\varepsilon \to u \text{ strongly in } W_0^{1,p(x)}(\Omega).$$

4.4. Proof of Theorem 4.1

Now, we can show that (u, w) satisfies (8). To do this, coming back to (10) and let $v \in W_0^{1,p(x)}(\Omega)$. For every $\frac{1}{\varepsilon} > C_\infty$ one has

$$\int_\Omega \Big(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\Big) v dx + \int_\Omega \frac{a(x, \nabla u_\varepsilon) \cdot \nabla v}{(1 + |u_\varepsilon|)^{\theta(p(x)-1)}} dx = \int_\Omega f v dx. \tag{27}$$

In view of (21), we have for a subsequence

$\nabla u_\varepsilon \to \nabla u$ a.e. in Ω,

which with

$u_\varepsilon \to u$ a.e. in Ω

yields, since $\frac{a(x, \nabla u_\varepsilon)}{(1 + |u_\varepsilon|)^{\theta(p(x)-1)}}$ is bounded in $(L^{p'(x)}(\Omega))^N$,

$$\frac{a(x, \nabla u_\varepsilon)}{(1 + |u_\varepsilon|)^{\theta(p(x)-1)}} \rightharpoonup \frac{a(x, \nabla u)}{(1 + |u|)^{\theta(p(x)-1)}} \text{ weakly in } (L^{p'(x)}(\Omega))^N.$$

Then, passing to the limit in (27), we get

$$\int_\Omega w v dx + \int_\Omega \frac{a(x, \nabla u)}{(1 + |u|)^{\theta(p(x)-1)}} \cdot \nabla v dx = \int_\Omega f v dx \text{ for any } v \in W_0^{1,p(x)}(\Omega).$$

4.5. Subdifferential argument

To demonstrate that $u(x) \in D(\gamma(x))$ and $w(x) \in \gamma(u(x))$ for almost all $x \in \Omega$, we use the same arguments as in [20].

Indeed, in view of *iii*) from Proposition 4.4, for each $0 < \varepsilon \leq 1$, it follows that

$$j_\varepsilon(s) \geq j_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + (s - T_{\frac{1}{\varepsilon}}(u_\varepsilon))\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \text{ a.e. in } \Omega, \text{ for all } s \in \mathbb{R}. \tag{28}$$

Fixing $\varepsilon_0 > 0$ and letting E be an arbitrary measurable subset of Ω, multiplying (28) by the function $h_\eta(u_\varepsilon)\chi_E$, then integrating over Ω and applying *iv*) from Proposition 4.4, we get that

$$j(s) \int_E h_\eta(u_\varepsilon) dx \geq \int_E \Big(j_{\varepsilon_0}(T_{\eta+1}(u_\varepsilon)) h_\eta(u_\varepsilon) + (s - T_{\eta+1}(u_\varepsilon)) h_\eta(u_\varepsilon)\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\Big) dx$$

for all $s \in \mathbb{R}$ and any $0 < \varepsilon < min\left(\varepsilon_0, \frac{1}{\eta}\right)$.

By letting ε goes to zero, having in the mind that E arbitrary, we get from the preceding inequality that

$$j(s) h_\eta(u) \geq j_{\varepsilon_0}(T_{\eta+1}(u)) h_\eta(u) + w h_\eta(u)(s - T_{\eta+1}(u)) \text{ a.e. in } \Omega, \text{ for all } s \in \mathbb{R}. \tag{29}$$

Letting η tends to infinity and then ε_0 goes to zero in (29), we deduce that

$$j(s) \geq j(u(x)) + w(x)(s - u(x)) \text{ a.e. in } \Omega, \text{ for all } s \in \mathbb{R}.$$

Therefore, $u \in D(\gamma)$ and $w \in \gamma(u)$ for a.e. in Ω.

We therefore complete the proof of Theorem 4.1 with this final step.

5. Proof of Theorem 3.5

5.1. *Approximation problems*

For $m, n \in \mathbb{N}$ and $f \in L^1(\Omega)$, let $f_{m,n} \in L^\infty(\Omega)$ be defined as in the beginning of section 4. In view of Theorem 4.1, there exists a solution $(u_{m,n}, w_{m,n})$ of $(S, f_{m,n})$ such that $u_{m,n} \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, $w_{m,n} \in L^\infty(\Omega)$ and

$$\int_\Omega w_{m,n} v dx + \int_\Omega \frac{a(x, \nabla u_{m,n})}{(1 + |u_{m,n}|)^{\theta(p(x)-1)}} \cdot \nabla v dx = \int_\Omega f_{m,n} v dx \tag{30}$$

holds for all $v \in W_0^{1,p(x)}(\Omega)$ and $m, n \in \mathbb{N}$.

5.2. *A priori estimates*

Claim 5.1. *Suppose $(u_{m,n}, w_{m,n})$ is a solution of $(S, f_{m,n})$. Then, for $k > 0$, there exists a constant $C(k) > 0$, not depending on $m, n \in \mathbb{N}$, such that*

$$\|T_k(u_{m,n})\|_{W_0^{1,p(x)}(\Omega)} \leq C(k), \tag{31}$$

and

$$\|w_{m,n}\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}. \tag{32}$$

Proof. By taking $T_k(u_{m,n})$ as a test function in (30), we obtain

$$\int_\Omega w_{m,n} T_k(u_{m,n}) dx + \int_\Omega \frac{a(x, \nabla u_{m,n})}{(1 + |u_{m,n}|)^{\theta(p(x)-1)}} \cdot \nabla T_k(u_{m,n}) dx = \int_\Omega f_{m,n} T_k(u_{m,n}) dx. \tag{33}$$

Since $w_{m,n}$ has the same sign as $u_{m,n}$ and from (4), we get

$$\alpha \int_\Omega |\nabla T_k(u_{m,n})|^{p(x)} dx \leq (1 + k)^{\theta(p^+-1)} k \|f\|_{L^1(\Omega)}. \tag{34}$$

Now, to obtain (32), we drop in (33) the positive term and keep

$$\int_\Omega w_{m,n} T_k(u_{m,n}) dx \leq \int_\Omega f_{m,n} T_k(u_{m,n}) dx \leq k \|f\|_{L^1(\Omega)},$$

since $w_{m,n} \in \gamma(u_{m,n})$ a.e. in Ω, it follows that

$$\int_{\{|u_{m,n}|>k\}} |w_{m,n}| dx \leq \|f\|_{L^1(\Omega)},$$

by passing to the limit with $k \downarrow 0$ and applying the Fatou's Lemma, one has

$$\int_\Omega |w_{m,n}| dx \leq \|f\|_{L^1(\Omega)}.$$

□

5.3. *Weak convergence of* $T_k(u_n)$ *in* $W_0^{1,p(x)}(\Omega)$

By definition, $f_{m,n}$ is a bi-monotone approximation of f in $L^1(\Omega)$,

$f_{m,n} \le f_{m+1,n}$ and $f_{m,n+1} \le f_{m,n}$.

According to Proposition 4.6 it follows that

$$u^{\varepsilon}_{m,n} \le u^{\varepsilon}_{m+1,n} \text{ and } u^{\varepsilon}_{m,n+1} \le u^{\varepsilon}_{m,n} \tag{35}$$

a.e. in Ω for all $m, n \in \mathbb{N}$ and any $\varepsilon > 0$.

Then, passing the limit as $\varepsilon \downarrow 0$ in (35) yields

$$u_{m,n} \le u_{m+1,n} \text{ and } u_{m,n+1} \le u_{m,n} \tag{36}$$

a.e. in Ω for all $m, n \in \mathbb{N}$.

Setting $w^{\varepsilon} := \gamma_{\varepsilon}(T_{\frac{1}{\varepsilon}}(u_{\varepsilon}))$, using (35), Remark 4.7, the fact that $w^{\varepsilon}_{m,n} \overset{*}{\rightharpoonup} w_{m,n}$ in $L^{\infty}(\Omega)$ and this convergence preserves order, we obtain

$$w_{m,n} \le w_{m+1,n} \text{ and } w_{m,n+1} \le w_{m,n} \tag{37}$$

a.e. in Ω for all $m, n \in \mathbb{N}$.

From (32) and (37), we get

$$w_{m,n} \uparrow_m w^n \downarrow_n w \text{ in } L^1(\Omega), \tag{38}$$

where $w^n, w : \Omega \to \mathbb{R}$ are measurable functions, finite a.e. in Ω. Here, we use the notation $\downarrow_n$ respectively $\uparrow_n$, to denote convergence of a sequence which is monotone decreasing, respectively increasing, in n.

In view of (36), we conclude that

$$u_{m,n} \uparrow_m u^n \downarrow_n u \text{ a.e. in } \Omega, \tag{39}$$

where $u^n, u : \Omega \to \overline{\mathbb{R}}$ are measurable functions.

So as to prove that u is finite almost everywhere, we will give an estimate on the level sets of $u_{m,n}$ in the following Claim 5.2.

Claim 5.2. *For $m, n \in \mathbb{N}$, let $(u_{m,n}, w_{m,n})$ be a solution of $(S, f_{m,n})$. Then, there exists a constant $C > 0$, not depending on $m, n \in \mathbb{N}$ such that*

$$|\{|u_{m,n}| \ge l\}| \le Cl^{-\Theta} \quad \text{for all } l \ge 1, \tag{40}$$

where $\Theta = (p^+ - 1)\left(\dfrac{p^- - 1}{p^+ - 1} - \theta\right)$.

Proof. By Chebyshev's inequality and Poincaré's inequality in $W_0^{1,p^-}(\Omega)$, also by (34), we have

$$\begin{aligned}
l^{p^-}|\{|u_{m,n}| \ge l\}| &\le \int_{\{|u_{m,n}|\ge l\}} |T_l(u_{m,n})|^{p^-} dx \\
&\le C\left(\int_{\{|\nabla T_l(u_{m,n})|<1\}} |\nabla T_l(u_{m,n})|^{p^-} dx + \int_{\{|\nabla T_l(u_{m,n})|\ge 1\}} |\nabla T_l(u_{m,n})|^{p(x)} dx\right) \\
&\le C\left(|\Omega| + \int_{\Omega} |\nabla T_l(u_{m,n})|^{p(x)} dx\right) \\
&\le Cl^{\theta(p^+-1)+1}.
\end{aligned}$$

Then, we obtain (40). □

Clearly (7) in particular implies $\Theta > 0$. Note that, as $(u_{m,n})_m$ is pointwise increasing with respect to m, we get

$$\lim_{m\to\infty} |\{u_{m,n} > l\}| = |\{u^n > l\}| \tag{41}$$

and

$$\lim_{m\to\infty} |\{u_{m,n} \le -l\}| = |\{u^n \le -l\}|. \tag{42}$$

Combining (40) with (41) and (42), we obtain

$$|\{u^n \le -l\}| + |\{u^n > l\}| \le Cl^{-\Theta} \tag{43}$$

for all $l \ge 1$, thus u^n is finite a.e. for any $n \in \mathbb{N}$.

Using similar justifications, we conclude that

$$|\{u \le -l\}| + |\{u > l\}| \le Cl^{-\Theta} \tag{44}$$

from (43), it follows that u is finite a.e. in Ω. Next, since $w_{m,n} \in \gamma(u_{m,n})$ almost everywhere in Ω we deduce, by a subdifferential argument, that $w^n \in \gamma(u^n)$ and $w \in \gamma(u)$ a.e. in Ω.

In the sequel of this section, for $m, n \in \mathbb{N}$, let $(u_{m,n}, w_{m,n})$ be a solution of $(S, f_{m,n})$. We construct a subsequence $(m(n))_n$, by applying the diagonal principle in $L^1(\Omega)$, such that

$$w_n := w_{m(n),n} \to w \text{ in } L^1\Omega) \text{ and a.e. in } \Omega,$$

$$f_n := f_{m(n),n} \to f \text{ in } L^1\Omega) \text{ and a.e. in } \Omega,$$

it follows that

$$u_n := u_{m(n),n} \to u \text{ converges a.e. in } \Omega. \tag{45}$$

Using (31) and (45) together, for any $k > 0$, we obtain

$$T_k(u_n) \rightharpoonup T_k(u) \text{ in } W_0^{1,p(x)}(\Omega). \tag{46}$$

5.4. *Some regularity results*

Claim 5.3. *There exists a subsequence of u_n such that, for each*

$$q(x) \in \left(1, \frac{(p(x) - 1 - \theta(p^+ - 1))N}{N - 1 - \theta(p^+ - 1)}\right), \tag{47}$$

we have

$$\|u_n\|_{W_0^{1,q(x)}(\Omega)} \le C. \tag{48}$$

Proof. Take $k > 0$ and denote $B_k = \{k < |u_n| \le k + 1\}$. We choose the test function $T_1(u_n - T_k(u_n))$ in the weak formulation (30), to obtain

$$\int_\Omega w_n T_1(u_n - T_k(u_n))dx + \int_\Omega \frac{a(x, \nabla u_n) \cdot \nabla T_1(u_n - T_k(u_n))}{(1 + |u_n|)^{\theta(p-1)}} dx = \int_\Omega f_n T_1(u_n - T_k(u_n))dx.$$

Since w_n has the same sign as u_n and from (4), we get

$$\alpha \int_{B_k} \frac{|\nabla u_n|^{p(x)}}{(1 + |u_n|)^{\theta(p(x)-1)}} dx \le \|f\|_{L^1(\Omega)},$$

particularly, there exists $C_1 > 0$ such that

$$\int_{B_k} \frac{|\nabla u_n|^{p^-}}{(1+|u_n|)^{\theta(p^+-1)}} dx \le C_1. \tag{49}$$

In a first step, let q^+ be a constant such that

$$1 \le q^+ < \frac{(p^- - 1 - \theta(p^+-1))N}{N - 1 - \theta(p^+-1)}. \tag{50}$$

Note that (50) in particular implies $q^+ < p^-$ and $\frac{p^- r}{p^- - q^+} < q^{+*}$ where $r = \frac{q^+\theta(p^+-1)}{p^-}$.

Next, we will estimate $\int_{B_k} |\nabla u_n|^{q^+} dx$ for all $k > 0$. According to (49) and Hölder's inequality, we conclude that

$$\begin{aligned}
\int_{B_k} |\nabla u_n|^{q^+} dx &= \int_{B_k} \frac{|\nabla u_n|^{q^+}}{(1+|u_n|)^r}(1+|u_n|)^r dx \\
&\le \left(\int_{B_k} \frac{|\nabla u_n|^{p^-}}{(1+|u_n|)^{\theta(p^+-1)}} dx\right)^{\frac{q^+}{p^-}} \left(\int_{B_k} (1+|u_n|)^{\frac{p^- r}{p^- - q^+}} dx\right)^{\frac{p^- - q^+}{p^-}} \\
&\le C_2 |B_k|^{\frac{p^- - q^+}{p^-}} + C_2 \left(\int_{B_k} |u_n|^{\frac{p^- r}{p^- - q^+}} dx\right)^{\frac{p^- - q^+}{p^-}} \\
&\le C_2 |B_k|^{\frac{p^- - q^+}{p^-}} + C_3 \left(\int_{B_k} |u_n|^{q^{+*}} dx\right)^{\frac{r}{q^{+*}}} |B_k|^{\frac{p^- - q^+}{p^-} - \frac{r}{q^{+*}}}.
\end{aligned}$$

Clearly $|B_k| \le \frac{1}{k^{q^{+*}}} \int_{B_k} |u_n|^{q^{+*}} dx$ for all $k \ge k_0 \ge 1$, one has

$$\begin{aligned}
\int_{B_k} |\nabla u_n|^{q^+} dx &\le C_2 \left(\frac{1}{k^{q^{+*}}} \int_{B_k} |u_n|^{q^{+*}} dx\right)^{\frac{p^- - q^+}{p^-}} + C_3 \frac{1}{k^{q^{+*}\left(\frac{p^- - q^+}{p^-} - \frac{r}{q^{+*}}\right)}} \left(\int_{B_k} |u_n|^{q^{+*}} dx\right)^{\frac{p^- - q^+}{p^-}} \\
&\le \frac{C_4}{k^{q^{+*}\left(\frac{p^- - q^+}{p^-} - \frac{r}{q^{+*}}\right)}} \left(\int_{B_k} |u_n|^{q^{+*}} dx\right)^{\frac{p^- - q^+}{p^-}}.
\end{aligned} \tag{51}$$

Now note that

$$\int_\Omega |\nabla u_n|^{q^+} dx = \sum_{k=0}^{k_0-1} \int_{B_k} |\nabla u_n|^{q^+} dx + \sum_{k=k_0}^{\infty} \int_{B_k} |\nabla u_n|^{q^+} dx.$$

From (49) and Hölder's inequality, we deduce

$$\begin{aligned}
\int_\Omega |\nabla u_n|^{q^+} dx &\le C_{k_0} + \sum_{k=k_0}^{\infty} \int_{B_k} |\nabla u_n|^{q^+} dx \\
&\le C_{k_0} + C_4 \left(\sum_{k=k_0}^{\infty} \frac{1}{k^{q^{+*}\left(\frac{p^- - q^+}{p^-} - \frac{r}{q^{+*}}\right)\frac{p^-}{q^+}}}\right)^{\frac{q^+}{p^-}} \left(\sum_{k=k_0}^{\infty} \int_{B_k} |u_n|^{q^{+*}} dx\right)^{\frac{p^- - q^+}{p^-}} \\
&\le C_{k_0} + C_4 \left(\sum_{k=k_0}^{\infty} \frac{1}{k^{q^{+*}\left(\frac{p^- - q^+}{p^-} - \frac{r}{q^{+*}}\right)\frac{p^-}{q^+}}}\right)^{\frac{q^+}{p^-}} \|u_n\|_{L^{q^{+*}}(\Omega)}^{\frac{q^{+*}(p^- - q^+)}{p^-}}.
\end{aligned} \tag{52}$$

Note that $\sum_{k=k_0}^{\infty} \frac{1}{k^{q^{+*}\left(\frac{p^- - q^+}{p^-} - \frac{r}{q^{+*}}\right)\frac{p^-}{q^+}}}$ converges since $q^{+*}\left(\frac{p^- - q^+}{p^-} - \frac{r}{q^{+*}}\right)\frac{p^-}{q^+} > 1$. Then

$$\int_\Omega |\nabla u_n|^{q^+}dx \leq C_{k_0} + C_5 \|u_n\|_{L^{q^{+*}}(\Omega)}^{\frac{q^{+*}(p^- - q^+)}{p^-}}.$$

Applying the Sobolev embedding $W_0^{1,q^+}(\Omega) \subset L^{q^{+*}}(\Omega)$, one gets

$$\begin{aligned}\left(\int_\Omega |u_n|^{q^{+*}}dx\right)^{\frac{q^+}{q^{+*}}} &\leq C_6 \int_\Omega |\nabla u_n|^{q^+}dx \\ &\leq C_{k_0} + C_7 \|u_n\|_{L^{q^{+*}}(\Omega)}^{\frac{q^{+*}(p^- - q^+)}{p^-}}.\end{aligned}$$

In other words,

$$\|u_n\|_{L^{q^{+*}}(\Omega)}^{q^+} \leq C_{k_0} + C_7 \|u_n\|_{L^{q^{+*}}(\Omega)}^{\varrho},$$

where $\varrho := \frac{q^{+*}(p^- - q^+)}{p^-}$. Since $p^- < N$ one has $\varrho < q^+$ and as a result, we get that

$$\|u_n\|_{W_0^{1,q^+}(\Omega)} \leq C,$$

for some constant $C > 0$. As a consequence, there exists a constant $C' > 0$ such that

$$\|u_n\|_{L^1(\Omega)} \leq C'. \tag{53}$$

Now let us consider a continuous variable exponent q on $\overline{\Omega}$ verifying the pointwise estimate (47). According to the continuity of $q(\cdot)$ and $p(\cdot)$ on $\overline{\Omega}$, there exists a constant $\delta > 0$ such that

$$\max_{y\in \overline{B(x,\delta)\cap\Omega}} q(y) < \min_{y\in \overline{B(x,\delta)\cap\Omega}} \frac{(p(y) - 1 - \theta(p^+ - 1))N}{N - 1 - \theta(p^+ - 1)} \text{ for all } x \in \Omega. \tag{54}$$

Note that $\overline{\Omega}$ is compact and therefore we can cover it with a finite number of balls $(B_i)_{i=1,...,k}$. Furthermore, there exists a constant $\rho > 0$ such that

$$|B_i \cap \Omega| > \rho \text{ for all } i = 1, ..., k. \tag{55}$$

We denote by q_i^- and q_i^+ (respectively p_i^- and p_i^+) the local minimum and the local maximum of q on $\overline{B_i \cap \Omega}$ (respectively the local minimum and the local maximum of p on $\overline{B_i \cap \Omega}$).

It is clear that

$$\begin{aligned}\int_{B_i\cap\Omega} |\nabla u_n|^{q_i^+}dx &= \sum_{k=0}^{k_0-1}\int_{B_i\cap B_k} |\nabla u_n|^{q_i^+}dx + \sum_{k=k_0}^{\infty}\int_{B_i\cap B_k} |\nabla u_n|^{q_i^+}dx \\ &\leq C_{k_0} + \sum_{k=k_0}^{\infty}\int_{B_i\cap B_k} |\nabla u_n|^{q_i^+}dx.\end{aligned}$$

Using now the same reasoning as before locally, we get that inequality (51) holds on $B_i \cap B_k$, i.e:

$$\int_{B_i\cap B_k} |\nabla u_n|^{q_i^+}dx \leq \frac{C_4'}{k^{q_i^{+*}\left(\frac{p_i^- - q_i^+}{p_i^-} - \frac{r_i}{q_i^{+*}}\right)}} \left(\int_{B_i\cap B_k} |u_n|^{q_i^{+*}}dx\right)^{\frac{p_i^- - q_i^+}{p_i^-}},$$

where $r_i = \dfrac{q_i^+ \theta(p_i^+ - 1)}{p_i^-}$. From this estimate and Hölder's inequality, we deduce

$$\begin{aligned}\int_{B_i\cap\Omega} |\nabla u_n|^{q_i^+} dx &\le C_{k_0} + C_4' \left(\sum_{k=k_0}^{\infty} \frac{1}{k^{q_i^{+*}\left(\frac{p_i^- - q_i^+}{p_i^-} - \frac{r_i}{q_i^{+*}}\right)\frac{p_i^-}{q_i^+}}} \right)^{\frac{q_i^+}{p_i^-}} \left(\sum_{k=k_0}^{\infty} \int_{B_i\cap B_k} |u_n|^{q_i^{+*}} dx \right)^{\frac{p_i^- - q_i^+}{p_i^-}} \\ &\le C_{k_0} + C_4' \left(\sum_{k=k_0}^{\infty} \frac{1}{k^{q_i^{+*}\left(\frac{p_i^- - q_i^+}{p_i^-} - \frac{r_i}{q_i^{+*}}\right)\frac{p_i^-}{q_i^+}}} \right)^{\frac{q_i^+}{p_i^-}} \|u_n\|_{L^{q_i^{+*}}(B_i\cap\Omega)}^{\frac{q_i^{+*}(p_i^- - q_i^+)}{p_i^-}}. \qquad (56)\end{aligned}$$

Denote by $\overline{u}_{n_i}$ the average of u_{n_i} over $B_i \cap \Omega$:

$$\overline{u}_{n_i} = \frac{1}{|B_i \cap \Omega|} \int_{B_i\cap\Omega} u_n dx.$$

In view of (53) and (55), we deduce that

$$|\overline{u}_{n_i}| \le \frac{C'}{\rho}. \qquad (57)$$

By Poincaré-Wirtinger's inequality, we obtain

$$\|u_n - \overline{u}_{n_i}\|_{L^{q_i^{+*}}(B_i\cap\Omega)} \le C_8 \|\nabla u_n\|_{L^{q_i^+}(B_i\cap\Omega)}, \qquad (58)$$

for some constant $C_8 > 0$.

Keeping in mind (54), from (57), (58) and (56), we conclude that

$$\|u_n\|_{L^{q_i^{+*}}(B_i\cap\Omega)}^{q_i^+} \le C_9 + C_{10} \|u_n\|_{L^{q_i^{+*}}(B_i\cap\Omega)}^{\varrho_i},$$

for some constants $C_9, C_{10} > 0$, where $\varrho_i := \frac{q_i^{+*}(p_i^- - q_i^+)}{p_i^-} < q_i^+$.

Clearly, this gives that, for some constant C_{11}, depending on $p(\cdot)$, $q(\cdot)$ and C_1,

$$\|u_n\|_{L^{q_i^{+*}}(B_i\cap\Omega)}^{q_i^+} \le C_{11} \qquad (59)$$

for all $i = 1, ..., k$.

Finally, since $q_i^{+*} \ge q^*(x) \ge q(x)$ and $q_i^+ \ge q(x)$ for all $x \in B_i \cap \Omega$ and for all $i = 1, ..., k$, we deduce from (56) and (59) that

$$\|u_n\|_{L^{q^*(\cdot)}(\Omega)} + \|u_n\|_{W_0^{1,q(\cdot)}(\Omega)} \le C_{12},$$

for some constant C_{12} depending on $p(\cdot)$, $q(\cdot)$ and Ω. So, the proof of the Claim 5.3 is concluded. □

In order to pass to the limit in the weak formulation (30), the almost everywhere of the ∇u_n to ∇u is required.

5.5. *Almost everywhere convergence of gradients*

Claim 5.4. *There exist a measurable function u and a subsequence of u_n, still denoted by u_n, such that*

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega. \tag{60}$$

Proof. The proof for this claim is practically similar to that establishing in [21, Lemma 4.1]. Due to small modifications, we simply outline the main processes in this proof.

Let λ be a real number between 0 and 1 which will be determined later, define

$$\tilde{a}(x,u,\xi) = \frac{a(x,\xi)}{(1+|u|)^{\theta(p(x)-1)}}$$

and

$$E_n = \int_\Omega ((\tilde{a}(x,u_n,\nabla u_n) - \tilde{a}(x,u_n,\nabla u)) \cdot \nabla(u_n - u))^\lambda dx.$$

It is clear that E_n is well defined and $E_n \geq 0$ thanks to (6). We fix $k > 0$ and split E_n on the sets $\{|u| \leq k\}$ and $\{|u| > k\}$, we obtain

$$E_n = E_1(n,k) + E_2(n,k),$$

where

$$E_1(n,k) = \int_{\{|u|>k\}} ((\tilde{a}(x,u_n,\nabla u_n) - \tilde{a}(x,u_n,\nabla u)) \cdot \nabla(u_n - u))^\lambda dx$$

and

$$E_2(n,k) = \int_{\{|u|\leq k\}} ((\tilde{a}(x,u_n,\nabla u_n) - \tilde{a}(x,u_n,\nabla u)) \cdot \nabla(u_n - u))^\lambda dx.$$

For $E_2(n,k)$, we have

$$E_2(n,k) \leq E_3(n,k) = \int_\Omega ((\tilde{a}(x,u_n,\nabla u_n) - \tilde{a}(x,u_n,\nabla T_k(u))) \cdot \nabla(u_n - T_k(u)))^\lambda dx.$$

For $h > k+1$, we split $E_3(n,k)$ on the sets $\{|u_n - T_k(u)| \leq h\}$ and $\{|u_n - T_k(u)| > h\}$, we get

$$E_4(n,k,h) = \int_{\{|u_n - T_k(u)|>h\}} ((\tilde{a}(x,u_n,\nabla u_n) - \tilde{a}(x,u_n,\nabla T_k(u))) \cdot \nabla(u_n - T_k(u)))^\lambda dx$$

and

$$\begin{aligned}
E_5(n,k,h) &= \int_{\{|u_n - T_k(u)|\leq h\}} ((\tilde{a}(x,u_n,\nabla u_n) - \tilde{a}(x,u_n,\nabla T_k(u))) \cdot \nabla(u_n - T_k(u)))^\lambda dx \\
&= \int_\Omega ((\tilde{a}(x,u_n,\nabla u_n) - \tilde{a}(x,u_n,\nabla T_k(u))) \cdot \nabla T_h(u_n - T_k(u)))^\lambda dx \\
&\leq |\Omega|^{1-\lambda} \left(\int_\Omega (\tilde{a}(x,u_n,\nabla u_n) - \tilde{a}(x,u_n,\nabla T_k(u))) \cdot \nabla T_h(u_n - T_k(u)) dx \right)^\lambda \\
&= |\Omega|^{1-\lambda} \left(E_6(n,k,h)\right)^\lambda .
\end{aligned}$$

The term $E_6(n,k,h)$ can be written as the difference $E_7(n,k,h) - E_8(n,k,h)$ such that

$$E_7(n,k,h) = \int_\Omega \tilde{a}(x,u_n,\nabla u_n) \cdot \nabla T_h(u_n - T_k(u)) dx$$

and

$$E_8(n,k,h) = \int_\Omega \tilde{a}(x, u_n, \nabla T_k(u)) \cdot \nabla T_h(u_n - T_k(u))dx.$$

Now we choose $\lambda < 1$ such that $\lambda p^+ < q^-$ where q is the same as in Claim 5.3. Due to (45), (46), (48) and in similar fashion as Lemma 4.1. in [21], it yields that

$$\lim_{k\to\infty} \limsup_{n\to\infty} E_1(n,k) = 0,$$

$$\lim_{k\to\infty} \lim_{h\to\infty} \limsup_{n\to\infty} E_4(n,k,h) = 0,$$

$$\lim_{n\to\infty} E_8(n,k,h) = 0.$$

For $E_7(n,k,h)$, let us take, for $h > 0$, $T_h(u_n - T_k(u))$ as a test function in (30), to get

$$\int_\Omega w_n T_h(u_n - T_k(u))dx + \int_\Omega \frac{a(x, \nabla u_n) \cdot \nabla T_h(u_n - T_k(u))}{(1 + |u_n|)^{\theta(p(x)-1)}}dx = \int_\Omega f_n T_h(u_n - T_k(u))dx.$$

By the strong convergence of f_n and w_n in $L^1(\Omega)$, we conclude that

$$\lim_{n\to\infty} E_7(n,k,h) = \int_\Omega f T_h(u - T_k(u))dx - \int_\Omega w T_h(u - T_k(u))dx.$$

As a result,

$$\lim_{k\to\infty} \lim_{n\to\infty} E_7(n,k,h) = 0.$$

Putting together all the limitations, one thus has

$$\lim_{n\to\infty} E_n = 0.$$

So, in the similar way as in [21], we obtain that

$$\nabla u_n(x) \to \nabla u(x) \text{ a.e. in } \Omega.$$

□

5.6. *Passage to the limit*

We are now able to prove the result of existence of entropy solution of the problem (S, f) announced in Theorem 3.5.

Let $v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ and $k > 0$ be fixed, by taking $T_k(u_n - v)$ as a test function in (30), one has

$$\int_\Omega w_n T_k(u_n - v)dx + \int_\Omega \frac{a(x, \nabla u_n) \cdot \nabla T_k(u_n - v)}{(1 + |u_n|)^{\theta(p(x)-1)}}dx = \int_\Omega f_n T_k(u_n - v)dx. \quad (61)$$

As w_n and f_n are strongly convergent in $L^1(\Omega)$, while $T_k(u_n - v)$ converges both weakly* in $L^\infty(\Omega)$ and a.e. to $T_k(u - v)$, we obtain

$$\lim_{n\to\infty} \int_\Omega w_n T_k(u_n - v)dx = \int_\Omega w T_k(u - v)dx$$

and

$$\lim_{n\to\infty} \int_\Omega f_n T_k(u_n - v)dx = \int_\Omega f T_k(u - v)dx.$$

Lastly, let us examine the second term of (61), we split it as the sum

$$\int_{\{|u_n-v|\leq k\}} \frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+|u_n|)^{\theta(p(x)-1)}}dx - \int_{\{|u_n-v|\leq k\}} \frac{a(x,\nabla u_n)\cdot\nabla v}{(1+|u_n|)^{\theta(p(x)-1)}}dx.$$

Observe that the second integral is on a subset of the set where $|u_n| \leq k + \|v\|_{L^\infty(\Omega)} = M$, thus, it can be written as

$$\int_{\{|u_n-v|\leq k\}} \frac{a(x,\nabla T_M(u_n))\cdot\nabla v}{(1+|T_M(u_n)|)^{\theta(p(x)-1)}}dx.$$

Since $\dfrac{a(x,\nabla T_M(u_n))}{(1+|T_M(u_n)|)^{\theta(p(x)-1)}}$ is bounded in $(L^{p'(x)}(\Omega))^N$. Thus, by (45) and (60), we obtain

$$\frac{a(x,\nabla T_M(u_n))}{(1+|T_M(u_n)|)^{\theta(p(x)-1)}} \rightharpoonup \frac{a(x,\nabla T_M(u))}{(1+|T_M(u)|)^{\theta(p(x)-1)}} \text{ weakly in } (L^{p'(x)}(\Omega))^N.$$

Then, the second integral converges, as n goes to infinity, to

$$\int_{\{|u-v|\leq k\}} \frac{a(x,\nabla T_M(u))\cdot\nabla v}{(1+|T_M(u)|)^{\theta(p(x)-1)}}dx = \int_{\{|u-v|\leq k\}} \frac{a(x,\nabla u)\cdot\nabla v}{(1+|u|)^{\theta(p(x)-1)}}dx.$$

Due to (4), the integrand function of the first integral is nonnegative, so, using Fatou's Lemma, (45) and (60), we obtain that

$$\int_{\{|u-v|\leq k\}} \frac{a(x,\nabla u)\cdot\nabla u}{(1+|u|)^{\theta(p(x)-1)}}dx \leq \liminf_{n\to\infty} \int_{\{|u_n-v|\leq k\}} \frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+|u_n|)^{\theta(p(x)-1)}}dx.$$

Finally, putting all the terms together, we get

$$\int_\Omega wT_k(u-v)dx + \int_{\{|u-v|\leq k\}} \frac{a(x,\nabla u)\cdot\nabla(u-v)}{(1+|u|)^{\theta(p(x)-1)}}dx \leq \int_\Omega fT_k(u-v)dx.$$

which is (9). As a result, (u,w) is an entropy solution of the problem (S,f). Claim 5.3 ensures the regularity of the entropy solution u.

So that the proof of the Theorem 3.5 is now completed.

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Conflict of interset

The authors declare that they have no conflict of interest.

Data Availability Statement

The manuscript has no associate data.

References

[1] Adams, R. A., Sobolev spaces, Pure and Applied Mathematics, 65, Academic Press, New York-London (1975).

[2] Akdim, Y., Belayachi, M., Ouboufettal, M., Existence of solution for nonlinear elliptic inclusion problems with degenerate coercivity and L^1-data, J. Elliptic Parabol. Equ., 8(1), 127–150 (2022). https://doi.org/10.1007/s41808-022-00145-0.

[3] Alvino, A., Boccardo, L., Ferone, V., Orsina, L., Trombetti, G., Existence results for nonlinear elliptic equations with degenerate coercivity, Ann. Mat. Pura Appl. (4), 182(1), 53-79 (2003).

[4] Benboubker, M. B., Azroul, E., Barbara, A., Quasilinear elliptic problems with nonstandard growth, Electron. J. Differential Equations, 62, 1-16 (2011).

[5] Bénilan, P., Boccardo, L., Gallouët, T., Gariepy, R., Pierre, M., Vazquez, J. L., An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 22(2), 241-273 (1995).

[6] Brézis, H., Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland Mathematics Studies, No. 5, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York (1973).

[7] Díaz, J. I., Nonlinear partial differential equations and free boundaries, Vol. I: Elliptic equations, Research Notes in Mathematics, 106, Pitman (Advanced Publishing Program), Boston, MA (1985).

[8] Diening, L., Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$, Math. Nachr., 268, 31-43 (2004).

[9] Diening, L., Harjulehto, P., Hästö, P., Růžička, M., Lebesgue and Sobolev spaces with variable exponents, Lecture Notes in Mathematics, 2017, Springer, Heidelberg (2011).

[10] Diening, L., Hästö, P., Nekvinda, A., Open problems in variable exponent Lebesgue and Sobolev spaces, Drabek, Rakosnik (Eds.), FSDONA04 Proceedings, Milovy, Czech Republic, 38-58 (2004).

[11] Eddine, N. C., Nguyen, P. D., Ragusa, M. A., Existence and multiplicity of solutions for a class of critical anisotropic elliptic equations of Schrödinger–Kirchhoff-type, Math. Methods Appl. Sci. 46(16), 16782–16801 (2023). https://doi.org/10.1002/mma.9474.

[12] Fan, X. L., Zhao, D., On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl., 263(2), 424-446 (2001).

[13] Fan, X. L., Zhao, D., On the generalised Orlicz-Sobolev Space $W^{k,p(x)}(\Omega)$, J. Gansu Educ. College, 12(1), 1-6 (1998).

[14] Harjulehto, P., Hästö, P., Sobolev inequalities with variable exponent attaining the values 1 and n, Publ. Mat., 52(2), 347-363 (2008).

[15] Leray, J., Lions, J.-L., Quelques résultats de Višik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty-Browder, Bull. Soc. Math. France, 93, 97-107 (1965).

[16] Lions, J.-L., Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris; Gauthier-Villars, Paris (1969).

[17] Panagiotopoulos, P. D., Inequality problems in mechanics and applications: Convex and nonconvex energy functions, Birkhäuser Boston, Inc., Boston, MA (1985).

[18] Sanchón, M., Urbano, J. M., Entropy solutions for the $p(x)$-Laplace equation, Trans. Amer. Math. Soc., 361(12), 6387-6405 (2009).

[19] Stampacchia, G., Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, Ann. Inst. Fourier (Grenoble), 15(1), 189-258 (1965).

[20] Wittbold, P., Zimmermann, A., Existence and uniqueness of renormalized solutions to nonlinear elliptic equations with variable exponents and L^1-data, Nonlinear Anal., 72(6), 2990-3008 (2010).

[21] Zhang, X., Fu, Y., Solutions for nonlinear elliptic equations with variable growth and degenerate coercivity, Ann. Mat. Pura Appl. (4), 193(1), 133-161 (2014).

[22] Zhao, D., Qiang, W. J., Fan, X. L., On generalized Orlicz spaces $L^{p(x)}(\Omega)$, J. Gansu Sci., 9(2), 1-7 (1997).

[23] Zuo, J., El Allali, Z., Taarabti, S., Repovš, D. D., On a fourth-order Neumann problem in variable exponent spaces, Filomat, 37 (7), 2027-2039 (2023). https://doi.org/10.1023/a:1026625615977

[24] Wang, L., The commutators of multilinear maximal and fractional-type operators on central Morrey spaces with variable exponent, J. Funct. Spaces, 2022, Art. ID 4875460, 13, (2022). https://doi.org/10.1155/2022/4875460.