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Some properties of (a,b,k)-critical graphs

Sizhong Zhou[a], Yuli Zhang[b,*], Hongxia Liu[c]

[a] *School of Science, Jiangsu University of Science and Technology, Zhenjiang, Jiangsu 212100, China*
[b] *School of Science, Dalian Jiaotong University, Dalian, Liaoning 116028, China*
[c] *School of Mathematics and Information Sciences, Yantai University, Yantai, Shandong 264005, China*

Abstract. Let a, b and k be nonnegative integers with $1 \leq a \leq b$, and let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Then a spanning subgraph F of G is called an $[a, b]$-factor if $a \leq d_F(v) \leq b$ for any $v \in V(G)$. A graph G is said to be (a, b, k)-critical if $G - D$ contains an $[a, b]$-factor for each subset D of k elements of $V(G)$. We use $|E(G)|$ and $\rho(G)$ to denote the size and spectral radius, respectively. In this paper, we establish a lower bound on the size and spectral radius of a graph G to ensure that G is (a, b, k)-critical, respectively.

1. Introduction

In this article, we only discuss finite undirected graphs without loops or multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. For a vertex $v \in V(G)$, let $N_G(v) = \{u \in V(G) : vu \in E(G)\}$. The cardinality of $N_G(v)$ is called the degree of v in G, which is denoted by $d_G(v)$. The minimum degree of a vertex of G is denoted by $\delta(G)$. For $X, Y \subseteq V(G)$, we denote by $E_G(X, Y)$ the set of edges with one end in X and the other in Y, and write $e_G(X, Y) = |E_G(X, Y)|$. For any $X \subseteq V(G)$, the subgraph of G induced by X is denoted by $G[X]$. We write $G - X$ for $G[V(G) \setminus X]$. For two graphs G_1 and G_2, we denote by $G_1 \cup G_2$ the disjoint union of G_1 and G_2. For any nonnegative integer t, let tG denote the disjoint union of t copies of G. The join $G_1 \vee G_2$ is derived from $G_1 \cup G_2$ by joining each vertex of G_1 with each vertex of G_2 by an edge. Let K_n denote the complete graph of order n.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix $A(G) = (a_{ij})$ of G is the $n \times n$ symmetric matrix, where $a_{ij} = 1$ if v_i and v_j are adjacent in G, zero otherwise. The largest eigenvalue of $A(G)$, denoted by $\rho(G)$, is called the spectral radius of G.

Let a and b denote two positive integers with $a \leq b$. Then a spanning subgraph F of G is called an $[a, b]$-factor if $a \leq d_F(v) \leq b$ for any $v \in V(G)$. In particular, when $a = b = r$, an $[a, b]$-factor is an r-factor. A 1-factor is also called a perfect matching. A graph G is said to be (a, b, k)-critical if $G - D$ contains an $[a, b]$-factor for each subset D of k elements of $V(G)$. An (r, r, k)-critical graph is called an (r, k)-critical graph.

In mathematical literature, the study on factors of graphs and factor-critical graphs attracted much attention. Some sufficient conditions for graphs with r-factors were obtained by Enomoto, Jackson, Katerinis and Saito [5], Katerinis [12], Nishimura [20], Niessen and Randerath [18], Gu [7]. Many researchers

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* Corresponding author: Yuli Zhang
Email addresses: zsz_cumt@163.com (Sizhong Zhou), zhangyuli_djtu@126.com (Yuli Zhang), liuhongxia@ytu.edu.cn (Hongxia Liu)

[11, 17, 23–25, 31, 33, 38–41] verified some conditions related to degree condition, neighborhood, the number of isolated vertices, binding number, sun toughness, etc., for a graph to possess a $[1,2]$-factor. Much effort has been devoted to finding sufficient conditions for the existence of $[a,b]$-factors in graphs by utilizing various graphic parameters such as neighborhood [10], independence number [29], eigenvalue [28] and others [6, 26, 27, 30, 32, 35–37]. Cai, Favaron and Li [2], Enomoto [3] established a connection between toughness and $(2,k)$-critical graphs, respectively. Enomoto and Hagita [4] showed a toughness condition for the existence of (r,k)-critical graphs. Li and Wang [15] provided a necessary and sufficient condition for a graph to be (a,b,k)-critical. Li [13, 14] derived some results on the existence of (a,b,k)-critical graphs.

Very recently, O [21] established a close connection between the spectral radius and perfect matchings of graphs. Zhou and Liu [34] gave a spectral radius for a graph with an odd $[1,b]$-factor. Motivated by [15, 21] directly, it is natural and interesting to present some sufficient conditions to guarantee that a graph is (a,b,k)-critical. Next, we focus on the sufficient conditions including the size or the spectral radius of graphs. Our main results will be provided in the following.

Theorem 1.1. Let a,b and k be nonnegative integers with $1 \le a \le b$ and $(a,b) \ne (1,1)$, and let G be a $(k+1)$-connected graph of order $n \ge a+k+1$ and minimum degree $\delta(G) \ge a+k$. If

$$|E(G)| \ge \binom{n-1}{2} + \frac{a+2k+1}{2}$$

and $b(n-k) \equiv 0 \pmod 2$ when $a=b$, then G is (a,b,k)-critical.

Theorem 1.2. Let a,b and k be nonnegative integers with $1 \le a \le b$ and $(a,b) \ne (1,1)$, and let G be a $(k+1)$-connected graph of order $n \ge a+2k+1$. If $\rho(G) > \rho(K_{a+2k-1} \vee (K_{n-a-2k} \cup K_1))$ and $b(n-k) \equiv 0 \pmod 2$ when $a=b$, then G is (a,b,k)-critical.

2. Some preliminaries

For a given positive integer r and a pair X,Y of disjoint subsets of $V(G)$, we call a component C of $G-(X\cup Y)$ odd if $r|V(C)| + e_G(Y,V(C))$ is odd. Let $\omega_1(X,Y)$ denote the number of odd components of $G-(X\cup Y)$. Write $\theta_G(X,Y) = r|X| - r|Y| + \sum_{v\in Y} d_{G-X}(v) - \omega_1(X,Y)$.

The following result is a necessary and sufficient condition for a graph to possess an r-factor and it was derived by Tutte [22].

Lemma 2.1 (Tutte [22]). (i) A graph G contains an r-factor if and only if $\theta_G(X,Y) \ge 0$ for every pair X,Y of disjoint subsets of $V(G)$.

(ii) $\theta_G(X,Y) \equiv r|V(G)| \pmod 2$ for every pair X,Y of disjoint subsets of $V(G)$.

Liu and Yu [16] presented the following characterization of (r,k)-critical graphs.

Lemma 2.2 (Liu and Yu [16]). Let r and k be integers with $r \ge 2$ and $k \ge 0$, and G be a graph of order $n \ge r+k+1$. Then G is (r,k)-critical if and only if $\theta_G(X,Y) \ge rk$ for every pair X,Y of disjoint subsets of $V(G)$ with $|X| \ge k$.

Liu and Wang [15] showed a necessary and sufficient condition for a graph to be (a,b,k)-critical.

Lemma 2.3 (Liu and Wang [15]). Let a,b and k be nonnegative integers with $1 \le a < b$, and G be a graph of order $n \ge a+k+1$. Then G is (a,b,k)-critical if and only if for any $X \subseteq V(G)$ with $|X| \ge k$,

$$\sum_{j=0}^{a-1}(a-j)p_j(G-X) \le b|X| - bk$$

or

$$\delta_G(X,Y) = b|X| - a|Y| + \sum_{v\in Y} d_{G-X}(v) \ge bk,$$

where $p_j(G-X) = |\{v : d_{G-X}(v) = j\}|$ and $Y = \{v : v \in V(G) \setminus X, d_{G-X}(v) \leq a-1\}$.

It is worthy to emphasize that Lemma 2.3 has its equivalent statement as follows.

Lemma 2.4. Let a, b and k be nonnegative integers with $1 \leq a < b$, and G be a graph of order $n \geq a+k+1$. Then G is (a, b, k)-critical if and only if $\delta_G(X, Y) \geq bk$ for every pair X, Y of disjoint subsets of $V(G)$ with $|X| \geq k$.

Hong, Shu and Fang [8], Nikiforov [19] presented an important upper bound on the spectral radius $\rho(G)$.

Lemma 2.5 (Hong, Shu and Fang [8], Nikiforov [19]). Let G be a graph of order n with minimum degree $\delta(G)$. Then

$$\rho(G) \leq \frac{\delta(G)-1}{2} + \sqrt{2e(G) - n\delta(G) + \frac{(\delta(G)+1)^2}{4}}.$$

The following observation is very useful when we use the above upper bound on $\rho(G)$.

Proposition 2.6 (Hong, Shu and Fang [8], Nikiforov [19]). For a graph G of order n with $e(G) \leq \binom{n}{2}$, the function

$$f(x) = \frac{x-1}{2} + \sqrt{2e(G) - nx + \frac{(x+1)^2}{4}}$$

is decreasing with respect to x for $0 \leq x \leq n-1$.

Let $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$. Define $A \leq B$ if for any $1 \leq i, j \leq n$, $a_{ij} \leq b_{ij}$ and $A < B$ if $A \leq B$ and $A \neq B$.

Lemma 2.7 (Berman and Plemmons [1], Horn and Johnson [9]). Let O be an $n \times n$ zero matrix, $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ matrices with the spectral radius $\rho(A)$ and $\rho(B)$, respectively. If $O \leq A \leq B$, then $\rho(A) \leq \rho(B)$. Furthermore, if B is irreducible and $O \leq A < B$, then $\rho(A) < \rho(B)$.

3. The proof of Theorem 1.1

In this section, we provide a proof of Theorem 1.1, which claims a lower bound on the size for a graph to be (a, b, k)-critical.

Proof of Theorem 1.1. For a pair X, Y of disjoint subsets of $V(G)$ with $|X| \geq k$, let

$$\gamma_G(X, Y) = \begin{cases} b|X| - a|Y| + \sum_{v\in Y} d_{G-X}(v) - \omega_1(X, Y), & \text{if } b = a; \\ b|X| - a|Y| + \sum_{v\in Y} d_{G-X}(v), & \text{if } b > a \geq 1. \end{cases} \tag{1}$$

Suppose to the contrary that G is not (a, b, k)-critical. Then it follows from (1), $b(n-k)$ even, Lemmas 2.1 (ii), 2.2 and 2.4 that

$$\gamma_G(X, Y) \leq \begin{cases} bk-2, & \text{if } b = a; \\ bk-1, & \text{if } b > a \geq 1 \end{cases} \tag{2}$$

for some pair disjoint subsets X, Y of $V(G)$. Subject to (2), we choose X and Y such that $|X \cup Y|$ is as large as possible. Then we shall prove the following claims.

Claim 1. Let $C_1, C_2, \ldots, C_l$ be the components of $G-(X\cup Y)$, where l is the number of components of $G-(X\cup Y)$. If $a = b$, then $|V(C_i)| \geq 2$ for any $i \in \{1, 2, \ldots, l\}$.

Proof. We verify Claim 1 by contradiction. Without loss of generality, let $|V(C_l)| = 1$. Write $V(C_l) = \{v_l\}$. If $e_G(v_l, Y) \le a - 1$, let $Y' = Y \cup \{v_l\}$, then it follows from $a = b$, (1) and (2) that

$$\begin{aligned}\gamma_G(X, Y') =& b|X| - a|Y'| + \sum_{v\in Y'} d_{G-X}(v) - \omega_1(X, Y')\\ \le& b|X| - a(|Y| + 1) + \sum_{v\in Y} d_{G-X}(v) + e_G(v_l, Y) - (\omega_1(X, Y) - 1)\\ =& \gamma_G(X, Y) + e_G(v_l, Y) - a + 1\\ \le& \gamma_G(X, Y) \le bk - 2,\end{aligned}$$

which contradicts the choice of $|X \cup Y|$.

If $e_G(v_l, Y) \ge a$, let $X' = X \cup \{v_l\}$, then it follows from $a = b$, (1) and (2) that

$$\begin{aligned}\gamma_G(X', Y) =& b|X'| - a|Y| + \sum_{v\in Y} d_{G-X'}(v) - \omega_1(X', Y)\\ \le& a(|X| + 1) - a|Y| + \sum_{v\in Y} d_{G-X}(v) - e_G(v_l, Y) - (\omega_1(X, Y) - 1)\\ =& \gamma_G(X, Y) - e_G(v_l, Y) + a + 1\\ \le& \gamma_G(X, Y) + 1 \le bk - 1.\end{aligned}$$

In terms of $b(n - k)$ even and Lemma 2.1 (ii), we possess $\gamma_G(X', Y) \le bk - 2$, which contradicts the choice of $|X \cup Y|$. Claim 1 is verified. □

Claim 2. $n \ge a + k + 2$.

Proof. By virtue of $\delta(G) \ge a + k$, it follows that G is a complete graph when $n = a + k + 1$. Thus if $n = a + k + 1$, then G is (a, b, k)-critical, which is a contradiction to the hypothesis. Hence we infer $n \ge a + k + 2$. This completes the proof of Claim 2. □

Claim 3. $|X| \ge k + 1$.

Proof. Assume that $|X| = k$. We first verify $Y \ne \emptyset$. Note that G is $(k + 1)$-connected. If $Y = \emptyset$, then $\omega_1(X, \emptyset) \le 1$. Together with (1), we deduce

$$\gamma_G(X, Y) \ge \begin{cases} bk - 1, & \text{if } b = a;\\ bk, & \text{if } b > a \ge 1,\end{cases}$$

which contradicts (2). Thus, we possess $Y \ne \emptyset$.

For $b > a$, it follows from (1) and $\delta(G) \ge a+k$ that $\gamma_G(X, Y) = b|X|-a|Y|+\sum_{v\in Y} d_{G-X}(v) \ge bk+(\delta(G)-k-a)|Y| \ge bk$, which is a contradiction to (2).

For $b = a$, we discuss the value of $\omega_1(X, Y)$. If $\omega_1(X, Y) \le 1$, then from (1) and $\delta(G) \ge a + k$, we derive $\gamma_G(X, Y) = b|X| - a|Y| + \sum_{v\in Y} d_{G-X}(v) - \omega_1(X, Y) \ge bk + (\delta(G) - k - a)|Y| - 1 \ge bk - 1$, which contradicts (2) when $b = a$. If $\omega_1(X, Y) \ge 2$, then $l \ge 2$. Recall that $|E(G)| \ge \binom{n-1}{2} + \frac{a+2k+1}{2}$. Then there exist at most $n - \frac{a+2k+3}{2} - \frac{1}{2}\sum_{i=1}^{l} |V(C_i)|(n - k - |Y| - |V(C_i)|)$ edges not in $E_G(V(G) \setminus (X \cup Y), Y) \cup E(G[Y])$. Combining this with (1), $|X| = k$, $|Y| \geq 1$, $b \geq 2$, Claims 1 and 2, we deduce

$$\begin{aligned}
\gamma_G(X,Y) =& b|X| - a|Y| + \sum_{v\in Y} d_{G-X}(v) - \omega_1(X,Y) \\
\geq& bk - a|Y| + (n-k-1)|Y| \\
&- 2\left(n - \frac{a+2k+3}{2} - \frac{1}{2}\sum_{i=1}^{l} |V(C_i)|(n-k-|Y|-|V(C_i)|)\right) - (n-k-|Y|) \\
\geq& bk - a|Y| + (n-k-1)|Y| - 2\left(n - \frac{a+2k+3}{2} - \sum_{i=1}^{l} |V(C_i)|\right) - (n-k-|Y|) \\
=& bk - a|Y| + (n-k-1)|Y| - 2\left(n - \frac{a+2k+3}{2} - (n-k-|Y|)\right) - (n-k-|Y|) \\
=& bk + (n-a-k-2)(|Y|-1) + 1 \\
\geq& bk + 1,
\end{aligned}$$

which contradicts (2) when $b = a$. Claim 3 is proved. □

The following proof will be divided into two cases by the value of $|Y|$.

Case 1. $0 \leq |Y| \leq b$.

For $b > a$, it follows from (1), $\delta(G) \geq a + k$ and Claim 3 that

$$\begin{aligned}
\gamma_G(X,Y) =& b|X| - a|Y| + \sum_{v\in Y} d_{G-X}(v) \\
\geq& b|X| - a|Y| + (\delta(G) - |X|)|Y| \\
=& (b - |Y|)|X| + (\delta(G) - a)|Y| \\
\geq& (b - |Y|)(k+1) + (\delta(G) - a)|Y| \\
=& bk + (b - |Y|) + (\delta(G) - a - k)|Y| \\
\geq& bk,
\end{aligned}$$

which contradicts (2) when $b > a$.

For $b = a$, we argue the value of $\omega_1(X,Y)$. If $\omega_1(X,Y) \leq 1$, then by (1), (2), $\delta(G) \geq a + k$ and Claim 3, we possess

$$\begin{aligned}
bk - 2 \geq& \gamma_G(X,Y) = b|X| - a|Y| + \sum_{v\in Y} d_{G-X}(v) - \omega_1(X,Y) \\
\geq& b|X| - a|Y| + (\delta(G) - |X|)|Y| - 1 \\
=& (b - |Y|)|X| + (\delta(G) - a)|Y| - 1 \\
\geq& (b - |Y|)(k+1) + (\delta(G) - a)|Y| - 1 \\
=& bk + (b - |Y|) + (\delta(G) - a - k)|Y| - 1 \\
\geq& bk - 1,
\end{aligned}$$

which is a contradiction. If $\omega_1(X,Y) \geq 2$, then $l \geq 2$. Next, we shall consider three subcases.

Subcase 1.1. $|Y| = 0$.

Note that $|E(G)| \geq \binom{n-1}{2} + \frac{a+2k+1}{2}$. Then there exist at least $\frac{1}{2}\sum_{i=1}^{l} |V(C_i)|(n - |X| - |V(C_i)|)$ edges not in

$G[V(G) \setminus X]$. In terms of Claim 1, we have

$$\begin{aligned}
0 \le & \binom{n}{2} - |E(G)| - \frac{1}{2}\sum_{i=1}^{l} |V(C_i)|(n - |X| - |V(C_i)|) \\
\le & \binom{n}{2} - \binom{n-1}{2} - \frac{a+2k+1}{2} - \frac{1}{2}\sum_{i=1}^{l} |V(C_i)|(n - |X| - |V(C_i)|) \\
= & n - \frac{a+2k+3}{2} - \frac{1}{2}\sum_{i=1}^{l} |V(C_i)|(n - |X| - |V(C_i)|) \\
\le & n - \frac{a+2k+3}{2} - \sum_{i=1}^{l} (n - |X| - |V(C_i)|) \\
= & (l-1)|X| - (l-2)n - \frac{a+2k+3}{2}.
\end{aligned}$$

If $l \ge 3$, then we derive $n \le 2|X| - \frac{a+2k+3}{2}$. Together with (1), $b \ge 2$ and Claim 3, we infer

$$\begin{aligned}
\gamma_G(X, \emptyset) &= b|X| - \omega_1(X, \emptyset) \\
&\ge \begin{cases} b|X| - 2 \ge bk + b - 2 \ge bk, & \text{if } l = 2; \\ b|X| - (n - |X|) \ge (b-1)|X| + \frac{a+2k+3}{2} \ge (b-1)(k+1) + \frac{a+2k+3}{2} > bk, & \text{if } l \ge 3, \end{cases}
\end{aligned}$$

which is a contradiction to (2) when $b = a$.

Subcase 1.2. $|Y| = 1$.

Recall that $b = a$, $|E(G)| \ge \binom{n-1}{2} + \frac{a+2k+1}{2}$ and $\omega_1(X, Y)$ is the number of odd components of $G - (X \cup Y)$. We deduce $|E(K_n)| - |E(G)| - (\omega_1(X, Y) - 1) \le \binom{n}{2} - \left(\binom{n-1}{2} + \frac{a+2k+1}{2}\right) - (\omega_1(X, Y) - 1) = n - \frac{b+2k+1}{2} - \omega_1(X, Y)$. Then there exist at most $n - \frac{b+2k+1}{2} - \omega_1(X, Y)$ edges not in $E_G(V(G) \setminus (X \cup Y), Y)$. Thus, we obtain

$$\begin{aligned}
\sum_{v \in Y} d_{G-X}(v) \ge & (n - 1 - |X|) - \left(n - \frac{b+2k+1}{2} - \omega_1(X, Y)\right) \\
= & \frac{b+2k-1}{2} - |X| + \omega_1(X, Y).
\end{aligned}$$

Together with (1), $b \ge 2$, $|Y| = 1$ and Claim 3, we deduce

$$\begin{aligned}
\gamma_G(X, Y) = & b|X| - a|Y| + \sum_{v \in Y} d_{G-X}(v) - \omega_1(X, Y) \\
\ge & b|X| - b + \frac{b+2k-1}{2} - |X| + \omega_1(X, Y) - \omega_1(X, Y) \\
= & (b-1)|X| - \frac{b-2k+1}{2} \\
\ge & (b-1)(k+1) - \frac{b-2k+1}{2} \\
= & bk + \frac{b-3}{2} \\
> & bk - 1,
\end{aligned}$$

which contradicts (2) when $b = a$.

Subcase 1.3. $|Y| \ge 2$.

Recall that $\omega_1(X, Y)$ is the number of odd components of $G - (X \cup Y)$. It follows from $b = a$ and $|E(G)| \ge \binom{n-1}{2} + \frac{a+2k+1}{2}$ that $|E(K_n)| - |E(G)| - (\omega_1(X, Y) - 1) \le \binom{n}{2} - \left(\binom{n-1}{2} + \frac{a+2k+1}{2}\right) - (\omega_1(X, Y) - 1) = n - \frac{b+2k+1}{2} - \omega_1(X, Y)$, and so there exist at most $n - \frac{b+2k+1}{2} - \omega_1(X,Y)$ edges not in $E_G(V(G) \setminus (X \cup Y), Y) \cup E(G[Y])$. Thus, we derive

$$\sum_{v\in Y} d_{G-X}(v) \geq (n-1-|X|)|Y| - 2\left(n - \frac{b+2k+1}{2} - \omega_1(X,Y)\right). \tag{3}$$

Using (1), (3), $b = a$, $\omega_1(X,Y) \geq 2$, Claims 2 and 3, we possess

$$\begin{aligned}
\gamma_G(X,Y) =& b|X| - a|Y| + \sum_{v\in Y} d_{G-X}(v) - \omega_1(X,Y)\\
\geq& b|X| - b|Y| + (n-1-|X|)|Y| - 2\left(n - \frac{b+2k+1}{2} - \omega_1(X,Y)\right) - \omega_1(X,Y)\\
=&(b-|Y|)|X| + (n-b-1)(|Y|-2) + \omega_1(X,Y) - b + 2k - 1\\
\geq&(b-|Y|)(k+1) + (k+1)(|Y|-2) - b + 2k + 1\\
=&bk-1,
\end{aligned}$$

which is a contradiction to (2) when $b = a$.
Case 2. $|Y| \geq b+1$.
Subcase 2.1. $b > a$, or $b = a$ and $\omega_1(X,Y) = 0$.
We easily see that

$$n \geq |X| + |Y| \geq |X| + b + 1. \tag{4}$$

In terms of $|E(G)| \geq \binom{n-1}{2} + \frac{a+2k+1}{2}$, there exist at most $n-1-\frac{a+2k+1}{2}$ edges not in $E_G(V(G)\setminus(X\cup Y), Y) \cup E(G[Y])$. Thus, we get

$$\sum_{v\in Y} d_{G-X}(v) \geq (n-1-|X|)|Y| - 2\left(n-1-\frac{a+2k+1}{2}\right). \tag{5}$$

It follows from (1), (4), (5), $b \geq 2$ and Claim 2 that

$$\begin{aligned}
\gamma_G(X,Y) =& b|X| - a|Y| + \sum_{v\in Y} d_{G-X}(v)\\
\geq& b|X| - a|Y| + (n-1-|X|)|Y| - 2\left(n-1-\frac{a+2k+1}{2}\right)\\
=& b|X| + (n-a-1-|X|)|Y| - 2n + a + 2k + 3\\
\geq& b|X| + (n-a-1-|X|)(b+1) - 2n + a + 2k + 3\\
=&(b-2)n + (n-|X|-b-1) - ab + 2k + 3\\
\geq&(b-2)(a+k+2) - ab + 2k + 3\\
=&bk + 2(b-a) - 1\\
\geq&\begin{cases} bk-1, & \text{if } b = a;\\ bk+1, & \text{if } b > a \geq 1,\end{cases}
\end{aligned}$$

which contradicts (2).
Subcase 2.2. $b = a$ and $\omega_1(X,Y) \geq 1$.
It is obvious that

$$n \geq |X| + |Y| + \omega_1(X,Y) \geq |X| + b + 1 + \omega_1(X,Y). \tag{6}$$

Recall that $\omega_1(X,Y)$ is the number of odd components of $G - (X\cup Y)$. According to $|E(G)| \geq \binom{n-1}{2} + \frac{a+2k+1}{2}$, We deduce $|E(K_n)| - |E(G)| - (\omega_1(X,Y)-1) \leq \binom{n}{2} - \left(\binom{n-1}{2} + \frac{a+2k+1}{2}\right) - (\omega_1(X,Y)-1) = n - \frac{a+2k+1}{2} - \omega_1(X,Y)$. Then there exist at most $n - \frac{a+2k+1}{2} - \omega_1(X,Y)$ edges not in $E_G(V(G) \setminus (X \cup Y), Y) \cup E(G[Y])$. Thus, we obtain

$$\sum_{v\in Y} d_{G-X}(v) \geq (n-1-|X|)|Y| - 2\left(n - \frac{a+2k+1}{2} - \omega_1(X,Y)\right). \tag{7}$$

By virtue of (1), (6), (7), $a = b \geq 2$, $|Y| \geq b+1 \geq 3$, $\omega_1(X,Y) \geq 1$ and Claim 3, we deduce

$$\begin{aligned}
\gamma_G(X,Y) =& b|X| - a|Y| + \sum_{v\in Y} d_{G-X}(v) - \omega_1(X,Y) \\
\geq& b|X| - a|Y| + (n-1-|X|)|Y| - 2\left(n - \frac{a+2k+1}{2} - \omega_1(X,Y)\right) - \omega_1(X,Y) \\
=& (b-2)|X| + (n-a-1-|X|)(|Y|-2) - a + 2k - 1 + \omega_1(X,Y) \\
\geq& (b-2)|X| + (b-a+\omega_1(X,Y))(|Y|-2) - a + 2k - 1 + \omega_1(X,Y) \\
\geq& (b-2)(k+1) - a + 2k + 1 \\
=& bk - 1,
\end{aligned}$$

which is a contradiction to (2) when $b = a$. This completes the proof of Theorem 1.1. □

4. The proof of Theorem 1.2

In this section, we are to verify Theorem 1.2, which provides a sufficient spectral condition to guarantee that a graph is (a, b, k)-critical.

Proof of Theorem 1.2. Since the graph K_{n-1} is a proper subgraph of the graph $K_{a+2k-1} \vee (K_{n-a-2k} \cup K_1)$ and the adjacency matrices of connected graphs are irreducible, it follows from Lemma 2.7 that

$$\rho(G) > \rho(K_{a+2k-1} \vee (K_{n-a-2k} \cup K_1)) > \rho(K_{n-1}) = n-2. \tag{8}$$

We are to prove the following claim.

Claim 1. $\delta(G) \geq a + 2k$.
Proof. Assume that $\delta(G) \leq a + 2k - 1$. Then there exists a vertex $v \in V(G)$ such that $d_G(v) \leq a + 2k - 1$, which implies that $G \subseteq K_{a+2k-1} \vee (K_{n-a-2k} \cup K_1)$. In terms of Lemma 2.7, we deduce

$$\rho(G) \leq \rho(K_{a+2k-1} \vee (K_{n-a-2k} \cup K_1)),$$

which is a contradiction to the condition that $\rho(G) > \rho(K_{a+2k-1} \vee (K_{n-a-2k} \cup K_1))$. Hence we possess $\delta(G) \geq a+2k$. This completes the proof of Claim 1. □

By virtue of Claim 1, Lemma 2.5 and Proposition 2.6, we derive

$$\begin{aligned}
\rho(G) \leq& \frac{\delta(G)-1}{2} + \sqrt{2|E(G)| - n\delta(G) + \frac{(\delta(G)+1)^2}{4}} \\
\leq& \frac{a+2k-1}{2} + \sqrt{2|E(G)| - n(a+2k) + \frac{(a+2k+1)^2}{4}}.
\end{aligned} \tag{9}$$

It follows from (8) and (9) that

$$n - 2 < \rho(G) \leq \frac{a+2k-1}{2} + \sqrt{2|E(G)| - n(a+2k) + \frac{(a+2k+1)^2}{4}},$$

which leads to

$$|E(G)| > \frac{(n-1)(n-2)}{2} + \frac{a+2k}{2} = \binom{n-1}{2} + \frac{a+2k}{2}.$$

In view of the integrity of $|E(G)|$, we have

$$|E(G)| \geq \binom{n-1}{2} + \frac{a+2k+1}{2}. \tag{10}$$

By means of (10), Claim 1 and Theorem 1.1, we know that G is (a, b, k)-critical. This completes the proof of Theorem 1.2. □

Data availability statement

My manuscript has no associated data.

Declaration of competing interest

The authors declare that they have no conflicts of interest to this work.

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References

[1] A. Berman, R. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979.
[2] M. Cai, O. Favaron, H. Li, *$(2, k)$-factor-critical graphs and toughness*, Graphs Combin. **15** (1999), 137–142.
[3] H. Enomoto, *Toughness and the existence of k-factors (III)*, Discrete Math. **189** (1998), 277–282.
[4] H. Enomoto, M. Hagita, *Toughness and the existence of k-factors (IV)*, Discrete Math. **216** (2000), 111–120.
[5] H. Enomoto, B. Jackson, P. Katerinis, A. Saito, *Toughness and the existence of k-factors*, J. Graph Theory **9** (1985), 87–95.
[6] W. Gao, W. Wang, Y. Chen, *Tight isolated toughness bound for fractional (k, n)-critical graphs*, Discrete Appl. Math. **322** (2022), 194–202.
[7] X. Gu, *Regular factors and eigenvalues of regular graphs*, European J. Combin. **42** (2014), 15–25.
[8] Y. Hong, J. Shu, K. Fang, *A sharp upper bound of the spectral radius of graphs*, J. Comb. Theory, Ser. B **81** (2001), 177–183.
[9] R. Horn, C. Johnson, *Matrix Analysis*, Cambridge University Press, 1986.
[10] M. Kano, *A sufficient condition for a graph to have $[a, b]$-factors*, Graphs Combin. **6** (1990), 245–251.
[11] M. Kano, G. Y. Katona, Z. Király, *Packing paths of length at least two*, Discrete Math. **283** (2004), 129–135.
[12] P. Katerinis, *Minimum degree of bipartite graphs and the existence of k-factors*, Graphs Combin. **6** (1990), 253–258.
[13] J. Li, *A new degree condition for graph to have $[a, b]$-factor*, Discrete Math. **290** (2005), 99–103.
[14] J. Li, *Sufficient conditions for graphs to be (a, b, n)-critical graphs*, Mathematica Applicata (Wuhan) **17** (2004), 450–455.
[15] G. Liu, J. Wang, *(a, b, k)-critical graphs*, Advances in Mathematics (China) **27** (1998), 536–540.
[16] G. Liu, Q. Yu, *k-factors and extendability with prescribed components*, Congressus Numerantium **139** (1999), 77–88.
[17] H. Liu, *Sun toughness and path-factor uniform graphs*, RAIRO Oper. Res. **56** (2022), 4057–4062.
[18] T. Niessen, B. Randerath, *Regular factors of simple regular graphs and factor-spectra*, Discrete Math. **185** (1998), 89–103.
[19] V. Nikiforov, *Some inequalities for the largest eigenvalue of a graph*, Comb. Probab. Comput. **11** (2002), 179–189.
[20] T. Nishimura, *Independence number, connectivity and r-factors*, J. Graph Theory **13** (1989), 63–69.
[21] S. O, *Spectral radius and matchings in graphs*, Linear Algebra Appl. **614** (2021), 316–324.
[22] W. T. Tutte, *The factors of graphs*, Can. J. Math. **4** (1952), 314–328.
[23] S. Wang, W. Zhang, *Degree conditions for the existence of a $\{P_2, P_5\}$-factor in a graph*, RAIRO Oper. Res. **57** (2023), 2231–2237.
[24] S. Wang, W. Zhang, *Independence number, minimum degree and path-factors in graphs*, Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci. **23** (2022), 229–234.
[25] S. Wang, W. Zhang, *Isolated toughness for path factors in networks*, RAIRO Oper. Res. **56** (2022), 2613–2619.
[26] S. Wang, W. Zhang, *On k-orthogonal factorizations in networks*, RAIRO Oper. Res. **55** (2021), 969–977.
[27] S. Wang, W. Zhang, *Some results on star-factor deleted graphs*, Filomat **38** (2024), 1101–1107.
[28] J. Wei, S. Zhang, *Proof of a conjecture on the spectral radius condition for $[a, b]$-factors*, Discrete Math. **346** (2023), 113269.
[29] J. Wu, *A sufficient condition for the existence of fractional (g, f, n)-critical covered graphs*, Filomat **38** (2024), 2177–2183.
[30] S. Zhou, *A neighborhood union condition for fractional (a, b, k)-critical covered graphs*, Discrete Appl. Math. **323** (2022), 343–348.
[31] S. Zhou, *Degree conditions and path factors with inclusion or exclusion properties*, Bull. Math. Soc. Sci. Math. Roumanie **66** (2023), 3–14.
[32] S. Zhou, Q. Bian, Z. Sun, *Two sufficient conditions for component factors in graphs*, Discuss. Math. Graph Theory **43** (2023), 761–766.

[33] S. Zhou, *Some results on path-factor critical avoidable graphs*, Discuss. Math. Graph Theory **43** (2023), 233–244.
[34] S. Zhou, H. Liu, *Two sufficient conditions for odd* $[1, b]$*-factors in graphs*, Linear Algebra Appl. **661** (2023), 149–162.
[35] S. Zhou, H. Liu, Y. Xu, *A note on fractional ID-*$[a, b]$*-factor-critical covered graphs*, Discrete Appl. Math. **319** (2022), 511–516.
[36] S. Zhou, Q. Pan, L. Xu, *Isolated toughness for fractional* $(2, b, k)$*-critical covered graphs*, Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci. **24** (2023), 11–18.
[37] S. Zhou, Z. Sun, H. Liu, *Distance signless Laplacian spectral radius for the existence of path-factors in graphs*, Aequationes Math. **98(3)** (2024), 727–737.
[38] S. Zhou, Z. Sun, Q. Bian, *Isolated toughness and path-factor uniform graphs (II)*, Indian J. Pure Appl. Math. **54** (2023), 689–696.
[39] S. Zhou, Z. Sun, H. Liu, *Some sufficient conditions for path-factor uniform graphs*, Aequationes Math. **97** (2023), 489–500.
[40] S. Zhou, Z. Sun, F. Yang, *A result on* $P_{\geq 3}$*-factor uniform graphs*, Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci. **23** (2022), 3–8.
[41] S. Zhou, J. Wu, Q. Bian, *On path-factor critical deleted (or covered) graphs*, Aequationes Math. **96** (2022), 795–802.