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On a new weighted Hermite-Hadamard inequality, application for weighted means

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Abstract. The purpose of this paper is to introduce a weighted Hermite-Hadamard inequality which generalizes the standard one. Some refinements and reverses of this weighted inequality are pointed out. As application, some new weighted means are derived and their related inequalities are investigated as well.

1. Introduction

Let C be a nonempty interval of $\mathbb{R}$. A function $f : C \to \mathbb{R}$ is called convex if the inequality

$$f\Big((1-\lambda)a+\lambda b\Big) \leq (1-\lambda)f(a)+\lambda f(b) \tag{1}$$

holds for any $a, b \in C$ and $\lambda \in [0,1]$. We say that f is concave if (1) is reversed.

The following double inequality

$$f\Big(\frac{a+b}{2}\Big) \leq \int_0^1 f\Big((1-t)a+tb\Big)dt \leq \frac{f(a)+f(b)}{2} \tag{2}$$

holds for any $a, b \in C$, whenever $f : C \to \mathbb{R}$ is convex. If f is concave then (2) are reversed. Inequality (2), known in the literature as the Hermite-Hadamard inequality (HHI), is useful in mathematical analysis and contributes as good tool for obtaining some interesting estimations. An enormous amount of efforts has been devoted in the literature for extending (2) from the case where the variables are real numbers to the case where the variables are bounded linear operators, see [1, 4–8] for instance.

The present manuscript contains four sections organized as follows: In Section 2, we collect some weighted means from the literature that will be needed along the paper. Section 3 displays with the so-called weighted Hermite-Hadamard inequality that refines (1) and generalizes (2). Some refinements and reverses of this weighted inequality are investigated. As application, Section 4 is focused to derive some new weighted means and so we study their elementary properties as well as their comparison with some known weighted means.

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2. Standard weighted means

Recently, the mean-theory attracts many mathematicians by virtue of its nice properties and various applications. As usual, we understand by (binary) mean a map m between positive numbers such that $\min(a,b) \le m(a,b) \le \max(a,b)$ for all $a,b > 0$. Among the standard means, we recall the following

$$a\nabla b = \frac{a+b}{2},\ a\sharp b = \sqrt{ab},\ a!b = \frac{2ab}{a+b},\ L(a,b) = \frac{b-a}{\log b - \log a},\ I(a,b) = e^{-1}\Big(b^b/a^a\Big)^{1/(b-a)},$$

with $L(a,a) = a$ and $I(a,a) = a$. These are known as the arithmetic mean, the geometric mean, the harmonic mean, the logarithmic mean and the identric mean, respectively. These means satisfy the chain of inequalities, $a!b \le a\sharp b \le L(a,b) \le I(a,b) \le a\nabla b$, see [2].

Let m_λ be a family of binary maps indexed by $\lambda \in [0,1]$. We recall the following, [10]

Definition 2.1. *We say that m_λ is a weighted mean if the following assertions hold:*
(i) $m_0(a,b) = a$ and $m_1(a,b) = b$,
(ii) m_λ is a mean, for any $\lambda \in [0,1]$,
(iii) $m_{1/2} =: m$ is a symmetric mean,
(iv) $m_\lambda(a,b) = m_{1-\lambda}(b,a)$ for any $a,b > 0$ and $\lambda \in [0,1]$.

It is obvious that, (iv) implies (iii). We say that the mean $m =: m_{1/2}$ is the associated symmetric mean of m_λ and m_λ is the weighted m-mean.

The standard weighted means are the following

$$a\nabla_\lambda b = (1-\lambda)a + \lambda b,\ a\sharp_\lambda b = a^{1-\lambda}b^\lambda,\ a!_\lambda b = \Big((1-\lambda)a^{-1} + \lambda b^{-1}\Big)^{-1},$$

which are the weighted arithmetic mean, the weighted geometric mean and the weighted harmonic mean, respectively. For $\lambda = 1/2$ they coincide with $a\nabla b$, $a\sharp b$ and $a!b$, respectively. These weighted means satisfy the following inequalities, $a!_\lambda b \le a\sharp_\lambda b \le a\nabla_\lambda b$.

Two other weighted means have been introduced in the literature [9], namely

$$L_\lambda(a,b) = \frac{1}{\log a - \log b}\left(\frac{1-\lambda}{\lambda}\Big(a - a^{1-\lambda}b^\lambda\Big) + \frac{\lambda}{1-\lambda}\Big(a^{1-\lambda}b^\lambda - b\Big)\right),\ a \ne b, \tag{3}$$

$$I_\lambda(a,b) = \frac{1}{e}\Big(a\nabla_\lambda b\Big)^{\frac{(1-2\lambda)(a\nabla_\lambda b)}{\lambda(1-\lambda)(b-a)}}\left(\frac{b^{\frac{\lambda b}{1-\lambda}}}{a^{\frac{(1-\lambda)a}{\lambda}}}\right)^{\frac{1}{b-a}},\ a \ne b. \tag{4}$$

One has $L_0(a,b) := \lim_{\lambda\downarrow 0} L_\lambda(a,b) = a$ and $L_1(a,b) := \lim_{\lambda\uparrow 1} L_\lambda(a,b) = b$, with similar equalities for $I_\lambda(a,b)$. One can see that L_λ and I_λ satisfy the conditions (ii),(iii) and (iv). For $\lambda = 1/2$, they coincide with $L(a,b)$ and $I(a,b)$, respectively. Therefore, following Definition 2.1, L_λ and I_λ can be called the weighted logarithmic mean and the weighted identric mean, respectively.

It has been shown in [9] that the following inequalities hold true

$$a\sharp_\lambda b \le L_\lambda(a,b) \le a\nabla_\lambda b,\quad a\sharp_\lambda b \le I_\lambda(a,b) \le a\nabla_\lambda b.$$

Remark 2.2. *When a weighted mean m_λ is given, its associated symmetric mean is of course unique. However, it is possible to have more one weighted mean whose the associated symmetric mean is the same. Section 4 below explains more this latter situation where we will define a weighted logarithmic mean and a weighted identric mean that are different from the previous ones.*

3. Weighted Hermite-Hadamard inequality

The following result gives a refinement of (1) and a generalization of (2).

Theorem 3.1. *Let $f : C \to \mathbb{R}$ be convex. For any $\lambda \in]0,1[$ and $a, b \in C$ the following inequalities hold*

$$f\Big((1-\lambda)a+\lambda b\Big) \le \int_0^1 f\Big((1-t)a+tb\Big)d\nu_\lambda(t) \le (1-\lambda)f(a)+\lambda f(b), \tag{5}$$

where ν_λ is the probability measure defined on $[0,1]$ by

$$d\nu_\lambda(t) = \Big((1-\lambda)(1-t)^{\frac{1-2\lambda}{\lambda}} + \lambda t^{\frac{2\lambda-1}{1-\lambda}}\Big)dt. \tag{6}$$

If $f : C \to \mathbb{R}$ is concave then (5) are reversed.

Proof. Applying the Jensen integral and discrete inequalities, we obtain

$$f\left(\int_0^1 \Big((1-t)a+tb\Big)d\nu_\lambda(t)\right) \le \int_0^1 f\Big((1-t)a+tb\Big)d\nu_\lambda(t) \le \int_0^1 \Big((1-t)f(a)+tf(b)\Big)d\nu_\lambda(t). \tag{7}$$

Using (6) and computing the involved integrals by elementary topics of real-integration, we get the desired inequalities. The details are simple and therefore omitted here. □

Remark 3.2. *(i) In what follows,* (5) *will be called the weighted Hermite-Hadamard inequality, (WHHI) is short. If $\lambda = 1/2$ then* (5) *coincides with* (2)*, i.e. $d\nu_{1/2}(t) = dt$. Further, $d\nu_{1-\lambda}(1-t) = d\nu_\lambda(t)$ for any $\lambda \in]0,1[$.*
(ii) For the sake of simplicity, we extend the weighted arithmetic mean from positive numbers to any real numbers by setting throughout the following

$$x\nabla_\lambda y := (1-\lambda)x+\lambda y, \ \ \text{for any } x, y \in \mathbb{R} \text{ and } \lambda \in [0,1].$$

With this notation, (5) *can be shortly written as*

$$f\Big(a\nabla_\lambda b\Big) \le \int_0^1 f\Big(a\nabla_t b\Big)d\nu_\lambda(t) \le f(a)\nabla_\lambda f(b). \tag{8}$$

(iii) Setting $x = (1-t)a+tb$, with $a < b$, (5) *and* (8) *are equivalent to*

$$f\Big(a\nabla_\lambda b\Big) \le \frac{1}{b-a}\int_a^b f(x)d\mu_\lambda(x) \le f(a)\nabla_\lambda f(b),$$

where we set

$$d\mu_\lambda(x) := \left((1-\lambda)\Big(\frac{b-x}{b-a}\Big)^{\frac{1-2\lambda}{\lambda}} + \lambda\Big(\frac{x-a}{b-a}\Big)^{\frac{2\lambda-1}{1-\lambda}}\right)dx.$$

The previous weighted Hermite-Hadamard inequality has many consequences. In particular, we have the following result which gives a refinement of the standard Hermite-Hadamard inequality.

Corollary 3.3. *If $f : C \to \mathbb{R}$ is convex then for any $a, b \in C$, there holds*

$$f\Big(a\nabla b\Big) \le \int_0^1 f\Big(a\nabla_t b\Big)dt \le J_f(a,b) \le f(a)\nabla f(b), \tag{9}$$

where we set

$$J_f(a,b) := \int_0^1\int_0^1 f\Big(a\nabla_t b\Big)\, d\nu_\lambda(t)d\lambda.$$

If f is concave then (9) are reversed.

Proof. Integrating (8) with respect to $\lambda \in]0,1[$ and using the left hand-side of (2) we get (9). □

For the sake of simplicity we set, for $a, b \in C$ and $\lambda \in]0,1[$,

$$\mathcal{M}_\lambda(f; a, b) = \int_0^1 f\Big((1-t)a + tb\Big) d\nu_\lambda(t). \tag{10}$$

The following corollary justifies that the map $\lambda \mapsto \mathcal{M}_\lambda(f; a, b)$ can be extended on the whole interval $[0,1]$.

Corollary 3.4. *Let $f : C \to \mathbb{R}$ be convex (resp. concave). For any $a, b \in C$ there holds*

$$\lim_{\lambda \downarrow 0} \mathcal{M}_\lambda(f; a, b) = f(a), \ \lim_{\lambda \uparrow 1} \mathcal{M}_\lambda(f; a, b) = f(b).$$

Proof. Follows from (5) by using the fact that if $f : C \to \mathbb{R}$ is convex then it is continuous on $[a, b] \subset C$. □

Proposition 3.5. *For any $a, b \in C$ and $\lambda \in [0,1]$ there holds*

$$\mathcal{M}_{1-\lambda}(f; a, b) = \mathcal{M}_\lambda(f; b, a) \tag{11}$$

Proof. By (6) and (10) we have

$$\mathcal{M}_{1-\lambda}(f; a, b) = \int_0^1 f\Big((1-t)a + tb\Big) d\nu_{1-\lambda}(t).$$

Making the change of variable $t = 1 - u$ and using the relationship $d\nu_{1-\lambda}(1-t) = d\nu_\lambda(t)$, with (10) again, we get the desired result. □

To state another result, we need the following lemma, see [3].

Lemma 3.6. *If $f : C \to \mathbb{R}$ is convex, the following inequalities*

$$r(s,t)\Big(f(a)\nabla_s f(b) - f\big(a\nabla_s b\big)\Big) \le f(a)\nabla_t f(b) - f\big(a\nabla_t b\big) \le R(s,t)\Big(f(a)\nabla_s f(b) - f\big(a\nabla_s b\big)\Big), \tag{12}$$

hold for any $a, b \in C$ and $s, t \in]0,1[$, where we set

$$r(s,t) := \min\left(\frac{t}{s}, \frac{1-t}{1-s}\right), \ \ R(s,t) := \max\left(\frac{t}{s}, \frac{1-t}{1-s}\right). \tag{13}$$

If f is concave then (12) are reversed.

It is clear that (12) refines and reverses (1). For $s = t$, (12) are equalities. We also need the following lemma.

Lemma 3.7. *For any $s, \lambda \in]0,1[$, the following equalities hold*

$$\int_0^1 r(s,t)\, d\nu_\lambda(t) = \alpha_{1-s,\lambda} + \alpha_{s,1-\lambda} \tag{14}$$

and

$$\int_0^1 R(s,t)\, d\nu_\lambda(t) = \frac{\lambda}{s} + \frac{1-\lambda}{1-s} - \alpha_{1-s,\lambda} - \alpha_{s,1-\lambda}, \tag{15}$$

where, for $a, b > 0$, we set

$$\alpha_{a,b} := b^2 \frac{1 - a^{\frac{1-b}{b}}}{1-a}.$$

Proof. It is easy to see that $\frac{t}{s} \leq \frac{1-t}{1-s}$ if and only if $0 \leq t \leq s$. Therefore, we write

$$\int_0^1 r(s,t)\, d\nu_\lambda(t) = \frac{1}{s}\int_0^s t\, d\nu_\lambda(t) + \frac{1}{1-s}\int_s^1 (1-t)\, d\nu_\lambda(t). \tag{16}$$

By using (6), we have

$$t d\nu_\lambda(t) = (1-\lambda)(1-t)^{\frac{1-2\lambda}{\lambda}} - (1-\lambda)(1-t)^{\frac{1-\lambda}{\lambda}} + \lambda t^{\frac{\lambda}{1-\lambda}}.$$

Using $d\nu_\lambda(t) = d\nu_{1-\lambda}(1-t)$ and some elementary computations we get

$$\int_0^s t.d\nu_\lambda(t) = -\lambda(1-s)^{\frac{1-\lambda}{\lambda}} + \lambda(1-\lambda)\left[(1-s)^{\frac{1}{\lambda}} + s^{\frac{1}{1-\lambda}}\right] + \lambda^2$$

and

$$\begin{aligned}\int_s^1 (1-t)\, d\nu_\lambda(t) &= \int_s^1 (1-t)\, d\nu_{1-\lambda}(1-t)\\ &= \int_0^{1-s} t\, d\nu_{1-\lambda}(t)\\ &= -(1-\lambda)s^{\frac{\lambda}{1-\lambda}} + \lambda(1-\lambda)\left[(1-s)^{\frac{1}{\lambda}} + s^{\frac{1}{1-\lambda}}\right] + (1-\lambda)^2.\end{aligned}$$

Substituting in (16), we obtain (14) after some algebraic manipulations.

For proving (15), it is sufficient to notice that

$$R(s,t) + r(s,t) = \frac{1}{1-s} + \frac{1-2s}{s(1-s)}t,$$

and then we have

$$\begin{aligned}\int_0^1 R(s,t)d\nu_\lambda(t) + \int_0^1 r(s,t)d\nu_\lambda(t) &= \frac{1}{1-s}\int_0^1 d\nu_\lambda(t) + \frac{1-2s}{s(1-s)}\int_0^1 t d\nu_\lambda(t)\\ &= \frac{1}{1-s} + \frac{1-2s}{s(1-s)}\lambda.\end{aligned}$$

Hence, the desired result is obtained. □

Now, we are in the position to state the following result which gives a refinement and a reverse of the right inequality in (5).

Theorem 3.8. *Let $f : C \to \mathbb{R}$ be convex. For any $s, \lambda \in (0,1)$ and $a, b \in C$ the following inequalities hold*

$$m(s,\lambda)\Big(f(a)\nabla_s f(b) - f\big(a\nabla_s b\big)\Big) \leq f(a)\nabla_\lambda f(b) - \int_0^1 f\big(a\nabla_t b\big)d\nu_\lambda(t) \leq M(s,\lambda)\Big(f(a)\nabla_s f(b) - f\big(a\nabla_s b\big)\Big), \tag{17}$$

where we set

$$m(s,\lambda) := (1-\lambda)^2\frac{1-s^{\frac{\lambda}{1-\lambda}}}{1-s} + \lambda^2\frac{1-(1-s)^{\frac{1-\lambda}{\lambda}}}{s}$$

$$M(s,\lambda) := \frac{1-\lambda}{1-s} + \frac{\lambda}{s} - m(s,\lambda)$$

If f is concave then (17) *are reversed.*

Proof. Multiplying all sides of (12) by $d\nu_\lambda(t)$ and then integrating with respect to $t \in [0,1]$, we obtain the desired inequalities by the use of (14) and (15). The details are simple and therefore omitted here for the reader. □

Taking $s = 1/2$ in Theorem 3.8, we get after some reductions the following result.

Corollary 3.9. *For $f : C \to \mathbb{R}$ convex, $\lambda \in (0,1)$ and $a, b \in C$, there holds*

$$l(\lambda)\Big(f(a)\nabla f(b) - f\Big(a\nabla b\Big)\Big) \le f(a)\nabla_\lambda f(b) - \int_0^1 f\Big(a\nabla_t b\Big)d\nu_\lambda(t) \le u(\lambda)\Big(f(a)\nabla f(b) - f\Big(a\nabla b\Big)\Big), \quad (18)$$

where we set

$$l(\lambda) := 2\left[(1-\lambda)^2\left(1 - 2^{\frac{\lambda}{1-\lambda}}\right) + \lambda^2\left(1 - 2^{\frac{1-\lambda}{\lambda}}\right)\right] \text{ and } u(\lambda) := 2 - l(\lambda).$$

If f is concave then (18) *are reversed.*

For the sake of convenience, we consider the function $x \mapsto \mathcal{N}_{f;\lambda,a,b}(x)$ defined on $[0,1]$ by

$$\mathcal{N}_{f;\lambda,a,b}(x) = \int_0^1 f\Big((a\nabla_\lambda b)\nabla_x (a\nabla_t b)\Big)d\nu_\lambda(t). \quad (19)$$

The following result concerns a refinement of the left inequality in (5).

Theorem 3.10. *Let $f : C \to \mathbb{R}$ be convex, $a, b \in C$ and $\lambda \in [0,1]$, there holds*

$$f(a\nabla_\lambda b) \le \int_0^1 f\Big((a\nabla_\lambda b)\nabla_\lambda (a\nabla_t b)\Big)d\nu_\lambda(t) \le \int_0^1 \mathcal{M}_\lambda(f; a\nabla_\lambda b, a\nabla_t b)d\nu_\lambda(t)$$
$$\le f\Big(a\nabla_\lambda b\Big)\nabla_\lambda \mathcal{M}_\lambda(f; a, b) \le \int_0^1 f\Big(a\nabla_t b\Big)d\nu_\lambda(t). \quad (20)$$

If f is concave then (20) *are reversed.*

Proof. By using the Jensen integral inequality, we get

$$\mathcal{N}_{f;\lambda,a,b}(x) \ge f\left(\int_0^1 (a\nabla_\lambda b)\nabla_x (a\nabla_t b)d\nu_\lambda(t)\right) = f(a\nabla_\lambda b).$$

By the convexity of f and the fact that $f(a\nabla_\lambda b) \le \mathcal{M}_\lambda(f; a, b)$, we get

$$\mathcal{N}_{f;\lambda,a,b}(x) \le \int_0^1 f\Big(a\nabla_\lambda b\Big)\nabla_x f\Big(a\nabla_t b\Big)d\nu_\lambda(t) = f\Big(a\nabla_\lambda b\Big)\nabla_x \mathcal{M}_\lambda(f; a, b)$$
$$\le \mathcal{M}_\lambda(f; a, b).$$

Thus,

$$f(a\nabla_\lambda b) \le \mathcal{N}_{f;\lambda,a,b}(x) \le f\Big(a\nabla_\lambda b\Big)\nabla_x \mathcal{M}_\lambda(f; a, b) \le \mathcal{M}_\lambda(f; a, b). \quad (21)$$

Noticing that $x \mapsto \mathcal{N}_{f,\lambda,a,b}(x)$ is a convex function, we can apply (8) for obtaining

$$\mathcal{N}_{f;\lambda,a,b}(\lambda) \le \int_0^1 \mathcal{N}_{f;\lambda,a,b}(t)d\nu_\lambda(t) \le (1-\lambda)\mathcal{N}_{f;\lambda,a,b}(0) + \lambda\mathcal{N}_{f;\lambda,a,b}(1).$$

Then,

$$\mathcal{N}_{f;\lambda,a,b}(\lambda) \le \int_0^1 \mathcal{M}_\lambda(f; a\nabla_\lambda b, a\nabla_t b)d\nu_\lambda(t) \le (1-\lambda)f(a\nabla_\lambda b) + \lambda \int_0^1 f(a\nabla_t b)d\nu_\lambda(t). \quad (22)$$

By combining (21) and (22) we get (20), so completing the proof. □

We have also the following result.

Theorem 3.11. *Let $f : C \longrightarrow \mathbb{R}$ be convex and differentiable. For all $a, b \in \mathring{C}$ with $a < b$, the following inequalities*

$$0 \le \int_0^1 f(a\nabla_t b) d\nu_\lambda(t) - \mathcal{N}_{f,\lambda,a,b}(u) \le (b-a)(1-u) \int_0^1 (\lambda - t) f'(a\nabla_t b) d\nu_\lambda(t) \tag{23}$$

hold for any $u \in [0, 1]$.

Proof. The left inequality is a straightforward deduction from (21). For the right inequality, we use the fact that f is a differentiable convex function to write $f(\alpha) - f(\beta) \ge (\alpha - \beta) f'(\beta)$, with $\alpha = (a\nabla_\lambda b)\nabla_u (a\nabla_t b)$ and $\beta = a\nabla_t b$. Hence,

$$f\left((a\nabla_\lambda b)\nabla_u (a\nabla_t b)\right) - f\left(a\nabla_t b\right) \ge (1-u)(b-a)(\lambda - t) f'\left(a\nabla_t b\right). \tag{24}$$

Multiplying both sides of (24) by $d\nu_\lambda(t)$ and integrating with respect to $t \in [0, 1]$, we find the right inequality in (23). □

If we take $u = 0$ in (23), we get the following result which gives a refinement and a reverse of the left inequality in (5).

Corollary 3.12. *With the same assumptions as in Theorem 3.11, we have*

$$0 \le \int_0^1 f(a\nabla_t b) d\nu_\lambda(t) - f\left(a\nabla_\lambda b\right) \le (b-a) \int_0^1 (\lambda - t) f'(a\nabla_t b) d\nu_\lambda(t). \tag{25}$$

4. Application: some new weighted means

As already stated, our aim in the ongoing section is to derive some new weighted means. We begin by stating the following.

Proposition 4.1. *For $a, b > 0$ and $\lambda \in [0, 1]$, we set*

$$\mathcal{L}_\lambda(a, b) := \left(\int_0^1 \frac{d\nu_\lambda(t)}{(1-t)a + tb}\right)^{-1}. \tag{26}$$

$$\mathcal{I}_\lambda(a, b) := \exp\left(\int_0^1 \log\Big((1-t)a + tb\Big) d\nu_\lambda(t)\right). \tag{27}$$

Then $\mathcal{L}_\lambda$ is a weighted logarithmic mean and $\mathcal{I}_\lambda$ is a weighted identric mean.

Proof. Take $C = (0, \infty)$ and $f(x) = 1/x$. Using the definition of $a!_\lambda b$ and $a\nabla_\lambda b$, (8) with (26) implies that $\min(a, b) \le a!_\lambda b \le \mathcal{L}_\lambda(a, b) \le a\nabla_\lambda b \le \max(a, b)$. We then deduce that $\mathcal{L}_\lambda$ is a mean for any $\lambda \in [0, 1]$ and $\mathcal{L}_0(a, b) = a, \mathcal{L}_1(a, b) = b$. Since $d\nu_{1/2}(t) = dt$ then a simple computation of integral gives $\mathcal{L}_{1/2}(a, b) = L(a, b)$ the standard logarithmic mean. Finally, the relationship $\mathcal{L}_{1-\lambda}(a, b) = \mathcal{L}_\lambda(b, a)$ follows from (11). Summarizing, $\mathcal{L}_\lambda$ satisfies all the conditions of Definition 2.1, with $\mathcal{L}_{1/2} = L$. So, $\mathcal{L}_\lambda$ is a weighted logarithmic mean.

For $\mathcal{I}_\lambda$, we choose $f(x) = \log(x)$ which is concave on $(0, \infty)$. The details are similar to those of $\mathcal{L}_\lambda$ and we left them here. □

The following result gives a chain of inequalities concerning a comparison between some of the previous weighted means.

Theorem 4.2. *For any $a, b > 0$ and $\lambda \in [0, 1]$ there holds*

$$a!_\lambda b \le \mathcal{L}_\lambda(a, b) \le \mathcal{I}_\lambda(a, b) \le a\nabla_\lambda b.$$

Proof. The inequalities $a!_\lambda b \le \mathcal{L}_\lambda(a,b)$ and $\mathcal{I}_\lambda(a,b) \le a\nabla_\lambda b$ were proved out before. We have to show $\mathcal{L}_\lambda(a,b) \le \mathcal{I}_\lambda(a,b)$. Since the real function $x \mapsto -\log(x)$ is convex on $(0,\infty)$ then the Jensen integral inequality gives

$$-\log \int_0^1 \frac{d\nu_\lambda(t)}{(1-t)a+tb} \le \int_0^1 \log\Big((1-t)a+tb\Big)d\nu_\lambda(t).$$

This, with (26) and (27), is equivalent to $\log\ \mathcal{L}_\lambda(a,b) \le \log\ \mathcal{I}_\lambda(a,b)$, which concludes the proof. □

Proposition 4.3. *If for $a,b>0$ and $\lambda \in [0,1]$, we set*

$$\mathbb{L}_\lambda(a,b) := \int_0^1 a^{1-t}b^t d\nu_\lambda(t),$$

then $\mathbb{L}_\lambda$ is also a weighted logarithmic mean that satisfies

$$a\sharp_\lambda b \le \mathbb{L}_\lambda(a,b) \le a\nabla_\lambda b. \tag{28}$$

Proof. If we multiply the weighted arithmetic-geometric mean inequality, namely $a^{1-t}b^t \le (1-t)a+tb$, by $d\nu_\lambda(t)$ and we integrate over $t \in [0,1]$ we obtain the right inequality in (28). Now, using the fact that $x \mapsto \log(x)$ is concave on $(0,\infty)$, the Jensen integral inequality implies that

$$\log \int_0^1 a^{1-t}b^t d\nu_\lambda(t) \ge \int_0^1 \Big((1-t)\log\ a + t\log\ b\Big)d\nu_\lambda(t) = (1-\lambda)\log\ a + \lambda\log\ b,$$

which yields the left inequality in (28). □

Now, we will justify that the previous weighted means $\mathcal{L}_\lambda$ (resp. $\mathbb{L}_\lambda$) and $\mathcal{I}_\lambda$ are different from L_λ and I_λ defined by (3) and (4) respectively. For this, we consider the following example

Example 4.4. *Let us take $\lambda = 1/3$ and $a=1, b=2$. Using real integration tools, we find*

$$\mathcal{L}_{1/3}(1,2) = 3\left(\int_0^1 \frac{2-2t+t^{-1/2}}{1+t}dt\right)^{-1} = \frac{6}{8\log 2+\pi-4};$$

$$\mathbb{L}_{1/3}(1,2) = \frac{1}{3}\int_0^1 (2-2t+t^{-1/2})2^t dt = \frac{4\log^{\frac{3}{2}}(2)\mathbf{D}\left(\sqrt{\log(2)}\right)+2-\log 4}{3\log^2(2)};$$

where $\mathbf{D}$ refers to the Dawson's integral defined by $\mathbf{D}(x) = \exp(-x^2)\int_0^x \exp(t^2)dt$.

$$L_{1/3}(1,2) = \frac{3\sqrt[3]{2}-2}{2\log 2};$$

$$\mathcal{I}_{1/3}(1,2) = \exp\left[\int_0^1 \left(\frac{2}{3}-\frac{2}{3}t+\frac{1}{3}t^{-1/2}\right)\log(1+t)dt\right] = 4\exp\left(-\frac{13}{6}+\frac{\pi}{3}\right);$$

$$I_{1/3}(1,2) = \frac{32}{9e}.$$

Numerical computations lead to

$\mathcal{L}_{1/3}(1,2) \simeq 1.28019934$; $\mathbb{L}_{1/3}(1,2) \simeq 1.28460020$; $L_{1/3}(1,2) \simeq 1.28382773$,
$\mathcal{I}_{1/3}(1,2) \simeq 1.30581223$; $I_{1/3}(1,2) \simeq 1.30801579$.

So, our claim is confirmed. See also Remark 2.2 which is in connection with this example.

Theorem 4.5. *For any $s, \lambda \in [0,1]$ and $a, b > 0$, there holds*

$$e^{m(s,\lambda)} \frac{a\sharp_s b}{a\nabla_s b} \le \frac{a\sharp_\lambda b}{\mathcal{I}_\lambda(a,b)} \le e^{M(s,\lambda)} \frac{a\sharp_s b}{a\nabla_s b}. \tag{29}$$

In particular,

$$e^{-M(\lambda,\lambda)} a\nabla_\lambda b \le \mathcal{I}_\lambda(a,b) \le e^{-m(\lambda,\lambda)} a\nabla_\lambda b \le a\nabla_\lambda b. \tag{30}$$

Proof. By applying (17) for the concave function $f(x) = \log x$ on $C = (0,+\infty)$, we find (29). Taking $s = \lambda$ in (29) we get (30). □

Now we can state the following result which concerns some refinements of the inequalities in Theorem 4.2.

Theorem 4.6. *For any $s, \lambda \in [0,1]$ and $a, b > 0$, the following inequalities hold*

$$m(s,\lambda)\left((a!_s b)^{-1} - (a\nabla_s b)^{-1}\right) \le (a!_\lambda b)^{-1} - \mathcal{L}_\lambda^{-1}(a,b) \le M(s,\lambda)\left((a!_s b)^{-1} - (a\nabla_s b)^{-1}\right).$$

Proof. We apply (17) to the convex function $f(x) = 1/x$ on $C = (0,+\infty)$. □

Corollary 4.7. *For any $\lambda \in [0,1]$ and $a, b > 0$, we have the following inequalities,*

$$\Big(1 - M(\lambda,\lambda)\Big)(a!_\lambda b)^{-1} + M(\lambda,\lambda)(a\nabla_\lambda b)^{-1} \le \mathcal{L}_\lambda^{-1}(a,b) \le (a!_\lambda b)^{-1} \nabla_{m(\lambda,\lambda)} (a\nabla_\lambda b)^{-1}.$$

Proof. We take $s = \lambda$ in Theorem 4.6. Noticing that

$$m(\lambda,\lambda) = 1 - \left[(1-\lambda)\lambda^{\frac{\lambda}{1-\lambda}} + \lambda(1-\lambda)^{\frac{1-\lambda}{\lambda}}\right] \le 1,$$

we find the desired result. □

Theorem 4.8. *For any $\lambda \in [0,1]$ and $a, b > 0$, the following inequalities hold*

$$\mathcal{I}_\lambda(a,b) \le (a\nabla_\lambda b)\sharp_\lambda \mathcal{I}_\lambda(a,b) \le \exp\left[\int_0^1 \log\Big(\mathcal{I}_\lambda(a\nabla_\lambda b, a\nabla_x b)\Big) d\nu_\lambda(x)\right]$$
$$\le \exp\left[\int_0^1 \log\Big((a\nabla_\lambda b)\nabla_\lambda (a\nabla_x b)\Big) d\nu_\lambda(x)\right] \le a\nabla_\lambda b.$$

Proof. We apply (20) for the concave function $f(x) = \log x$ on $(0,+\infty)$. □

Theorem 4.9. *For any $\lambda \in [0,1]$ and any $a, b > 0$, we have*

$$\mathcal{L}_\lambda(a,b) \le (a\nabla_\lambda b)!_\lambda \mathcal{L}_\lambda(a,b) \le \left[\int_0^1 \mathcal{L}_\lambda^{-1}(a\nabla_\lambda b, a\nabla_x b)\, d\nu_\lambda(x)\right]^{-1}$$
$$\le \left[\int_0^1 ((a\nabla_\lambda b)\nabla_\lambda (a\nabla_x b))^{-1}\, d\nu_\lambda(x)\right]^{-1} \le a\nabla_\lambda b.$$

Proof. We apply (20) for the convex function $f(x) = 1/x$ on $(0,+\infty)$. □

Theorem 4.10. *For any $s, \lambda \in [0,1]$ and $a, b > 0$, it holds*

$$m(s,\lambda)\Big(a\nabla_s b - a\sharp_s b\Big) \le a\nabla_\lambda b - \mathbb{L}_\lambda(a,b) \le M(s,\lambda)\Big(a\nabla_s b - a\sharp_s b\Big). \tag{31}$$

In particular, we have

$$\Big(1 - M(\lambda,\lambda)\Big)(a\nabla_\lambda b) + M(\lambda,\lambda)(a\sharp_\lambda b) \le \mathbb{L}_\lambda(a,b) \le (a\nabla_\lambda b)\nabla_{m(\lambda,\lambda)} (a\sharp_\lambda b) \le a\nabla_\lambda b \tag{32}$$

Proof. To get the desired result, we apply (17) to the convex function $f(x) = \exp(x)$ on $C = (0, +\infty)$ and we replace a and b respectively by $\log(a)$ and $\log(b)$. Taking $s = \lambda$ in (31) and noticing that

$$(a\nabla_\lambda\, b)\nabla_{m(\lambda,\lambda)}\,(a\sharp_\lambda\, b) \le (a\nabla_\lambda\, b)\nabla_{m(\lambda,\lambda)}\,(a\nabla_\lambda\, b) = a\nabla_\lambda\, b,$$

we get (32) after simple manipulation. □

Finally, the following result provides a refinement of (28).

Theorem 4.11. *For any $\lambda \in [0,1]$ and $a, b > 0$, there hold*

$$a\sharp_\lambda\, b \le \int_0^1 \Big(a\sharp_\lambda\, b\Big)\sharp_\lambda\Big(a\sharp_x\, b\Big)d\nu_\lambda(x) \le \int_0^1 \mathbb{L}_\lambda(a\sharp_\lambda\, b, a\sharp_x\, b)d\nu_\lambda(x) \le (a\sharp_\lambda\, b)\sharp_\lambda\mathbb{L}_\lambda(a,b) \le \mathbb{L}_\lambda(a,b).$$

Proof. We apply (20) to the convex function $f(x) = \exp(x)$ on $C = (0, +\infty)$ and we replace a and b by $\log(a)$ and $\log(b)$ respectively. □

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