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Boundness of some multilinear fractional integral operators in generalized Morrey spaces on stratified Lie groups

Jianglong Wu[a,*], Xiaojiao Tian[a]

[a] *Department of Mathematics, Mudanjiang Normal University, Mudanjiang 157011, China*

Abstract. In this paper, the main aim is to consider the Spanne-type boundedness of the multilinear fractional integral operator $\mathcal{I}_{\alpha,m}$ and multilinear fractional maximal operator $\mathcal{M}_{\alpha,m}$ in the generalized Morrey spaces over some stratified Lie group $\mathbb{G}$.

1. Introduction and main results

Stratified groups appear in quantum physics and many parts of mathematics, including several complex variables, Fourier analysis, geometry, and topology [6, 24]. The geometry structure of stratified groups is so good that it inherits a lot of analysis properties from the Euclidean spaces [9, 23]. Apart from this, the difference between the geometry structures of Euclidean spaces and stratified groups makes the study of function spaces on them more complicated. However, many harmonic analysis problems on stratified Lie groups deserve a further investigation since most results of the theory of Fourier transforms and distributions in Euclidean spaces cannot yet be duplicated [13, 19, 26].

Nowadays, more and more attention has been paid to the study of function spaces which arise in the context of groups, such as variable Lebesgue spaces[19, 20], Orlicz spaces [13] and generalized Morrey spaces [12] et al. And Morrey spaces were originally introduced by Morrey in [21] to study the local behavior of solutions to second-order elliptic partial differential equations.

The multilinear fractional integral operators were first studied by Grafakos [8], followed by Kenig and Stein [18] et al. The importance of fractional integral operators is due to the fact that they have been widely used in various areas, such as potential analysis, harmonic analysis, and partial differential equations and so on[25].

Let $\mathbb{G}$ be a stratified group. According to the definition of the classical multilinear fractional integral operator, the multilinear fractional integral operator $\mathcal{I}_{\alpha,m}$ on stratified groups (also see [19]) can be defined by

$$\mathcal{I}_{\alpha,m}(\vec{f})(x) = \int_{\mathbb{G}^m} \frac{f_1(y_1)\cdots f_m(y_m)}{(\rho(y_1^{-1}x)+\cdots+\rho(y_m^{-1}x))^{mQ-\alpha}}d\vec{y}, \ \ 0 \leq \alpha < mQ,$$

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* Corresponding author: Jianglong Wu
Email addresses: jl-wu@163.com (Jianglong Wu), 2981311965@qq.com (Xiaojiao Tian)

and the multilinear fractional maximal operator $\mathcal{M}_{\alpha,m}$ is defined as follows

$$\mathcal{M}_{\alpha,m}(\vec{f})(x) = \sup_{\substack{B\ni x\\ B\subset\mathbb{G}}} |B|^{\frac{\alpha}{Q}} \prod_{i=1}^{m} \frac{1}{|B|}\int_B |f_i(y_i)|dy_i, \ \ 0 \le \alpha < mQ,$$

where the supremum is taken over all $\mathbb{G}$-balls B (see the notion in Section 2) containing x with radius $r > 0$, $|B|$ is the Haar measure of the $\mathbb{G}$-ball B, Q is the homogeneous dimension of $\mathbb{G}$, and vector function $\vec{f} = (f_1, f_2, \dots, f_m)$ is locally integrable on $\mathbb{G}$.

If we take $m = 1$, then $\mathcal{I}_{\alpha,m}$ is a natural generalization of the classical fractional integral operator $I_\alpha \equiv \mathcal{I}_{\alpha,1}$, and the classical fractional maximal function $M_\alpha \equiv \mathcal{M}_{\alpha,1}$ coincides for $\alpha = 0$ with the Hardy-Littlewood maximal function $M \equiv \mathcal{M}_{0,1}$.

A Carnot group, which also known as stratified Lie group, is a connected, simply connected, nilpotent Lie group with stratified Lie algebra [7] (see Section 2 below).

In 2013, Guliyev et al. [14] proved the boundedness of the fractional maximal operator $\mathcal{M}_{\alpha,m}$ with $m = 1$ in the generalized Morrey spaces $\mathcal{L}^{p,\varphi}(\mathbb{G})$ (see the definition below) on any Carnot group $\mathbb{G}$. In 2017, Eroglu et al. [3] studied the boundedness of the fractional integral operator $\mathcal{I}_{\alpha,m}$ with $m = 1$ in the generalized Morrey spaces $\mathcal{L}^{p,\varphi}(\mathbb{G})$ on Carnot group $\mathbb{G}$. In 2014, Guliyev and Ismayilova [15] obtained the boundedness of $\mathcal{M}_{\alpha,m}$ and $\mathcal{I}_{\alpha,m}$ on product generalized Morrey spaces with $\mathbb{G} = \mathbb{R}^n$. And in recently, Liu et al. [19] considered the multilinear fractional integral $\mathcal{I}_{\alpha,m}$ in variable Lebesgue spaces on stratified groups.

Inspired by the above literature, the purpose of this paper is to study the Spanne-type boundedness of the multilinear fractional integral operator $\mathcal{I}_{\alpha,m}$ and multilinear fractional maximal operator $\mathcal{M}_{\alpha,m}$ in the generalized Morrey spaces over some stratified Lie group $\mathbb{G}$. In all cases, the conditions for the boundedness of $\mathcal{I}_{\alpha,m}$ are given in terms of Zygmund-type integral inequalities on $(\varphi_1, \dots, \varphi_m, \psi)$ and the conditions for the boundedness of $\mathcal{M}_{\alpha,m}$ are given in terms of supremal type inequalities on $(\varphi_1, \dots, \varphi_m, \psi)$, which do not assume any assumption on monotonicity of $\varphi_1, \dots, \varphi_m$ and ψ in r.

Our main result can be stated as follows.

The first result gives the Spanne-type boundedness of multilinear fractional integral operator $\mathcal{I}_{\alpha,m}$ on product generalized Morrey space.

Theorem 1.1. *Let $\mathbb{G}$ be a stratified Lie group. Suppose that $m \in \mathbb{Z}^+$, $0 < \alpha_i < Q$, $1 < p_i < Q/\alpha_i$ $(i = 1, 2, \dots, m)$ and $\alpha = \sum_{i=1}^{m} \alpha_i$. Let q satisfy*

$$\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \frac{\alpha}{Q} < 1,$$

and $(\varphi_1, \dots, \varphi_m, \psi)$ satisfy

$$\prod_{i=1}^{m} \int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_i(x,s)s^{Q/p_i}}{t^{Q/q_i}} \frac{dt}{t} \le C\psi(x,r), \tag{1.1}$$

where $C > 0$ does not depend on $r > 0$ and $x \in \mathbb{G}$, and $\frac{1}{q} = \sum_{i=1}^{m} \frac{1}{q_i}$ with $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha_i}{Q}$ $(i = 1, 2, \dots, m)$. Then the operator $\mathcal{I}_{\alpha,m}$ is bounded from product space $\mathcal{L}^{p_1,\varphi_1}(\mathbb{G}) \times \cdots \times \mathcal{L}^{p_m,\varphi_m}(\mathbb{G})$ to $\mathcal{L}^{q,\psi}(\mathbb{G})$.

The following result gives the Spanne-type boundedness of multilinear fractional maximal operator $\mathcal{M}_{\alpha,m}$ on product generalized Morrey space.

Theorem 1.2. *Let $\mathbb{G}$ be a stratified Lie group. Suppose that $m \in \mathbb{Z}^+$, $0 < \alpha_i < Q$, $1 < p_i < Q/\alpha_i$ $(i = 1, 2, \dots, m)$ and $\alpha = \sum_{i=1}^{m} \alpha_i$. Let q satisfy*

$$\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \frac{\alpha}{Q} < 1,$$

and $(\varphi_1, \ldots, \varphi_m, \psi)$ *satisfy*

$$\prod_{i=1}^{m} \sup_{r<t<\infty} \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_i(x,s) s^{Q/p_i}}{t^{Q/q_i}} \leq C\psi(x,r), \tag{1.2}$$

where $C > 0$ *does not depend on* $r > 0$ *and* $x \in \mathbb{G}$, *and* $\frac{1}{q} = \sum_{i=1}^{m} \frac{1}{q_i}$ *with* $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha_i}{Q}$ $(i = 1, 2, \ldots, m)$. *Then the operator* $\mathcal{M}_{\alpha,m}$ *is bounded from product space* $\mathcal{L}^{p_1,\varphi_1}(\mathbb{G}) \times \cdots \times \mathcal{L}^{p_m,\varphi_m}(\mathbb{G})$ *to* $\mathcal{L}^{q,\psi}(\mathbb{G})$.

Remark 1.1. *(i) When* $\mathbb{G} = \mathbb{R}^n$, *the conclusion in Theorem 1.1 can be found from [15, Theorem 4.3], and the one in Theorem 1.2 can be found in [15, Theorem 3.3].*

(ii) In Theorem 1.1, when $m = 1$, *the result has been considered in [3](see Theorem 4.3).*

(iii) In Theorem 1.2, when $m = 1$, *the conclusion can be found in [14](see Theorem 3.2).*

(iv) In the case $\alpha = 0$ *from Theorem 1.2, the above result is also true. And when* $m = 1$, *the conclusion coincides with the Corollary 3.1 in [14].*

(v) Similar to [15], it is easy to check that the condition (1.2) is weaker than (1.1).

Throughout this paper, the letter C always stands for a constant independent of the main parameters involved and whose value may differ from line to line. In addition, we give some notations. Here and hereafter $|E|$ will always denote the Haar measure of a measurable set E of $\mathbb{G}$ and by χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{G}$. Let L^p $(1 \leq p \leq \infty)$ be the standard L^p-space with respect to the Haar measure dx. For a measurable set $E \subset \mathbb{G}$ and a positive integer m, we will use the notation $(E)^m = \underbrace{E \times \cdots \times E}_{m}$ sometimes. And we will occasionally use the notational $\vec{f} = (f_1, \ldots, f_m)$, $T(\vec{f}) = T(f_1, \ldots, f_m)$, $\mathrm{d}\vec{y} = dy_1 \cdots dy_m$ and $(x, \vec{y}) = (x, y_1, \ldots, y_m)$ for convenience.

2. Preliminaries and lemmas

To prove the main results of this paper, we need some necessary notions and remarks. Firstly, we recall some preliminaries concerning stratified Lie groups (or so-called Carnot groups). We refer the reader to [1, 6, 23].

2.1. Lie group $\mathbb{G}$

Definition 2.1. *Let* $m \in \mathbb{Z}^+$, $\mathcal{G}$ *be a finite-dimensional Lie algebra,* $[X, Y] = XY - YX \in \mathcal{G}$ *be Lie bracket with* $X, Y \in \mathcal{G}$.

(1) If $Z \in \mathcal{G}$ *is an* m^{th} *order Lie bracket and* $W \in \mathcal{G}$, *then* $[Z, W]$ *is an* $(m+1)^{st}$ *order Lie bracket.*

(2) We say $\mathcal{G}$ *is a nilpotent Lie algebra of step* m *if* m *is the smallest integer for which all Lie brackets of order* $m+1$ *are zero.*

(3) We say that a Lie algebra $\mathcal{G}$ *is stratified if there is a direct sum vector space decomposition*

$$\mathcal{G} = \oplus_{j=1}^{m} V_j = V_1 \oplus \cdots \oplus V_m \tag{2.1}$$

such that $\mathcal{G}$ *is nilpotent of step* m, *that is,*

$$[V_1, V_j] = \begin{cases} V_{j+1} & 1 \leq j \leq m-1 \\ 0 & j \geq m \end{cases}$$

holds.

It is not difficult to find that the above V_1 generates the whole of the Lie algebra $\mathcal{G}$ by taking Lie brackets since each element of V_j $(2 \le j \le m)$ is a linear combination of $(j-1)^{\text{th}}$ order Lie bracket of elements of V_1.

With the help of the related notions of Lie algebra (see Definition 2.1), the following definition can be obtained.

Definition 2.2. *Let $\mathbb{G}$ be a finite-dimensional, connected and simply-connected Lie group associated with Lie algebra $\mathcal{G}$. Then*

(1) $\mathbb{G}$ is called nilpotent if its Lie algebra $\mathcal{G}$ is nilpotent.

(2) $\mathbb{G}$ is said to be stratified if its Lie algebra $\mathcal{G}$ is stratified.

(3) $\mathbb{G}$ is called homogeneous if it is a nilpotent Lie group whose Lie algebra $\mathcal{G}$ admits a family of dilations $\{\delta_r\}$, namely, for $r > 0$, $X_k \in V_k$ $(k = 1, \ldots, m)$,

$$\delta_r\Big(\sum_{k=1}^{m} X_k\Big) = \sum_{k=1}^{m} r^k X_k,$$

which are Lie algebra automorphisms.

Remark 2.1. *Let $\mathcal{G} = \mathcal{G}_1 \supset \mathcal{G}_2 \supset \cdots \supset \mathcal{G}_{m+1} = \{0\}$ denote the lower central series of $\mathcal{G}$, and $X = \{X_1, \ldots, X_n\}$ be a basis for V_1 of $\mathcal{G}$.*

(i) (see [27]) The direct sum decomposition (2.1) can be constructed by identifying each $\mathcal{G}_j$ as a vector subspace of $\mathcal{G}$ and setting $V_m = \mathcal{G}_m$ and $V_j = \mathcal{G}_j \setminus \mathcal{G}_{j+1}$ for $j = 1, \ldots, m-1$.

(ii) (see [5]) The number $Q = \operatorname{trace} A = \sum_{j=1}^{m} j \dim(V_j)$ is called the homogeneous dimension of $\mathcal{G}$, where A is a diagonalizable linear transformation of $\mathcal{G}$ with positive eigenvalues.

(iii) (see [27] or [5]) The number Q is also called the homogeneous dimension of $\mathbb{G}$ since $\mathrm{d}(\delta_r x) = r^Q \mathrm{d}x$ for all $r > 0$, and

$$Q = \sum_{j=1}^{m} j \dim(V_j) = \sum_{j=1}^{m} \dim(\mathcal{G}_j).$$

By the Baker-Campbell-Hausdorff formula for sufficiently small elements X and Y of $\mathcal{G}$ one has

$$\exp(X)\exp(Y) = \exp(H(X, Y)),$$

where $\exp : \mathcal{G} \to \mathbb{G}$ is the exponential map, $H(X, Y) = X + Y + \frac{1}{2}[X, Y] + \cdots$ is an infinite linear combination of X and Y and their Lie brackets, and the dots denote terms of order higher than two. And the above equation is finite in the case of $\mathcal{G}$ is a nilpotent Lie algebra.

The following properties can be found in [22](see Proposition 1.1.1, or Proposition 1.2 in [6]).

Proposition 2.1. *Let $\mathcal{G}$ be a nilpotent Lie algebra, and let $\mathbb{G}$ be the corresponding connected and simply-connected nilpotent Lie group. Then we have*

(1) The exponential map $\exp : \mathcal{G} \to \mathbb{G}$ is a diffeomorphism. Furthermore, the group law $(x, y) \mapsto xy$ is a polynomial map if $\mathbb{G}$ is identified with $\mathcal{G}$ via $\exp$.

(2) If λ is a Lebesgue measure on $\mathcal{G}$, then $\exp \lambda$ is a bi-invariant Haar measure on $\mathbb{G}$ (or a bi-invariant Haar measure $\mathrm{d}x$ on $\mathbb{G}$ is just the lift of Lebesgue measure on $\mathcal{G}$ via $\exp$).

Thereafter, we use Q to denote the homogeneous dimension of $\mathbb{G}$, y^{-1} represents the inverse of $y \in \mathbb{G}$, $y^{-1}x$ stands for the group multiplication of y^{-1} by x and the group identity element of $\mathbb{G}$ will be referred to as the origin denotes by e.

A homogeneous norm on $\mathbb{G}$ is a continuous function $x \to \rho(x)$ from $\mathbb{G}$ to $[0, \infty)$, which is C^∞ on $\mathbb{G} \setminus \{e\}$ and satisfies

$$\begin{cases} \rho(x^{-1}) = \rho(x), \\ \rho(\delta_t x) = t\rho(x) \ \text{ for all } x \in \mathbb{G} \text{ and } t > 0, \\ \rho(e) = 0. \end{cases}$$

Moreover, there exists a constant $c_0 \geq 1$ such that $\rho(xy) \leq c_0(\rho(x) + \rho(y))$ for all $x, y \in \mathbb{G}$.

With the norm above, we define the $\mathbb{G}$ ball centered at x with radius r by $B(x,r) = \{y \in \mathbb{G} : \rho(y^{-1}x) < r\}$, and by λB denote the ball $B(x, \lambda r)$ with $\lambda > 0$, let $B_r = B(e,r) = \{y \in \mathbb{G} : \rho(y) < r\}$ be the open ball centered at e with radius r, which is the image under δ_r of $B(e,1)$. And by ${}^{\complement}B(x,r) = \mathbb{G} \setminus B(x,r) = \{y \in \mathbb{G} : \rho(y^{-1}x) \geq r\}$ denote the complement of $B(x,r)$. Let $|B(x,r)|$ be the Haar measure of the ball $B(x,r) \subset \mathbb{G}$, and there exists $c_1 = c_1(\mathbb{G})$ such that

$$|B(x,r)| = c_1 r^Q, \quad x \in \mathbb{G}, r > 0.$$

In addition, the Haar measure of a homogeneous Lie group $\mathbb{G}$ satisfies the doubling condition (see [4, pages 140 and 501]), i.e. $\forall\, x \in \mathbb{G}$, $r > 0$, $\exists\, C$, such that

$$|B(x,2r)| \leq C|B(x,r)|.$$

The most basic partial differential operator in a stratified Lie group is the sub-Laplacian associated with $X = \{X_1, \ldots, X_n\}$, i.e., the second-order partial differential operator on $\mathbb{G}$ given by

$$\mathfrak{L} = \sum_{i=1}^{n} X_i^2.$$

The part (1) in following lemma is known as the Hölder's inequality on Lebesgue spaces over Lie groups $\mathbb{G}$, it can be found in [26]. And by simple calculations, the part (2) can be deduced from the part (1).

Lemma 2.1 (Hölder's inequality on $\mathbb{G}$). *Let $\Omega \subset \mathbb{G}$ be a measurable set.*

(1) Suppose that $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, and measurable functions $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then there exists a positive constant C such that

$$\int_\Omega |f(x)g(x)| \mathrm{d}x \leq C\|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

(2) Suppose that $1 < q_i < \infty$ $(i = 1, 2, \ldots, m)$ and q satisfy $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. Then there exists a positive constant C such that the inequality

$$\|f_1 \cdots f_m\|_{L^q(\Omega)} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{q_i}(\Omega)}.$$

holds for all $f_i \in L^{q_i}(\Omega)$ $(i = 1, 2, \ldots, m)$.

The following property can be found in [26].

Lemma 2.2 (Norms of characteristic functions). *Let $0 < p < \infty$ and $\Omega \subset \mathbb{G}$ be a measurable set with finite Haar measure. Then*

$$\|\chi_\Omega\|_{L^p(\mathbb{G})} = \|\chi_\Omega\|_{WL^p(\mathbb{G})} = |\Omega|^{1/p}.$$

2.2. Morrey spaces on $\mathbb{G}$

Morrey spaces, named after C. B. Morrey, seem to describe the boundedness property of the classical fractional integral operators more precisely than Lebesgue spaces [17].

Definition 2.3 (Morrey-type spaces on $\mathbb{G}$). *Let $1 \le p < \infty$, and $B = B(x,r)$ be a $\mathbb{G}$-ball centered at x with radius $r > 0$.*

(1) When $0 \le \lambda \le Q$. The Morrey-type space $L^{p,\lambda}(\mathbb{G})$ is defined by

$$L^{p,\lambda}(\mathbb{G}) = \{f \in L^p_{\mathrm{loc}}(\mathbb{G}) : \|f\|_{L^{p,\lambda}(\mathbb{G})} < \infty\}$$

with

$$\|f\|_{L^{p,\lambda}(\mathbb{G})} = \sup_{\substack{x\in\mathbb{G}\\ r>0}} \Big(\frac{1}{|B|^{\lambda/Q}} \int_B |f(y)|^p \mathrm{d}y\Big)^{1/p}.$$

(2) Set $\varphi(x,r)$ be a positive measurable function on $\mathbb{G} \times (0,\infty)$. The generalized Morrey space $\mathcal{L}^{p,\varphi}(\mathbb{G})$ is defined for all functions $f \in L^p_{\mathrm{loc}}(\mathbb{G})$ with the finite quasinorm

$$\|f\|_{\mathcal{L}^{p,\varphi}(\mathbb{G})} = \sup_{\substack{x\in\mathbb{G}\\ r>0}} \frac{1}{\varphi(x,r)}\Big(\frac{1}{|B|} \int_B |f(y)|^p \mathrm{d}y\Big)^{1/p}.$$

Remark 2.2 (see [3] or [11]). *(i) It is well known that if $1 \le p < \infty$ then*

$$L^{p,\lambda}(\mathbb{G}) = \begin{cases} L^p(\mathbb{G}) & \text{if } \lambda = 0, \\ L^\infty(\mathbb{G}) & \text{if } \lambda = Q, \\ \Theta & \text{if } \lambda < 0 \text{ or } \lambda > Q, \end{cases}$$

where Θ is the set of all functions equivalent to 0 on $\mathbb{G}$.

(ii) In (2), when $1 \le p < \infty$ and $0 \le \lambda \le Q$, we have $\mathcal{L}^{p,\varphi}(\mathbb{G}) = L^{p,\lambda}(\mathbb{G})$ if $\varphi(x,r) = |B|^{(\lambda/Q-1)/p}$ and $B \subset \mathbb{G}$ denotes the ball with radius r and containing x.

Now, we give some necessary notation and notions. Let w be a weight function and u be a continuous and non-negative function on $(0,\infty)$.

- We denote by $L^p_w(\Omega)$ $(1 < p < \infty)$ the weight $L^p(\Omega)$ space of all functions f measurable on a measurable set $\Omega \subset \mathbb{G}$ with $\|f\|_{L^p_w(\Omega)} = \|wf\|_{L^p(\Omega)} < \infty$.
- By $L^\infty_w(0,\infty)$ denotes the weight $L^\infty(0,\infty)$ space of all functions $g(t)$ measurable on $(0,\infty)$ with finite norm

$$\|g\|_{L^\infty_w(0,\infty)} = \operatorname*{ess\,sup}_{t>0} w(t)|g(t)|.$$

- $L^\infty_1(0,\infty) \overset{w=1}{=\!=} L^\infty(0,\infty)$.
- Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all non-negative functions on $(0,\infty)$.
- Denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and set

$$\mathbb{A} = \{\varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0^+} \varphi(t) = 0\}.$$

- We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) = \|ug\|_{L^\infty(t,\infty)}, \qquad t \in (0,\infty).$$

3. Proofs of the main results

Now we give the proofs of the Theorem 1.1 and Theorem 1.2.

3.1. *Proof of Theorem 1.1*

In order to prove Theorem 1.1, we also need some auxiliary results. The following lemma can be obtained from [19] with $p_i(x)$ $(i = 1, 2, \dots, m)$ is constant (see P.111–114). And in the case $m = 1$, the following result can be founded in [16] (see theorem 2.5 or [6]).

Lemma 3.1. *Suppose that* $m \in \mathbb{Z}^+$, $0 < \alpha_i < Q$, $1 < p_i < Q/\alpha_i$ $(i = 1, 2, \dots, m)$ *and* $\alpha = \sum_{i=1}^{m} \alpha_i$. *Let* q *satisfy*

$$\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \frac{\alpha}{Q} < 1.$$

Then the operator $\mathcal{I}_{\alpha,m}$ *is bounded from product space* $L^{p_1}(\mathbb{G}) \times \cdots \times L^{p_m}(\mathbb{G})$ *to* $L^q(\mathbb{G})$, *namely*

$$\|\mathcal{I}_{\alpha,m}(\vec{f})\|_{L^q(\mathbb{G})} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mathbb{G})}.$$

The following pointwise estimate is also necessary, and it is proved in [10] (or [3]).

Lemma 3.2. *Let* v_1, v_2 *and* w *be positive weight functions on* $(0, \infty)$, *and let* v_1 *be bounded outside a neighborhood of the origin. Then the inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H_\omega g(t) \le C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \tag{3.1}$$

holds for some $C > 0$ *and all nonnegative and nondecreasing function* g *on* $(0, \infty)$ *if and only if*

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{\omega(s)\mathrm{d}s}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

where

$$H_\omega g(t) = \int_t^\infty g(s)\omega(s)\mathrm{d}s, \quad 0 < t < \infty.$$

Moreover, the value $C = B$ *is the best constant for* (3.1).

In addition, the following local estimates are valid.

Lemma 3.3. *Suppose that* $m \in \mathbb{Z}^+$, $0 < \alpha_i < Q$, $1 < p_i < Q/\alpha_i$ $(i = 1, 2, \dots, m)$ *and* $\alpha = \sum_{i=1}^{m} \alpha_i$. *Let* q *satisfy*

$$\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \frac{\alpha}{Q} < 1.$$

Then the inequality

$$\|\mathcal{I}_{\alpha,m}(\vec{f})\|_{L^q(B(x,r))} \le C r^{Q/q} \prod_{i=1}^{m} \int_{2r}^{\infty} t^{\alpha_i - \frac{Q}{p_i} - 1} \|f_i\|_{L^{p_i}(B(x,t))} \mathrm{d}t$$

holds for any ball $B(x, r)$ *and for all* $\vec{f} \in L^{p_1}_{\mathrm{loc}}(\mathbb{G}) \times \cdots \times L^{p_m}_{\mathrm{loc}}(\mathbb{G})$.

Proof. For arbitrary $x \in \mathbb{G}$, set $B = B(x, r)$ be the $\mathbb{G}$ ball centered at x with radius r, and $2B = B(x, 2r)$. For each j, we decompose $f_j = f_j^0 + f_j^\infty$ with $f_j^0 = f_j \chi_{2B}$. Then

$$\begin{aligned}\prod_{j=1}^m f_j = \prod_{j=1}^m \left(f_j^0 + f_j^\infty\right) &= \sum_{\beta_1,\dots,\beta_m \in \{0,\infty\}} f_1^{\beta_1} \cdots f_m^{\beta_m} \\ &= \prod_{j=1}^m f_j^0 + \sum_{(\beta_1,\dots,\beta_m)\in \ell} f_1^{\beta_1} \cdots f_m^{\beta_m},\end{aligned}$$

where $\ell = \{(\beta_1, \dots, \beta_m) : \text{there is at least one } \beta_j \neq 0\}$. Thus, for arbitrary $y \in B(x, r)$, we obtain

$$\mathcal{I}_{\alpha,m}(\vec{f})(y) = \mathcal{I}_{\alpha,m}(f_1^0, \dots, f_m^0)(y) + \sum_{(\beta_1,\dots,\beta_m)\in \ell} \mathcal{I}_{\alpha,m}(f_1^{\beta_1}, \dots, f_m^{\beta_m})(y).$$

Then,

$$\begin{aligned}\|\mathcal{I}_{\alpha,m}(\vec{f})\|_{L^q(B(x,r))} &\leq \|\mathcal{I}_{\alpha,m}(f_1^0, \dots, f_m^0)\|_{L^q(B(x,r))} + \left\| \sum_{(\beta_1,\dots,\beta_m)\in \ell} \mathcal{I}_{\alpha,m}(f_1^{\beta_1}, \dots, f_m^{\beta_m}) \right\|_{L^q(B(x,r))} \\ &= E_1 + E_2.\end{aligned}$$

For E_1, applying the boundedness of $\mathcal{I}_{\alpha,m}$ (see Lemma 3.1), we have

$$\begin{aligned}E_1 &= \|\mathcal{I}_{\alpha,m}(f_1^0, \dots, f_m^0)\|_{L^q(B(x,r))} \leq \|\mathcal{I}_{\alpha,m}(f_1^0, \dots, f_m^0)\|_{L^q(\mathbb{G})} \\ &\leq C \prod_{i=1}^m \|f_i^0\|_{L^{p_i}(\mathbb{G})} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(B(x,2r))}.\end{aligned}$$

Applying the continuous version of Minkowski's inequality and doubling condition of Haar measure, it can obtain the following fact

$$\|f_i\|_{L^{p_i}(B(x,2r))} \leq C r^{\frac{Q}{q_i}} \int_{2r}^\infty t^{\alpha_i - \frac{Q}{p_i} - 1} \|f_i\|_{L^{p_i}(B(x,t))} \mathrm{d}t, \qquad i = 1, 2, \dots, m. \tag{3.2}$$

Therefore, we get

$$\begin{aligned}E_1 &\leq C \prod_{i=1}^m r^{\frac{Q}{q_i}} \int_{2r}^\infty t^{\alpha_i - \frac{Q}{p_i} - 1} \|f_i\|_{L^{p_i}(B(x,t))} \mathrm{d}t \\ &\leq C r^{\frac{Q}{q}} \prod_{i=1}^m \int_{2r}^\infty t^{\alpha_i - \frac{Q}{p_i} - 1} \|f_i\|_{L^{p_i}(B(x,t))} \mathrm{d}t.\end{aligned}$$

To estimate E_2, we consider first the case $\beta_1 = \beta_2 = \cdots = \beta_m = \infty$. When $y \in B(x, r)$ and $z_i \in {}^{\complement}B(x, 2r) = \mathbb{G} \setminus B(x, 2r)$ $(i = 1, 2, \dots, m)$, the conditions imply $\frac{1}{2}\rho(z_i^{-1}y) \leq \rho(z_i^{-1}x) \leq \frac{3}{2}\rho(z_i^{-1}y)$. Thus, we can obtain

$$\begin{aligned}|\mathcal{I}_{\alpha,m}(f_1^\infty, \dots, f_m^\infty)(y)| &= \left| \int_{\mathbb{G}^m} \frac{f_1^\infty(z_1) \cdots f_m^\infty(z_m)}{(\rho(z_1^{-1}y) + \cdots + \rho(z_m^{-1}y))^{mQ-\alpha}} \mathrm{d}\vec{z} \right| \\ &\leq C \int_{(\mathbb{G}\setminus B(x,2r))^m} \frac{|f_1(z_1) \cdots f_m(z_m)|}{|\rho(z_1^{-1}y) + \cdots + \rho(z_m^{-1}y)|^{mQ-\alpha}} \mathrm{d}\vec{z} \\ &\leq C \prod_{i=1}^m \int_{\mathbb{G}\setminus B(x,2r)} \frac{|f_i(z_i)|}{\rho(z_i^{-1}y)^{Q-\alpha_i}} \mathrm{d}z_i \\ &\leq C \prod_{i=1}^m \int_{\mathbb{G}\setminus B(x,2r)} \frac{|f_i(z_i)|}{\rho(z_i^{-1}x)^{Q-\alpha_i}} \mathrm{d}z_i.\end{aligned}$$

Further, by applying Fubini's theorem, Hölder's inequality (see Lemma 2.1) and Lemma 2.2, we have that

$$\begin{aligned}\int_{\mathbb{G}\setminus B(x,2r)} \frac{|f_i(z_i)|}{\rho(z_i^{-1}x)^{Q-\alpha_i}}\mathrm{d}z_i &= \int_{\mathbb{G}\setminus B(x,2r)} |f_i(z_i)|\rho(z_i^{-1}x)^{\alpha_i-Q}\mathrm{d}z_i\\ &\le C\int_{\mathbb{G}\setminus B(x,2r)} |f_i(z_i)|\Big(\int_{\rho(z_i^{-1}x)}^{\infty} t^{\alpha_i-Q-1}\mathrm{d}t\Big)\mathrm{d}z_i\\ &\approx C\int_{2r}^{\infty}\Big(\int_{2r<\rho(z_i^{-1}x)<t}|f_i(z_i)|\mathrm{d}z_i\Big)t^{\alpha_i-Q-1}\mathrm{d}t\\ &\le C\int_{2r}^{\infty}\Big(\int_{B(x,t)}|f_i(z_i)|\mathrm{d}z_i\Big)t^{\alpha_i-Q-1}\mathrm{d}t\\ &\le C\int_{2r}^{\infty} t^{\alpha_i-Q-1}\|f_i\|_{L^{p_i}(B(x,t))}\|\chi_{B(x,t)}\|_{L^{p_i'}(B(x,t))}\mathrm{d}t\\ &\le C\int_{2r}^{\infty} t^{\alpha_i-\frac{Q}{p_i}-1}\|f_i\|_{L^{p_i}(B(x,t))}\mathrm{d}t.\end{aligned}$$

Consequently, taking into account the obtained estimate above, we conclude that

$$\begin{aligned}E_{2\infty} &= \|\mathcal{I}_{\alpha,m}(f_1^{\infty},\dots,f_m^{\infty})\|_{L^q(B(x,r))} = \Big(\int_{B(x,r)}|\mathcal{I}_{\alpha,m}(f_1^{\infty},\dots,f_m^{\infty})(y)|^q\mathrm{d}y\Big)^{1/q}\\ &\le Cr^{\frac{Q}{q}}\prod_{i=1}^{m}\int_{2r}^{\infty} t^{\alpha_i-\frac{Q}{p_i}-1}\|f_i\|_{L^{p_i}(B(x,t))}\mathrm{d}t.\end{aligned}$$

Now, for $(\beta_1,\dots,\beta_m)\in\ell$, let us consider the terms $E_{2(\beta_1,\dots,\beta_m)}$ such that at least one $\beta_i=0$ and one $\beta_j=\infty$. Without loss of generality, we assume that $\beta_1=\cdots=\beta_k=0$ and $\beta_{k+1}=\cdots=\beta_m=\infty$ with $1\le k<m$. It is easy to check that $\rho(z_i^{-1}x)\approx\rho(z_i^{-1}y)$ since $y\in B(x,r)$ and $z_i\in {}^{\complement}B(x,2r)=\mathbb{G}\setminus B(x,2r)$ $(i=1,2,\dots,m)$. Thus, we have

$$\begin{aligned}|\mathcal{I}_{\alpha,m}(f_1^0,\dots,f_k^0,f_{k+1}^{\infty},\dots,f_m^{\infty})(y)| &= \Big|\int_{\mathbb{G}^m}\frac{f_1^0(z_1)\cdots f_k^0(z_k)f_{k+1}^{\infty}(z_{k+1})\cdots f_m^{\infty}(z_m)}{(\rho(z_1^{-1}y)+\cdots+\rho(z_m^{-1}y))^{mQ-\alpha}}\mathrm{d}\vec{z}\Big|\\ &\le C\int_{\mathbb{G}^m}\frac{|f_1^0(z_1)\cdots f_k^0(z_k)f_{k+1}^{\infty}(z_{k+1})\cdots f_m^{\infty}(z_m)|}{|\rho(z_1^{-1}y)+\cdots+\rho(z_m^{-1}y)|^{mQ-\alpha}}\mathrm{d}\vec{z}\\ &\le C\Big(\prod_{i=1}^{k}\int_{B(x,2r)}\frac{|f_i(z_i)|}{\rho(z_i^{-1}y)^{Q-\alpha_i}}\mathrm{d}z_i\Big)\Big(\prod_{j=k+1}^{m}\int_{\mathbb{G}\setminus B(x,2r)}\frac{|f_i(z_j)|}{\rho(z_j^{-1}y)^{Q-\alpha_j}}\mathrm{d}z_j\Big)\\ &\le C\Big(\prod_{i=1}^{k}\int_{B(x,2r)}\frac{|f_i(z_i)|}{\rho(z_i^{-1}x)^{Q-\alpha_i}}\mathrm{d}z_i\Big)\Big(\prod_{j=k+1}^{m}\int_{\mathbb{G}\setminus B(x,2r)}\frac{|f_i(z_j)|}{\rho(z_j^{-1}x)^{Q-\alpha_j}}\mathrm{d}z_j\Big).\end{aligned}$$

Similar to the estimate above, using Fubini's theorem, Hölder's inequality (see Lemma 2.1), Lemma 2.2 and (3.2), we get

$$\begin{aligned}E_{2(\beta_1,\dots,\beta_m)} &= \|\mathcal{I}_{\alpha,m}(f_1^0,\dots,f_k^0,f_{k+1}^{\infty},\dots,f_m^{\infty})\|_{L^q(B(x,r))}\\ &= \Big(\int_{B(x,r)}|\mathcal{I}_{\alpha,m}(f_1^0,\dots,f_k^0,f_{k+1}^{\infty},\dots,f_m^{\infty})(y)|^q\mathrm{d}y\Big)^{1/q}\\ &\le Cr^{\frac{Q}{q}}\Big(\prod_{i=1}^{k}\int_{B(x,2r)}\frac{|f_i(z_i)|}{\rho(z_i^{-1}x)^{Q-\alpha_i}}\mathrm{d}z_i\Big)\Big(\prod_{j=k+1}^{m}\int_{\mathbb{G}\setminus B(x,2r)}\frac{|f_i(z_j)|}{\rho(z_j^{-1}x)^{Q-\alpha_j}}\mathrm{d}z_j\Big)\\ &\le Cr^{\frac{Q}{q}}\prod_{i=1}^{m}\int_{2r}^{\infty} t^{\alpha_i-\frac{Q}{p_i}-1}\|f_i\|_{L^{p_i}(B(x,t))}\mathrm{d}t.\end{aligned}$$

Combining the above estimates we get the desired result. The proof is completed. □

Now, we give the proof of Theorem 1.1.

Proof. [Proof of Theorem 1.1] Let $1 < p_i < \infty$ $(i = 1, 2, \dots, m)$ and $\vec{f} = (f_1, f_2, \dots, f_m) \in \mathcal{L}^{p_1,\varphi_1}(\mathbb{G}) \times \cdots \times \mathcal{L}^{p_m,\varphi_m}(\mathbb{G})$. According to the assumption (1.1), and using Lemma 3.3 and Lemma 3.2 with $w(r) = r^{-Q/q_i - 1}$, $v_2(r) = \psi(x,r)^{-1/m}$ and $v_1(r) = \varphi_i(x,r)^{-1} r^{-Q/p_i}$ $(i = 1, 2, \dots, m)$, we have

$$\begin{aligned}
\|\mathcal{I}_{\alpha,m}(\vec{f})\|_{\mathcal{L}^{q,\psi}(\mathbb{G})} &= \sup_{\substack{x\in\mathbb{G}\\ r>0}} \frac{1}{\psi(x,r)}\Big(\frac{1}{|B(x,r)|}\int_{B(x,r)} |\mathcal{I}_{\alpha,m}(\vec{f})(y)|^q \mathrm{d}y\Big)^{1/q} \\
&\le C \sup_{\substack{x\in\mathbb{G}\\ r>0}} \prod_{i=1}^m \psi(x,r)^{-1/m} \int_r^\infty t^{-\frac{Q}{q_i}-1} \|f_i\|_{L^{p_i}(B(x,t))} \mathrm{d}t \\
&\le C \sup_{\substack{x\in\mathbb{G}\\ r>0}} \prod_{i=1}^m \varphi_i(x,r)^{-1} r^{-Q/p_i} \|f_i\|_{L^{p_i}(B(x,r))} \\
&\le C \prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,\varphi_i}(\mathbb{G})}
\end{aligned}$$

This completes the proof of Theorem 1.1. □

3.2. *Proof of Theorem 1.2*

In order to prove Theorem 1.2, we also need the follow auxiliary results.

Similar to the pointwise relation between fraction integral operator and fractional maximal operator, by elementary calculations, we can obtain the following lemma, and omit the proof.

Lemma 3.4. *Suppose that $m \in \mathbb{Z}^+$, $0 < \alpha_i < Q$, $1 < p_i < Q/\alpha_i$ $(i = 1, 2, \dots, m)$ and $\alpha = \sum_{i=1}^m \alpha_i$. Let $\vec{f} \in L^{p_1}(\mathbb{G}) \times \cdots \times L^{p_m}(\mathbb{G})$, then there exists a positive constant C such that the pointwise inequality*

$$\mathcal{M}_{\alpha,m}(\vec{f})(x) \le C\mathcal{I}_{\alpha,m}(|f_1|, \dots, |f_m|)(x)$$

holds for any $x \in \mathbb{G}$.

According to Lemma 3.1 and Lemma 3.4, the following result can be obtained, and we omit the proof.

Lemma 3.5. *Suppose that $m \in \mathbb{Z}^+$, $0 < \alpha_i < Q$, $1 < p_i < Q/\alpha_i$ $(i = 1, 2, \dots, m)$ and $\alpha = \sum_{i=1}^m \alpha_i$. Let q satisfy*

$$\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \frac{\alpha}{Q} < 1.$$

Then the operator $\mathcal{M}_{\alpha,m}$ is bounded from product space $L^{p_1}(\mathbb{G}) \times \cdots \times L^{p_m}(\mathbb{G})$ to $L^q(\mathbb{G})$, namely

$$\|\mathcal{M}_{\alpha,m}(\vec{f})\|_{L^q(\mathbb{G})} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{G})}.$$

In addition, the following local estimates are true. Specifically, when m=1, the result can be found in [14] (see Lemma 3.2).

Lemma 3.6. *Suppose that $m \in \mathbb{Z}^+$, $0 < \alpha_i < Q$, $1 < p_i < Q/\alpha_i$ $(i = 1, 2, \ldots, m)$ and $\alpha = \sum_{i=1}^{m} \alpha_i$. Let q satisfy*

$$\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \frac{\alpha}{Q} < 1.$$

Then the inequality

$$\|\mathcal{M}_{\alpha,m}(\vec{f})\|_{L^q(B(x,r))} \le C r^{Q/q} \prod_{i=1}^{m} \sup_{t>2r} t^{\alpha_i - \frac{Q}{p_i}} \|f_i\|_{L^{p_i}(B(x,t))}$$

holds for any ball $B(x, r)$ and for all $\vec{f} \in L^{p_1}_{\mathrm{loc}}(\mathbb{G}) \times \cdots \times L^{p_m}_{\mathrm{loc}}(\mathbb{G})$.

Proof. Similar to the proof of Lemma 3.3, for each j, we decompose $f_j = f_j^0 + f_j^\infty$ with $f_j^0 = f_j \chi_{2B}$, and

$$\prod_{j=1}^{m} f_j = \prod_{j=1}^{m} f_j^0 + \sum_{(\beta_1,\ldots,\beta_m)\in\ell} f_1^{\beta_1} \cdots f_m^{\beta_m},$$

where $\ell = \{(\beta_1, \ldots, \beta_m) :$ there is at least one $\beta_j \neq 0\}$. Thus, for arbitrary $y \in B(x, r)$, we obtain

$$\mathcal{M}_{\alpha,m}(\vec{f})(y) = \mathcal{M}_{\alpha,m}(f_1^0, \ldots, f_m^0)(y) + \sum_{(\beta_1,\ldots,\beta_m)\in\ell} \mathcal{M}_{\alpha,m}(f_1^{\beta_1}, \ldots, f_m^{\beta_m})(y).$$

Then,

$$\begin{aligned}\|\mathcal{M}_{\alpha,m}(\vec{f})\|_{L^q(B(x,r))} &\le \|\mathcal{M}_{\alpha,m}(f_1^0, \ldots, f_m^0)\|_{L^q(B(x,r))} + \Big\| \sum_{(\beta_1,\ldots,\beta_m)\in\ell} \mathcal{M}_{\alpha,m}(f_1^{\beta_1}, \ldots, f_m^{\beta_m}) \Big\|_{L^q(B(x,r))} \\ &= E_1 + E_2.\end{aligned}$$

For E_1, applying the boundedness of $\mathcal{M}_{\alpha,m}$ (see Lemma 3.5) and the doubling condition of Haar measure, we have

$$\begin{aligned}E_1 &= \|\mathcal{M}_{\alpha,m}(f_1^0, \ldots, f_m^0)\|_{L^q(B(x,r))} \le \|\mathcal{M}_{\alpha,m}(f_1^0, \ldots, f_m^0)\|_{L^q(\mathbb{G})} \\ &\le C \prod_{i=1}^{m} \|f_i^0\|_{L^{p_i}(\mathbb{G})} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(B(x,2r))} \\ &\le C r^{\frac{Q}{q}} \prod_{i=1}^{m} \sup_{t>2r} t^{\alpha_i - \frac{Q}{p_i}} \|f_i\|_{L^{p_i}(B(x,t))}.\end{aligned}$$

To estimate E_2, we first consider the case $\beta_1 = \beta_2 = \cdots = \beta_m = \infty$. Let y be an arbitrary point from $B(x, r)$. If $B(y, t) \bigcap {}^{\complement}B(x, 2r) \neq \emptyset$, then $t > r$. In fact, when $z_i \in B(y, t) \bigcap {}^{\complement}B(x, 2r)$ $(i = 1, 2, \ldots, m)$, we have

$$t > \rho(z_i^{-1} y) \ge \rho(z_i^{-1} x) - \rho(y^{-1} x) > 2r - r = r.$$

On the other hand, $B(y, t) \bigcap {}^{\complement}B(x, 2r) \subset B(x, 2t)$. Indeed, when $z_i \in B(y, t) \bigcap {}^{\complement}B(x, 2r)$ $(i = 1, 2, \ldots, m)$, we get

$$\rho(z_i^{-1} x) \le \rho(z_i^{-1} y) + \rho(y^{-1} x) < t + r < 2t.$$

Then, for all $y \in B(x,r)$ and any $z_i \in B(y,t) \bigcap {}^{\complement}B(x,2r)$ $(i = 1,2,\dots,m)$, using Hölder's inequality (see Lemma 2.1) and Lemma 2.2, we have

$$\begin{aligned}
\mathcal{M}_{\alpha,m}(f_1^{\infty},\dots,f_m^{\infty})(y) &= \sup_{\substack{B(x,r)\ni y\\ t>0}} |B(y,t)|^{\frac{\alpha}{Q}} \prod_{i=1}^{m} \frac{1}{|B(y,t)|} \int_{B(y,t)} |f_i^{\infty}(z_i)|dz_i \\
&= \sup_{\substack{B(x,r)\ni y\\ t>0}} |B(y,t)|^{\frac{\alpha}{Q}} \prod_{i=1}^{m} \frac{1}{|B(y,t)|} \int_{B(y,t)\bigcap {}^{\complement}B(x,2r)} |f_i(z_i)|dz_i \\
&\le C \sup_{t>r} |B(x,2t)|^{\frac{\alpha}{Q}} \prod_{i=1}^{m} \frac{1}{|B(x,2t)|} \int_{B(x,2t)} |f_i(z_i)|dz_i \\
&\le C \sup_{t>2r} |B(x,t)|^{\frac{\alpha}{Q}} \prod_{i=1}^{m} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f_i(z_i)|dz_i \\
&\le C \sup_{t>2r} |B(x,t)|^{\frac{\alpha}{Q}} \prod_{i=1}^{m} \frac{1}{|B(x,t)|} \|f_i\|_{L^{p_i}(B(x,t))} \|\chi_{B(x,t)}\|_{L^{p_i'}(B(x,t))} \\
&\le C \sup_{t>2r} |B(x,t)|^{\frac{\alpha}{Q}} \prod_{i=1}^{m} |B(x,t)|^{-1/p_i} \|f_i\|_{L^{p_i}(B(x,t))} \\
&\le C \prod_{i=1}^{m} \sup_{t>2r} t^{\alpha_i - \frac{Q}{p_i}} \|f_i\|_{L^{p_i}(B(x,t))}.
\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}
E_{2\infty} &= \|\mathcal{M}_{\alpha,m}(f_1^{\infty},\dots,f_m^{\infty})\|_{L^q(B(x,r))} = \Big(\int_{B(x,r)} |\mathcal{M}_{\alpha,m}(f_1^{\infty},\dots,f_m^{\infty})(y)|^q dy \Big)^{1/q} \\
&\le C r^{\frac{Q}{q}} \prod_{i=1}^{m} \sup_{t>2r} t^{\alpha_i - \frac{Q}{p_i}} \|f_i\|_{L^{p_i}(B(x,t))}.
\end{aligned}$$

Now, for $(\beta_1,\dots,\beta_m) \in \ell$, let us consider the terms $E_{2(\beta_1,\dots,\beta_m)}$ such that at least one $\beta_i = 0$ and one $\beta_j = \infty$. Without loss of generality, we assume that $\beta_1 = \cdots = \beta_k = 0$ and $\beta_{k+1} = \cdots = \beta_m = \infty$ with $1 \le k < m$. Then, for all $y \in B(x,r)$, using Hölder's inequality (see Lemma 2.1) and Lemma 2.2, we obtain that

$$\begin{aligned}
\mathcal{M}_{\alpha,m}(f_1^0,\dots,f_k^0,f_{k+1}^{\infty},\dots,f_m^{\infty})(y) &= \mathcal{M}_{\alpha,k}(f_1^0,\dots,f_k^0)(y)\mathcal{M}_{\alpha,m-k}(f_{k+1}^{\infty},\dots,f_m^{\infty})(y) \\
&\le C\Big(\prod_{i=1}^{k} M_{\alpha}(f_i^0)(y)\Big)\Big(\prod_{j=k+1}^{m} \sup_{t>2r} t^{\alpha_j - \frac{Q}{p_j}} \|f_j\|_{L^{p_j}(B(x,t))}\Big).
\end{aligned}$$

Similar to the estimates E_1 and $E_{2\infty}$, using Lemma 3.5 with $m = 1$, the doubling condition of Haar measure, Hölder's inequality (see Lemma 2.1) and Lemma 2.2, we get

$$\begin{aligned}
E_{2(\beta_1,\dots,\beta_m)} &= \|\mathcal{M}_{\alpha,m}(f_1^0,\dots,f_k^0,f_{k+1}^{\infty},\dots,f_m^{\infty})\|_{L^q(B(x,r))} \\
&= \Big(\int_{B(x,r)} |\mathcal{M}_{\alpha,m}(f_1^0,\dots,f_k^0,f_{k+1}^{\infty},\dots,f_m^{\infty})(y)|^q dy \Big)^{1/q} \\
&\le C r^{\frac{Q}{q}} \Big(\prod_{i=1}^{k} \sup_{t>2r} t^{\alpha_i - \frac{Q}{p_i}} \|f_i\|_{L^{p_i}(B(x,t))} \Big)\Big(\prod_{j=k+1}^{m} \sup_{t>2r} t^{\alpha_j - \frac{Q}{p_j}} \|f_j\|_{L^{p_j}(B(x,t))} \Big) \\
&\le C r^{\frac{Q}{q}} \prod_{i=1}^{m} \sup_{t>2r} t^{\alpha_i - \frac{Q}{p_i}} \|f_i\|_{L^{p_i}(B(x,t))}.
\end{aligned}$$

Combining the above estimates we get the desired result. The proof is completed. □

The following supremal type inequality plays a key role in the proof of Theorem 1.2, which can be founded in [2](see Theorem 5.4 or Theorem 3.1 in [14]).

Lemma 3.7. *Let v_1 and v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L^\infty(t,\infty)} < \infty$, $0 < \|v_2\|_{L^\infty(0,t)} < \infty$ for any $t \in (0,\infty)$. And let u be a continuous non-negative function on $(0,\infty)$. Then the supremal operator $\overline{S}_u$ is bounded from $L^\infty_{v_1}(0,\infty)$ to $L^\infty_{v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L^\infty(\cdot,\infty)} \right) \right\|_{L^\infty(0,\infty)} < \infty.$$

Now, we give the proof of Theorem 1.2.

Proof. [Proof of Theorem 1.2] Let $u(r) = r^{-Q/q_i}$, $v_2(r) = \psi(x,r)^{-1/m}$ and $v_1(r) = \varphi_i(x,r)^{-1} r^{-Q/p_i}$ $(i = 1,2,\ldots,m)$, According to the hypothesis (1.2), it follows that

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L^\infty(\cdot,\infty)} \right) \right\|_{L^\infty(0,\infty)} \le C.$$

Set $1 < p_i < \infty$ $(i = 1,2,\ldots,m)$ and $\vec{f} = (f_1, f_2, \ldots, f_m) \in \mathcal{L}^{p_1,\varphi_1}(\mathbb{G}) \times \cdots \times \mathcal{L}^{p_m,\varphi_m}(\mathbb{G})$. Using Lemma 3.6 and Lemma 3.7, we obtain

$$\begin{aligned}
\|\mathcal{M}_{\alpha,m}(\vec{f})\|_{\mathcal{L}^{q,\psi}(\mathbb{G})} &= \sup_{\substack{x\in G\\ r>0}} \frac{1}{\psi(x,r)} \Big(\frac{1}{|B(x,r)|} \int_{B(x,r)} |\mathcal{M}_{\alpha,m}(\vec{f})(y)|^q \mathrm{d}y \Big)^{1/q} \\
&\le C \prod_{i=1}^m \sup_{\substack{x\in G\\ r>0}} \psi(x,r)^{-1/m} \sup_{t>2r} t^{\alpha_i - \frac{Q}{p_i}} \|f_i\|_{L^{p_i}(B(x,t))} \\
&\le C \prod_{i=1}^m \sup_{\substack{x\in G\\ r>0}} \varphi_i(x,r)^{-1} r^{-Q/p_i} \|f_i\|_{L^{p_i}(B(x,r))} \\
&\le C \prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,\varphi_i}(\mathbb{G})}.
\end{aligned}$$

This completes the proof of Theorem 1.2. □

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