Filomat 38:16 (2024), 5605–5613
https://doi.org/10.2298/FIL2416605K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Specification of the Malliavin weights under stochastic volatility and stochastic interest rates processes for American option evaluation

Mohamed Kharrat[a]

[a]*Mathematics Department, College of Science, Jouf University, Sakaka, Saudi Arabia*

Abstract. In this paper, using the Malliavin calculus, we compute the conditional expectation related to the pricing problem of an American put option through considering the volatility and the interest rates, both stochastic and generated by the Cox-Ingersoll-Ross process.

1. Introduction

Malliavin calculus [10] is a highly useful tool for calculating the value of the conditional expectation in order to solve a wide range of financial mathematics problems [2, 5, 12]. The papers elaborated by Fournié et al. [6] and [7] served as the basic cornerstone for those that followed.

One of the most thorny problems in option pricing literature is the evaluation of American options. The American options pricing model, which is based on constant parameters, cannot account for the reality of financial markets. As the dynamics of the volatility or the interest rates are intrinsic to develop strategies for hedging as well as arbitrage, the pricing of options under stochastic parameters models are largely needed. The incorporation of a stochastic parameters factor significantly complicates the pricing of American options. Numerous works on option pricing using stochastic volatility models have been conducted.

The significance of using Malliavin calculus has been reinforced in recent years for settling the American options pricing problem. [1, 3, 8, 9].

In [3], Bally et al. invested Malliavin calculus to develop a representation formula for the conditional expectation in order to assess the American option for constant volatility. Abbas-Turki and Lapeyre [1], Kharrat [8] and Kharrat and Bastin [9] introduced new methods to price American option, under stochastic volatility for different models. Mallivin calculus was also applied successfully to other option problems. For example, Mancino [11] set forward a methodology for calculating the Malliavin weight for Delta hedging under a local volatility model, and Saporito [13] created a multiscale stochastic volatility model approximation for the price of path-dependent derivatives. Yamada [15] developed a Malliavin calculus approximation scheme for multidimensional Stratonovich stochastic differential equations and applied it to the SABR model.

Our main contribution resides in supposing that the dynamics of the volatility and the interest rates are

2020 *Mathematics Subject Classification*. 91Gxx, 26A33, 34A08.

Keywords. American option, Stochastic Volatility, Stochastic Interest Rate, Malliavin Calculus.

Received: 09 July 2023; Revised: 07 February 2024; Accepted: 26 April 2024

Communicated by Miljana Jovanović

This work was funded by the Deanship of Graduate Studies and Scientific Research at Jouf University under grant No. (DGSSR-2023-02-02011).

Email address: mkharrat@ju.edu.sa (Mohamed Kharrat)

stochastic in order to identify and compute theoretically the value of the conditional expectation related to the pricing problem of an American option:

$$\mathbb{E}\left[P_t(X_t, V_t, r_t) \,|\, (X_l = \gamma, V_l = \delta, r_l = \beta)\right] \tag{1}$$

for any $0 \leq l < t$, using Malliavin calculus, where V_t and r_t are generated by the Cox-Ingersoll-Ross (CIR) process [4], i.e., through following a mean reverting and a square-root diffusion process generated respectively by:

$$dV_t = k_V(\theta_V - V_t)dt + \sigma_V \sqrt{V_t} dW_t^V \tag{2}$$

$$dr_t = k_r(\theta_r - r_t)dt + \sigma_r \sqrt{r_t} dW_t^r \tag{3}$$

and where X_t is expressed through the geometric Brownian process :

$$dX_t = r_t X_t dt + X_t \sqrt{V_t} dW_t^S \tag{4}$$

or, by integration

$$X_t = X_l exp\left(\int_l^t r_s ds + \int_l^t \sqrt{V_s} dW_s^S - \frac{1}{2}\int_l^t V_s ds\right), \tag{5}$$

where W_t^S, W_t^V and W_t^r are correlated standard Brownian motions. The parameters θ_V, k_V and σ_V are, respectively, the long-term mean, the rate of mean reversion, and the volatility of the stochastic process V_t. The parameters θ_r, k_r and σ_r are, respectively, the long-term mean, the rate of mean reversion, and the volatility of the stochastic interest rates r_t. We assume that the volatility process V_t and stochastic interest rates r_t are almost surely strictly positive.

We initially introduce basic notations and definitions that will be needed to introduce the Malliavin derivative operator. T represents the expiration time of the financial product to price, which which has been normalized to 1 for simplicity.

We subdivide the horizon in 2^k dyadic intervals,

$$I_k^i =]t_k^{i-1}, t_k^i], \; t_k^i = \frac{i}{2^k},$$

for $i = 1, \dots, 2^k$, $k \in \mathbb{N}$, and $t_k^0 = 0$. We denote by $x_k = (x_k^1, \dots, x_k^{2^k})$ in $\mathbb{R}^{2^k}$, and $\forall\, t \in]0, T]$, let $i_k(t)$ be the only element $i \in \{1, \dots, 2^k\}$ such that $t \in I_k^i$. In the following, we present the notion of simple functionals.

Definition 1.1. *Given $k \in \mathbb{N}$, the family of simple k-th order functionals is defined by*

$$\mathcal{S}_k := \left\{\varphi(\Delta_k) \,|\, \varphi \in C_{pol}^{\infty}(\mathbb{R}^{2^k}; \mathbb{R})\right\}$$

where C_{pol}^{∞} is the family of infinitely differentiable functions which, together with their derivatives of any order, have at most a polynomial growth, and $\Delta_k := \left(\Delta_k^1, \dots, \Delta_k^{2^k}\right)$ is the vector of Brownian increments $\Delta_k^i = W_{t_k^i} - W_{t_k^{i-1}}$, $i = 1, \dots, 2^k$.

We may now explicitly define the Malliavin derivative.

Definition 1.2. *For every $X = \varphi(\Delta_k) \in \mathcal{S}$, the stochastic (or Malliavin) derivative of X at time t is defined by the $i_k(t)$-th partial derivative of $\varphi(\Delta_k)$:*

$$D_t X := \frac{\partial \varphi}{\partial x_k^{i_k(t)}}(\Delta_k),$$

and we denote by DX the associated stochastic process on $[0, T]$.

In the definitions below, we may also allow us to create the space of Malliavin differentiable variables and the family of k-th order simple processes.

Definition 1.3. *The space $\mathbb{D}^{1,2}$ of the Malliavin-differentiable random variables is the closure of $\mathcal{S}$ with respect to the norm $\|.\|_{1,2}$, defined as*

$$\|X\|_{1,2} = \sqrt{\mathbb{E}[X^2]} + \sqrt{\int_0^T (D_s X)^2 ds}$$

In other words, $X \in \mathbb{D}^{1,2}$ if and only if, there exists a sequence (X_k), $k \in \mathbb{N}$, in S such that X_k converges in distribution to a square integrable random variable X as $k \to \infty$, the limit $\lim_{k\to\infty} DX_k$ exists and is square integrable.

Definition 1.4. *The family $\mathcal{P}_k$, $k \in \mathbb{N}$, of the k-th order simple processes consists of the processes U of the form*

$$U_t = \sum_{i=1}^{2^k} \varphi_i(\Delta_k)\mathbf{1}_{I_k^i}(t) = \varphi_{i_k(t)}(\Delta_k), \tag{6}$$

where $\varphi_i \in C^\infty_{pol}(\mathbb{R}^{2^k}; \mathbb{R})$ for $i = 1, \ldots, 2^k$.

It is clear to see that $\mathcal{P}_k \subseteq \mathcal{P}_{k+1}$, $k \in \mathbb{N}$. Let $\mathcal{P} := \bigcup_{k\in\mathbb{N}} \mathcal{P}_k$ the family of simple functionals, and note that $DX \in \mathcal{P}$, for $X \in \mathcal{S}$. In other words, $D : \mathcal{S} \to \mathcal{P}$. We are now ready to introduce the Malliavin derivative's adjoint operator.

Definition 1.5. *Given a simple process $U \in \mathcal{P}$ of the form* (6)*, the Skorokhod-integral $D * U$ of U is defined as [14]*

$$D * U = \sum_{i=1}^{2^k} \left(\varphi_i(\Delta_k)\Delta_k^i - \partial_{x_k^i}\varphi_i(\Delta_k)\frac{1}{2^k} \right).$$

We also write $D * U = \int_0^T U_t \diamond dW_t$.

The following two technical lemmas, are proved in [12, Chapter 16].

Lemma 1.6. *Let $X \in \mathbb{D}^{1,2}$ and let U be a second-order Skorokhod-integrable process. Then,*

$$\int_0^T XU_t \diamond dW_t = X \int_0^T U_t \diamond dW_t - \int_0^T (D_t X)U_t dt \tag{7}$$

and, when U_t is adapted, the above equation can be expressed as

$$\int_0^T XU_t \diamond dW_t = X \int_0^T U_t dW_t - \int_0^T (D_t X)U_t dt. \tag{8}$$

Lemma 1.7 (Stochastic integration by parts). *Let $F \in C_b^1$, the space of functions in C^1 bounded together with their derivatives, and let $X \in \mathbb{D}^{1,2}$. Hence, the following integration by parts*

$$\mathbb{E}[F'(X)Y] = \mathbb{E}\left[F(X) \int_0^T \frac{u_t Y}{\int_0^T u_s D_s X ds} \diamond dW_t \right] \tag{9}$$

holds for every random variable Y and for every stochastic process u for which (9) *is well defined.*

To improve notation clarity, we expand the concept of Malliavin derivative to a multidimensional process by introducing the concept of partial derivative.

Definition 1.8 (Partial Malliavin derivative). *Let $W = (W^1, \ldots, W^d)$ be a $d-$dimensional Brownian motion. For $s \le t$, the partial Malliavin derivative at time s with respect to the $n-th$ component of W, denoted by D_s^n, is*

$$D_s^n W_t^j = \begin{cases} 1 & \text{if } n = j, \\ 0 & \text{otherwise.} \end{cases}$$

2. Main Results

2.1. Computation of Malliavin weights related to the interest rates

Our basic target is to compute (1) for any $0 \leq l < t$, with γ, δ and β being positive real numbers. Therefore, at first, we need to compute the Malliavin weights related to the interest rates, then the ones related to the volatility and finally the Malliavin weights related to the underlining asset price.
Using the result reported by Kharrat and Bastin in [9], we can compute explicitly $D_s^r V_l$. Additionally, we can deduce immediately $D_s^r r_l$. This result will be presented in the next proposition.

For $0 < s < l < t$, let:
by the integration of the process (3), r_l be the solution of the following Stochastic Differential Equation (SDE):

$$r_l = r_q + \int_q^l k_r(\theta_r - r_{s'})ds' + \int_q^l \sigma_r \sqrt{r_{s'}} dW_{s'}^r \ , \tag{10}$$

Y_l be the unique strong solution of the following SDE:

$$Y_l = 1 - k_r \int_0^l Y_{s'} ds' + \int_0^l \frac{\sigma_r}{2\sqrt{r_{s'}}} Y_{s'} dW_{s'}^r \ , \tag{11}$$

and Z_l be the unique strong solution of the following SDE:

$$Z_l = 1 + \int_0^l (\frac{\sigma_r^2}{4 r_{s'}} + k_r) Z_{s'} ds' - \int_0^l \frac{\sigma_r}{2\sqrt{r_{s'}}} Z_{s'} dW_{s'}^r \ . \tag{12}$$

Proposition 2.1. *Let r_l , Y_l and Z_l be as previously defined. For $0 < s < l < t$, we get:*

$$D_s^r r_l \ = \ \sigma_r Y_l Z_s \sqrt{r_s}.$$

In the next proposition, we shall use the stochastic integration by parts in order to provide the expression of $\mathbb{E}(\Psi'(r_l)P(X_t, V_t, r_t))$ for any $0 \leq l < t$ and for any function $\Psi \ \in \ C_b^{+\infty}(\mathbb{R})$, where $C_b^{+\infty}(\mathbb{R})$ is the space of bounded and infinitely differentiable functions.

Proposition 2.2. *Let r_l, Y_l and Z_l be as previously defined. For any $0 \leq l < t$ and for any function $\Psi \ \in \ C_b^{+\infty}(\mathbb{R})$, we have:*

$$\mathbb{E}(\Psi'(r_l)P_t(X_t, V_t, r_t)) = \mathbb{E}\left(\frac{\Psi(r_l)P_t(X_t, V_t, r_t)}{t-l}\left[\frac{\int_l^t \frac{dW_s^r}{Z_s\sqrt{r_s}}}{Y_l\sigma_r} - \int_l^t \left(\sigma_r Y_l + \frac{D_s^r M_l}{Z_s\sqrt{r_s}}\right)ds \int_l^t \frac{dW_s^r}{Y_l Z_s \sigma_r \sqrt{r_s} + D_s^r M_l}\right]\right) \tag{13}$$

where $M_l = \int_l^t k(\theta - r_{s'})ds' + \int_l^t \sigma_r \sqrt{r_{s'}} dW_{s'}^r$, and $C_b^{+\infty}(\mathbb{R})$ corresponds to the space of bounded and infinitely differentiable functions.

In order to prove Proposition 2.2, we need to define and prove two Lemmas .

Lemma 2.3. *Let r_l, Y_l, Z_l and M_l be as previously defined. We therefore have:*

$$\begin{aligned}\mathbb{E}(\Psi'(r_l)P_t(X_t,V_t,r_t)) &= \mathbb{E}\left(\frac{\Psi(r_l)P_t(X_t,V_t,r_l+M_l)}{Y_l\sigma_r(t-l)}\int_l^t \frac{dW_s^r}{Z_s\sqrt{r_s}}\right)\\ &- \mathbb{E}\left(\Psi(r_l)P_t'(X_t,V_t,r_l+M_l)\int_l^t\left(\sigma_r Y_l+\frac{D_s^r M_l}{Z_s\sqrt{r_s}}\right)ds\right).\end{aligned}$$

Proof. : Using the stochastic integration by parts, we have:

$$\mathbb{E}(\Psi'(V_l)P_t(X_t,V_t,r_t)) = \mathbb{E}(\Psi'(r_l)P_t(X_t,V_t,r_l+M_l)) = \mathbb{E}\left(\Psi(r_l)\int_l^t \frac{u_sP_t(X_t,V_t,r_l+M_l)}{\int_l^t u_{s'}D_{s'}^r r_l ds'}\diamond dW_s^r\right)$$

Knowing that $D_{s'}^r r_l = Y_l Z_{s'}\sigma_r\sqrt{V_{s'}}$ and let $u_s = \frac{1}{Z_s\sqrt{r_s}}$. Hence, we get:

$$\mathbb{E}(\Psi'(r_l)P_t(X_t,V_t,r_t)) = \mathbb{E}\left(\frac{\Psi(r_l)}{Y_l\sigma_r(t-l)}\int_l^t \frac{P_t(X_t,V_t,r_l+M_l)}{Z_s\sqrt{r_s}}\diamond dW_s^r\right),$$

where $\frac{1}{Z_s\sqrt{r_s}}$ is adapted. Thus, by using the Malliavin derivative, we have:

$$\mathbb{E}(\Psi'(r_l)P_t(X_t,V_t,r_t)) = \mathbb{E}\left(\frac{\Psi(r_l)}{Y_l\sigma_r(t-l)}\left(P_t(X_t,V_t,r_l+M_l)\int_l^t\frac{dW_s^r}{Z_s\sqrt{r_s}} - \int_l^t\frac{D_s^r(P_t(X_t,V_t,r_l+M_l))}{Z_s\sqrt{r_s}}ds\right)\right).$$

Applying the Malliavin derivative, we obtain:

$$\begin{aligned}\mathbb{E}(\Psi'(r_l)P_t(X_t,V_t,r_t)) &= \mathbb{E}\left(\frac{\Psi(r_l)P_t(X_t,V_t,r_l+M_l)}{Y_l\sigma_r(t-l)}\int_l^t \frac{dW_s^r}{Z_s\sqrt{r_s}}\right) &(14)\\ &- \mathbb{E}\left(\Psi(r_l)P_t'(X_t,V_t,r_l+M_l)\int_l^t\left(\sigma_r Y_l+\frac{D_s^r M_l}{Z_s\sqrt{r_s}}\right)ds\right). &(15)\end{aligned}$$

□

Lemma 2.4. *: Let r_l, Y_l, Z_l and M_l be as previously defined. For any $0 \le l < t$, we have:*

$$\mathbb{E}(P_t'(X_t,V_t,r_l+M_l)) = \mathbb{E}\left(\frac{P_t(X_t,V_t,r_l+M_l)}{t-l}\int_l^t\frac{1}{Y_lZ_s\sigma_r\sqrt{r_s}+D_s^rM_l}dW_s^r\right). \quad (16)$$

Proof. We have:

$$\begin{aligned}&\mathbb{E}\left(\Psi(r_l)P_t'(X_t,V_t,r_l+M_l)\int_l^t\left(\sigma_rY_l+\frac{D_s^rM_l}{Z_s\sqrt{r_s}}\right)ds\right)\\ &= \mathbb{E}\left(\mathbb{E}\left(\Psi(z')P_t'(X_t,V_t,r_l+M_l)\int_l^t\left(\sigma_rY_l+\frac{D_s^rM_l}{Z_s\sqrt{r_s}}\right)ds\right)\Big|_{z'=r_l}\right)\\ &= \mathbb{E}\left(\Psi(z')\int_l^t\left(\sigma_rY_l+\frac{D_s^rM_l}{Z_s\sqrt{r_s}}\right)ds\,\mathbb{E}(P_t'(X_t,V_t,z'+M_l))\Big|_{z'=r_l}\right).\end{aligned}$$

Using the Malliavin derivative, we get:

$$\mathbb{E}(P_t'(X_t,V_t,r_l+M_l)) = \mathbb{E}\left(P_t(X_t,V_t,r_l+M_l)\int_l^t\frac{u_s}{\int_l^t u_{s'}D_{s'}^r(r_l+M_l)ds'}\diamond dW_s^r\right). \quad (17)$$

Let $u_s = \frac{1}{Y_l Z_s \eta \sqrt{r_s} + D_s^r M_l}$, which is adapted. Then, we have:

$$\mathbb{E}(P_t'(X_t, V_t, r_l + M_l)) = \mathbb{E}\left(\frac{P_t(X_t, V_t, r_l + M_l)}{t-l} \int_l^t \frac{1}{Y_l Z_s \sigma_r \sqrt{r_s} + D_s^r M_l} dW_s^r\right).$$

□

After elaborating these two previous Lemmas, we may now prove Proposition 2.2.

Proof. : Relying on both Lemma 2.3 and Lemma 2.4, we obtain:

$$\mathbb{E}(\Psi'(r_l) P_t(X_t, V_t, r_t)) =$$

$$\mathbb{E}\left(\frac{\Psi(r_l) P_t(X_t, V_t, r_t)}{t-l}\left[\frac{\int_l^t \frac{dW_s^r}{Z_s \sqrt{r_s}}}{Y_l \sigma_r} - \int_l^t \left(\sigma_r Y_l + \frac{D_s^r M_l}{Z_s \sqrt{r_s}}\right) ds \int_l^t \frac{dW_s^r}{Y_l Z_s \sigma_r \sqrt{r_s} + D_s^r M_l}\right]\right),$$

where $M_l = \int_l^t k(\theta - r_{s'}) ds' + \int_l^t \sigma_r \sqrt{r_{s'}} dW_{s'}^r$. □

Using the Malliavin calculus in the following theorem, we provide the expression of the conditional expectation: $\mathbb{E}(P_t(X_t, V_t, r_t) | r_l = \beta)$.

Theorem 2.5. *Let r_l, Y_l, Z_l and M_l be as previously defined. Assuming that β is a positive real, for any $0 \leq l < t$, we get:*

$$\mathbb{E}(P_t(X_t, V_t, r_t) | r_l = \beta) = \frac{\mathbb{E}(H(r_l - \beta) \Upsilon_{r_l}(P_t(X_t, V_t, r_t)))}{\mathbb{E}(H(r_l - \beta) \Upsilon_{r_l}(1))}$$

where:

$$\Upsilon_{r_l}(P_t(X_t, V_t, r_t)) = \frac{P_t(X_t, V_t, r_t)}{t-l}\left(\frac{\int_l^t \frac{dW_s^r}{Z_s \sqrt{r_s}}}{Y_l \sigma_r} - \int_l^t \left(\sigma_r Y_l + \frac{D_s^r M_l}{Z_s \sqrt{r_s}}\right) ds \int_l^t \frac{dW_s^r}{Y_l Z_s \sigma_r \sqrt{r_s} + D_s^r M_l}\right)$$

and

$$\Upsilon_{r_l}(1) = \frac{1}{t-l}\left(\frac{\int_l^t \frac{dW_s^r}{Z_s \sqrt{r_s}}}{Y_l \eta} - \int_l^t \left(\sigma_r Y_l + \frac{D_s^r M_l}{Z_s \sqrt{r_s}}\right) ds \int_l^t \frac{dW_s^r}{Y_l Z_s \sigma_r \sqrt{r_s} + D_s^r M_l}\right)$$

where H is the Heaviside function with the convention that $\mathbb{E}(P_t(X_t, V_t, r_t) | r_l = \beta) = 0$ when, $\mathbb{E}(H(r_l - \beta) \Upsilon_{r_l}(1)) = 0$.

Proof. Using a basic result of Malliavin approach, we have:

$$\mathbb{E}(\Psi'(r_l) P_t(X_t, r_t)) = \mathbb{E}(\Psi(r_l) \Upsilon_{r_l}(P_t(X_t, V_t, r_t))).$$

Grounded on Proposition 2.2, for any $\Psi \in C_b^\infty(\mathbb{R})$, we get:

$$\mathbb{E}(\Psi(r_l) \Upsilon_{r_l}(P_t(X_t, V_t, r_t))) =$$

$$\mathbb{E}\left(\frac{\Psi(r_l) P_t(X_t, V_t, r_t)}{t-l}\left(\frac{\int_l^t \frac{dW_s^r}{Z_s \sqrt{r_s}}}{Y_l \sigma_r} - \int_l^t \left(\sigma_r Y_l + \frac{D_s^r M_l}{Z_s \sqrt{r_s}}\right) ds \int_l^t \frac{dW_s^r}{Y_l Z_s \sigma_r \sqrt{r_s} + D_s^r M_l}\right)\right).$$

By identification, the square integrable weight $\Upsilon_{r_l}(P_t(X_t, V_t, r_t))$ is defined as follows:

$$\Upsilon_{r_l}(P_t(X_t, V_t, r_t)) = \frac{P_t(X_t, V_t, r_t)}{t-l}\left(\frac{\int_l^t \frac{dW_s^r}{Z_s \sqrt{r_s}}}{Y_l \sigma_r} - \int_l^t \left(\sigma_r Y_l + \frac{D_s^r M_l}{Z_s \sqrt{r_s}}\right) ds \int_l^t \frac{dW_s^r}{Y_l Z_s \sigma_r \sqrt{r_s} + D_s^r M_l}\right)$$

Proceeding with the same logic, the square integrable weight $\Upsilon_{r_l}(1)$ is calculated as follows:

$$\Upsilon_{r_l}(1) = \frac{1}{t-l}\left(\frac{\int_l^t \frac{dW_s^r}{Z_s\sqrt{r_s}}}{Y_l\eta} - \int_l^t \left(\sigma_r Y_l + \frac{D_s^r M_l}{Z_s\sqrt{r_s}}\right)ds \int_l^t \frac{dW_s^r}{Y_l Z_s \sigma_r \sqrt{r_s} + D_s^r M_l}\right)$$

□

2.2. Malliavin weights related to the volatility

Based on the result of [8], we provide the following lemma to determine the expression of the Malliavin derivative $D_s^V V_l$ associated with the stochastic process V_t.

Lemma 2.6. *For $0 < s < l < t$, by integration of the process (2), V_l stands for the solution of the following stochastic differential equation:*

$$V_l = V_s + \int_s^l k(\theta - V_r)dr + \int_s^l \eta\sqrt{V_r}dW_r^V, \tag{18}$$

S_l corresponds to the solution of the following stochastic differential equation:

$$S_l = 1 - k\int_0^l S_r dr + \int_0^l \frac{\eta}{2\sqrt{V_r}} S_r dW_r^V, \tag{19}$$

and Q_l is the solution of the following stochastic differential equation:

$$Q_l = 1 + \int_0^l (\frac{\eta^2}{4V_r} + k)Q_r dr - \int_0^l \frac{\eta}{2\sqrt{V_r}} Q_r dW_r^V, \tag{20}$$

therefor, for every l, we have $S_l Q_l = 1$ and $D_s^V V_l = S_l Q_s \eta\sqrt{V_s}$.

Subsequently, we set forward the expression of the Malliavin weights of the conditional expectation: $\mathbb{E}(P_t(X_t, V_t, r_t)|(V_l = \delta, r_l = \beta))$.

Theorem 2.7. *Let V_l, S_l, Q_l and N_l be as previously defined and let δ , β be two positive real numbers. For any $0 \le l < t$,*

$$\mathbb{E}(P_t(X_t, V_t, r_t)|(V_l = \delta, r_l = \beta)) = \frac{\mathbb{E}(H(V_l - \delta)\Upsilon_{V_l}(G_{X_t,V_t}(\beta)))}{\mathbb{E}(H(V_l - \delta)\Upsilon_{V_l}(1))},$$

where the Malliavin weights are written as follows:

$$\Upsilon_{V_l}(G_{X_t,V_t}(\beta)) = \frac{G_{X_t,V_t}(\beta)}{t-l}\left(\frac{\int_l^t \frac{dW_s^V}{Q_s\sqrt{V_s}}}{S_l\eta} - \int_l^t \left(\eta S_l + \frac{D_s^V N_l}{Q_s\sqrt{V_s}}\right)ds \int_l^t \frac{dW_s^V}{S_l Q_s \eta\sqrt{V_s} + D_s^V N_l}\right) \tag{21}$$

and

$$\Upsilon_{V_l}(1) = \frac{1}{t-l}\left(\frac{\int_l^t \frac{dW_s^V}{Q_s\sqrt{V_s}}}{S_l\eta} - \int_l^t \left(\eta Y_l + \frac{D_s^V N_l}{Q_s\sqrt{V_s}}\right)ds \int_l^t \frac{dW_s^V}{S_l Q_s \eta\sqrt{V_s} + D_s^V N_l}\right), \tag{22}$$

where $G_{X_t,V_t}(\beta) = \mathbb{E}(P_t(X_t, V_t, r_t)|r_l = \beta)$ and H is the Heaviside function with the convention that $\mathbb{E}(P_t(X_t, V_t, r_t)|(V_l = \delta, r_l = \beta)) = 0$, when $\mathbb{E}(H(V_l - \delta)\Upsilon_{V_l}(1)) = 0$.

Proof. Referring to Theorem 2.3 of [8], we have the square integrable weight $\Upsilon_{V_l}(G_{X_t,V_t}(\beta))$ which is equal to:

$$\Upsilon_{V_l}(G_{X_t,V_t}(\beta)) = \frac{G_{X_t,V_t}(\beta)}{t-l}\left(\frac{\int_l^t \frac{dW_s^V}{Q_s\sqrt{V_s}}}{S_l\eta} - \int_l^t\left(\eta Y_l + \frac{D_s^V N_l}{Q_s\sqrt{V_s}}\right)ds\int_l^t \frac{dW_s^V}{S_lQ_s\eta\sqrt{V_s}+D_s^VN_l}\right).$$

and the square integrable weight $\Upsilon_{V_l}(1)$ which is equal to:

$$\Upsilon_{V_l}(1) = \frac{1}{t-l}\left(\frac{\int_l^t \frac{dW_s^V}{Q_s\sqrt{V_s}}}{S_l\eta} - \int_l^t\left(\eta Y_l + \frac{D_s^V N_l}{Q_s\sqrt{V_s}}\right)ds\int_l^t \frac{dW_s^V}{S_lQ_s\eta\sqrt{V_s}+D_s^VN_l}\right).$$

□

2.3. Computation of Malliavin weights related to the underlying asset price

Based on Theorem 2.5 of [8], we make an extension in order to consider three stochastic processes. The following Theorem exhibits the analytic expression of
$\mathbb{E}\left[P_t(X_t,V_t,r_t)\,|\,(X_l=\gamma,V_l=\delta,r_l=\beta)\right]$.

Theorem 2.8. *Let* $X_t = X_l exp\left(\int_l^t r_s ds + \int_l^t \sqrt{V_s}dW_s^S - \frac{1}{2}\int_l^t V_s ds\right)$*, with* $0 \le l \le t$*. We therefore have*

$$\mathbb{E}\left[P_t(X_t,V_t,r_t)\,|\,(X_l=\gamma,V_l=\delta,r_l=\beta)\right] = \frac{\mathbb{E}\left[H(X_l-\gamma)\Upsilon_{X_l}(F_{X_l}(\delta,\beta))\right]}{\mathbb{E}\left[H(X_l-\gamma)\Upsilon_{X_l}(1)\right]},$$

where

$$\Upsilon_{X_l}(g(X_t)) = \frac{g(X_t)}{\sqrt{1-\rho^2}X_l}\left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \frac{1}{t-l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}} + 1\right)$$

with $F_{X_l}(\delta,\beta) = \mathbb{E}(G_{X_t,V_t}(\beta)|V_l=\delta)$*, H representing the Heaviside function with the convention that* $\mathbb{E}\left[P_t(X_t,V_t,r_t)\,|\,(X_l=\gamma,V_l=\delta,r\right.$
0 *when* $\mathbb{E}\left[H(X_l-\gamma)\Upsilon_{X_l}(1)\right]=0$.

Example 2.9 (American call option pricing). *Let* $0 = t_0 < t_1 < \ldots < t_k = T$*, be a discretization of the time interval* $[0,T]$*. Assessing an American call option can be specified using the following backward iterations*

$$\begin{cases} P_T(X_T,V_T,r_T) = & \max\{X_T-K,0\},\\ P_{t_i}(X_{t_i},V_{t_i},r_{t_i}) = & \max\Big\{\max\{X_{t_i}-K,0\},\\ & e^{-r\frac{T}{k}}\mathbb{E}\left[P_{t_{i+1}}(X_{t_{i+1}},V_{t_{i+1}},r_{t_{i+1}})\,|\,(X_{t_i},V_{t_i},r_{t_i})\right]\Big\},\ i=k-1,\ldots,0.\end{cases}$$

Using Monte Carlo simulations, the Malliavin weights indicated in Theorem 2.8 are approximated to compute the previous conditional expectations.

Example 2.10 (American put option pricing). *Let* $0 = t_0 < t_1 < \ldots < t_k = T$*, be a discretization of the time interval* $[0,T]$*. Evaluating an American put option can be estimated using the following backward iterations*

$$\begin{cases} P_T(X_T,V_T,r_T) = & \max\{K-X_T,0\},\\ P_{t_i}(X_{t_i},V_{t_i},r_{t_i}) = & \max\Big\{\max\{K-X_{t_i},0\},\\ & e^{-r\frac{T}{k}}\mathbb{E}\left[P_{t_{i+1}}(X_{t_{i+1}},V_{t_{i+1}},r_{t_{i+1}})\,|\,(X_{t_i},V_{t_i},r_{t_i})\right]\Big\},\ i=k-1,\ldots,0.\end{cases}$$

3. Conclusion

In this research work, using the Malliavin calculus, we elaborated the expression of the expectation conditionally to the interest rates, the volatility and the underlying asset price that we supposed to be stochastic. We equally specified the Malliavin weights of these conditional expectations.

References

[1] L. A. Abbas-Turki, B. Lapeyre, *American Options by Malliavin Calculus and Nonparametric Variance and Bias Reduction Methods*, Siam J Financial Math. **3** (2012), 479–510.

[2] E. Alòs , D.G. Lorite, *Malliavin Calculus in Finance Theory and Practice*, Boca Raton, FL, USA: Taylor and Francis, 2021.

[3] V. Bally, L. Caramellino, A. Zanette, *Pricing and hedging American options by Monte Carlo methods using a Malliavin calculus approach*, Monte Carlo Methods and Applications **11** (2005), 97–133.

[4] J.C. Cox, J.E. Ingersoll, S.A. Ross, *A Theory of the Term Structure of Interest Rates*, Econometrica **53** (1985), 385–407.

[5] J.P. Fouque, G. Papanicolaou, S.K. Ronnie, *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge, United Kingdom: Cambridge University Press, 2000.

[6] E. Fournié, J. M. Lasry, J. Lebouchoux, P. L. Lions, N. Touzi, *Applications of Malliavin calculus to Monte Carlo Methods in Finance*, Finance Stoch. **5(2)** (1999), 201–236.

[7] E. Fournié, J.M. Lasry, J. Lebouchoux, P.L. Lions, *Applications of Malliavin Calculus to Monte Carlo Methods in Finance II*, Finance Stoch. **2** (2001), 73–88.

[8] M. Kharrat, *Pricing American put option using Malliavin derivatives under stochastic volatility*, Revista de la Unión Matemática Argentina **60(1)** (2019), 137–147.

[9] M. Kharrat, F. Bastin, *Continuation value computation using Malliavin calculus under general volatility stochastic process for American option pricing*, Turkish Journal of Mathematics **46** (2022), 71–86.

[10] P. Malliavin, *Stochastic calculus of variations and hypoelliptic operators*. In Proceedings of the International Symposium on Stochastic Differential Equations; Kyoto, Japan; (1978). Wiley and Sons, New York, NY, USA. pp. 195–263.

[11] M.E. Mancino, *Nonparametric Malliavin Monte Carlo Computation of Hedging Greeks*, Risks **8(4)** (2020), 1–17.

[12] A. Pascucci, *PDE and Martingale Methods in Option Pricing*, Milan, Italy: Springer-Verlag, 2010.

[13] Y.F. Saporito, *Pricing Path-Dependent Derivatives under Multiscale Stochastic Volatility Models: A Malliavin Representation*, SIAM Journal on Financial Mathematics **11(3)** (2020), 14–25.

[14] A.V. Skorohod, *On a generalization of the stochastic integral*, Teor. Verojatnost. i Primenen **20(2)** (1975), 223–238.

[15] T. Yamada, *An Arbitrary High Order Weak Approximation of SDE and Malliavin Monte Carlo: Analysis of Probability Distribution Functions*, SIAM Journal on Numerical Analysis **50(2)** (2019) 563–591.