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# Differences of the Stević-Sharma operators from $\alpha$-Bloch spaces to Zygmund-type spaces

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**Abstract.** We characterize the boundedness and compactness of the differences of the Stević-Sharma operators from the $\alpha$-Bloch spaces to the Zygmund-type spaces. Two examples related to the main results are also given.

## 1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. Denote by $S(\mathbb{D})$ the set of all analytic self-maps of $\mathbb{D}$. For any $a \in \mathbb{D}$, we let $\varphi_a(z) = (a-z)/(1-\bar{a}z)$ be the involutive automorphism that exchanges 0 and $a$. The pseudo-hyperbolic metric on $\mathbb{D}$ is defined by

$$\rho(z,w) = \left|\varphi_z(w)\right|, \quad z, w \in \mathbb{D},$$

which is essential in the proof of our main results. For simplicity, denote $\rho(z) := \rho(\varphi(z), \psi(z))$, for $\varphi, \psi \in S(\mathbb{D})$. By $W(\mathbb{D})$ we denote the set of all positive continuous and bounded functions on $\mathbb{D}$ (weights).

For $0 < \alpha < \infty$, the $\alpha$-Bloch space $\mathcal{B}^\alpha$ consists of all $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|f'(z)\right| < \infty.$$

The little $\alpha$-Bloch space $\mathcal{B}_0^\alpha$ is a closed subspace of $\mathcal{B}^\alpha$ consisting those $f \in \mathcal{B}^\alpha$ for which

$$\lim_{|z|\to 1} \left(1-|z|^2\right)^\alpha \left|f'(z)\right| = 0.$$

For a strictly positive continuous and bounded function $\mu$ on $\mathbb{D}$, the Zygmund-type space $\mathcal{Z}_\mu$ is the set of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(z) \left|f''(z)\right| < \infty.$$

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The little Zygmund-type space $\mathcal{Z}_{\mu,0}$ is the closed subspace of $\mathcal{Z}_\mu$ consisting those $f \in \mathcal{Z}_\mu$ such that

$$\lim_{|z|\to 1} \mu(z) \left|f''(z)\right| = 0.$$

Some investigations on these classical spaces can be found, e.g., in [1, 11, 12, 15, 16, 21, 27, 29] (see also the related reference therein).

Given $\varphi \in S(\mathbb{D})$, associate to $\varphi$ is the composition operator $C_\varphi$ defined on $H(\mathbb{D})$ by

$$C_\varphi f(z) = f(\varphi(z)), \quad z \in \mathbb{D}.$$

Let $u \in H(\mathbb{D})$, the multiplication operator $M_u$ is defined on $H(\mathbb{D})$ by

$$M_u f(z) = u(z)f(z), \quad z \in \mathbb{D}.$$

The extensive research on the above operators and their differences can be found e.g., in [3, 4, 6, 7, 9, 23, 25, 26, 41]. Inspired by the relevant research results in [13] and [24], we consider the combination of composition operator, multiplication operator and differential operator of order $n \in \mathbb{N}_0$,

$$\left(D^n_{\varphi,u} f\right)(z) = u(z)f^{(n)}(\varphi(z)), \quad z \in \mathbb{D}.$$

If $n = 0$, then $D^0_{\varphi,u}$ is denoted as $uC_\varphi$, the weighted composition operator. The interested readers can refer to [1, 8, 11, 17, 28, 31, 32, 45–47] and their reference therein. A generalization of the operator acting on spaces of holomorphic functions on the unit ball in $\mathbb{C}^n$ was introduced in [30], and studied also in [33–35].

Now let $n \in \mathbb{N}$, $u, v \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, the Stević-Sharma operator $T^n_{u,v,\varphi}$ (see, e.g., [39]) is defined on $H(\mathbb{D})$ by

$$T^n_{u,v,\varphi} f(z) = u(z)f^{(n)}(\varphi(z)) + v(z)f^{(n+1)}(\varphi(z)), \quad z \in \mathbb{D}.$$

When $n = 0$, $T^0_{u,v,\varphi}$ is denoted by $T_{u,v,\varphi}$.

Recently, Stević et al. [37, 38] characterized the boundedness and compactness of $T_{u,v,\varphi}$ on the weighted Bergman space. In [36] was considered the Stević-Sharma operator between the Hardy and $\alpha$-Bloch spaces on the upper half-plane. Liu et al. [19, 20, 42, 43] described the boundedness and compactness of $T_{u,v,\varphi}$ from several specific holomorphic function spaces to the weighted-type space or the Bloch-type space. Wang et al. [40] investigated some properties of differences of two Stević-Sharma operators from $H^\infty$, $\mathcal{A}^{-\alpha}$, $A^p_\alpha$ or $H^p$ to the weighted-type space $H^\infty_v$. For some other results on these and related Stević-Sharma operators see, for example, [10, 21, 36, 39, 48, 49].

Especially inspired by [2, 18, 25, 32] and the characterizations of the (single) operator $T^n_{u,v,\varphi} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$, we wish to investigate the boundedness and compactness of the differences of the Stević-Sharma operators from the (little) $\alpha$-Bloch space to the (little) Zygmund-type space. This paper is organized as follows. In Section 2, we prepared some relevant lemmas to pave the way for the proof of the main results. In Section 3, our theorem provides a natural and intrinsic characterization for the boundedness and compactness of the difference $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ as the main result. Section 4 presents some characterizations for the boundedness and compactness of the difference from $\mathcal{B}^\alpha$ (or $\mathcal{B}^\alpha_0$) to $\mathcal{Z}_{\mu,0}$ and establishes the equivalent relation between them. Some corollaries are presented in Section 5.

Throughout this paper, let $C$ denote a positive constant and its exact value may change in different circumstances. We write $A \lesssim B$ (or equivalently $B \gtrsim A$), if there exists a positive constant $C$ independent of the argument such that $A \le CB$(or equivalently $B \ge CA$). And $A \approx B$ means both $A \lesssim B$ and $B \lesssim A$.

## 2. Preliminaries

In this section, we provide several lemmas to ensure the rigour of further proofs.

**Lemma 2.1.** ([44, Propositon 8]) Let $n \in \mathbb{N}$, $\alpha > 0$, and $f \in \mathcal{B}^\alpha$, then

$$\|f\|_{\mathcal{B}^\alpha} \approx \sum_{i=0}^{n-1} \left|f^{(i)}(0)\right| + \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^{\alpha+n-1} \left|f^{(n)}(z)\right|.$$

**Lemma 2.2.** ([18, Lemma 2.1]) Let $n \in \mathbb{N}$, $\alpha > 0$, and $f \in \mathcal{B}^\alpha$, then

$$\left|\left(1-|z|^2\right)^{\alpha+n-1} f^{(n)}(z) - \left(1-|w|^2\right)^{\alpha+n-1} f^{(n)}(w)\right| \lesssim \|f\|_{\mathcal{B}^\alpha}\rho(z,w),$$

for all $z, w \in \mathbb{D}$.

The proof of the following lemma can be similarly traced back to [2, Lemma 2.1].

**Lemma 2.3.** *Let $n \in \mathbb{N}$, $\alpha > 0$. Then for any $a \in \mathbb{D}$ with $a \neq 0$, there exist functions $g_{n,i,a} \in \mathcal{B}_0^\alpha$ such that*

$$g_{n,i,a}^{(n+i)}(z) = \frac{\left(1-|a|^2\right)^{\alpha+n-1+i}}{(1-\bar{a}z)^{2(\alpha+n-1+i)}}, \quad i \in \{0,1,2,3\}. \tag{1}$$

*Proof.* Let

$$g_{n,i,a}(z) = \int_0^z \int_0^{t_n} \cdots \int_0^{t_2} \frac{\left(1-|a|^2\right)^{\alpha+n-1+i}}{(1-\bar{a}t_1)^{2(\alpha+n-1)+i}} \frac{1}{\bar{a}^i} \frac{\Gamma(2\alpha+2n+i-2)}{\Gamma(2\alpha+2n+2i-2)} dt_1 dt_2 \cdots dt_n,$$

where $a \in \mathbb{D}$. By calculation, it is easy to confirm that it satisfies (1). Furthermore, for $i \in \{0,1,2,3\}$,

$$\begin{aligned}
\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\alpha+n-1+i} \left|g_{n,i,a}^{(n+i)}(z)\right| &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\alpha+n-1+i} \left|\frac{(1-|a|^2)^{\alpha+n-1+i}}{(1-\bar{a}z)^{2(\alpha+n-1+i)}}\right| \\
&\leqslant \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\alpha+n-1+i} \frac{\left(1-|a|^2\right)^{\alpha+n-1+i}}{(1-|\bar{a}|)^{\alpha+n-1+i}(1-|z|)^{\alpha+n-1+i}} \\
&\leqslant 2^{2(\alpha+n-1+i)} < \infty,
\end{aligned}$$

and using the method in [14], it is easy to prove that

$$\lim_{|z|\to 1} \left(1-|z|^2\right)^{\alpha+n-1+i} \left|g_{n,i,a}^{(n+i)}(z)\right| = 0,$$

which shows that $g_{n,i,a} \in \mathcal{B}_0^\alpha$. □

The following Lemma can be proved similar to Lemma 2.3.

**Lemma 2.4.** *Let $n \in \mathbb{N}$, $\alpha > 0$. For any $a \in \mathbb{D} \setminus \{0\}$.*

(1) *Define*

$$\begin{aligned}
h_{n,i,a}(z) = \int_0^z \int_0^{t_n} \cdots \int_0^{t_2} \left(1-|a|^2\right)^{\alpha+n-1+i} \cdot \Bigg[ &\frac{a-t_1}{(1-\bar{a}t_1)^{2\alpha+2n+i-1}} \frac{1}{\bar{a}^i} \frac{\Gamma(2\alpha+2n+i-1)}{\Gamma(2\alpha+2n+2i-1)} \\
&+ \frac{i}{(1-\bar{a}t_1)^{2\alpha+2n+i-2}} \frac{1}{\bar{a}^{i+1}} \frac{\Gamma(2\alpha+2n+i-2)}{\Gamma(2\alpha+2n+2i-1)} \Bigg] dt_1 dt_2 \cdots dt_n,
\end{aligned}$$

*we get functions $h_{n,i,a} \in \mathcal{B}_0^\alpha$ such that*

$$h_{n,i,a}^{(n+i)}(z) = \frac{\left(1-|a|^2\right)^{\alpha+n-1+i}(a-z)}{(1-\bar{a}z)^{2\alpha+2n+2i-1}}, \quad i \in \{0,1,2,3\}. \tag{2}$$

(2) *Let*

$$r_{n,i,a}(z) = \frac{1}{(n+i)!} z^{n+i},$$

*it yields that functions $r_{n,i,a} \in \mathcal{B}_0^\alpha$ satisfying*

$$r_{n,i,a}^{(n+i)}(z) = 1, \quad i \in \{0,1,2,3\}. \tag{3}$$

(3) *Set*

$$p_{n,0,a}(z) = \int_0^z \int_0^{t_n} \cdots \int_0^{t_2} \frac{t_1}{(1-\bar{a}t_1)^{\alpha+n-1}} dt_1 dt_2 \cdots dt_n,$$

*and for $i \in \{1,2,3\}$,*

$$p_{n,i,a}(z) = \begin{cases} \int_0^z \int_0^{t_n} \cdots \int_0^{t_2} \frac{1}{\bar{a}^{i+1}} \left[ \frac{\Gamma(\alpha+n-1)}{\Gamma(\alpha+n+i-1)} \frac{1}{(1-\bar{a}t_1)^{\alpha+n-1}} - \frac{\Gamma(\alpha+n-2)}{\Gamma(\alpha+n+i-2)} \frac{1}{(1-\bar{a}t_1)^{\alpha+n-2}} \right] dt_1 dt_2 \cdots dt_n, & \alpha+n>2, \\ \int_0^z \int_0^{t_n} \cdots \int_0^{t_2} \frac{1}{\bar{a}^{i+1}} \left[ \frac{1}{i!} \frac{1}{1-\bar{a}t_1} - \frac{1}{(i-1)!} \ln(1-\bar{a}t_1) \right] dt_1 dt_2 \cdots dt_n, & \alpha+n=2, \\ \int_0^z \int_0^{t_n} \cdots \int_0^{t_2} \frac{1}{\bar{a}^{i+1}} \left[ \frac{\Gamma(\alpha+n-1)}{\Gamma(\alpha+n+i-1)} \frac{1}{(1-\bar{a}t_1)^{\alpha+n-1}} - \frac{\Gamma(\alpha+n-1)}{\Gamma(\alpha+n+i-2)} \frac{(1-\bar{a}t_1)^{2-\alpha-n}}{(2-\alpha-n)} \right] dt_1 dt_2 \cdots dt_n, & \alpha+n<2, \end{cases}$$

*we thus obtain functions $p_{n,i,a} \in \mathcal{B}_0^\alpha$ such that*

$$p_{n,i,a}^{(n+i)}(z) = \frac{z}{(1-\bar{a}z)^{\alpha+n-1+i}}, \quad i \in \{0,1,2,3\}. \tag{4}$$

(4) *By choosing*

$$q_{n,i,a}(z) = \frac{a}{(n+i)!} z^{n+i} - \frac{1}{(n+i+1)!} z^{n+i+1},$$

*we get $q_{n,i,a} \in \mathcal{B}_0^\alpha$ and*

$$q_{n,i,a}^{(n+i)}(z) = a - z, \quad i \in \{0,1,2,3\}. \tag{5}$$

The following lemma is a slight modification of [5, Proposition 3.11].

**Lemma 2.5.** *Let $\alpha > 0$, $n \in \mathbb{N}_0$, $u_1, u_2, v_1, v_2 \in H(\mathbb{D})$, $\varphi, \psi \in S(\mathbb{D})$ and $\mu \in W(\mathbb{D})$. Then $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ is compact if and only if $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}$ is bounded and whenever $\{f_n\}$ is bounded in $\mathcal{B}^\alpha$ with $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\left(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}\right) f_n \to 0$ in $\mathcal{Z}_\mu$.*

The following lemma will be used to show the compactness of operators to $\mathcal{Z}_{\mu,0}$. It is proved similar to Theorem 4.1 in [22], so we omit the proof.

**Lemma 2.6.** *A closed subset $\mathcal{L}$ in $\mathcal{Z}_{\mu,0}$ is compact if and only if it is a bounded subset and satisfies*

$$\lim_{|z|\to 1} \sup_{f \in \mathcal{L}} \mu(z) \left| f''(z) \right| = 0.$$

## 3. Boundedness and Compactness of $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$

Let $\alpha > 0$, $n \in \mathbb{N}$, $i \in \{0,1,2,3\}$, $u_1, u_2, v_1, v_2 \in H(\mathbb{D})$, $\varphi, \psi \in S(\mathbb{D})$ and $\mu \in W(\mathbb{D})$. In this section, we present some necessary and sufficient conditions for the boundedness and compactness of $T^n_{u_1,v_1,\varphi} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$. Then we investigate the boundedness and compactness of the difference $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$. For simplicity, we use the following notations.

$$\begin{cases} \widetilde{A_0}(\varphi(z)) = \mu(z) u_1''(z), \\ \widetilde{A_1}(\varphi(z)) = \mu(z)[2u_1'(z)\varphi'(z) + u_1(z)\varphi''(z) + v_1''(z)], \\ \widetilde{A_2}(\varphi(z)) = \mu(z)[u_1(z)\varphi'^2(z) + 2v_1'(z)\varphi'(z) + v_1(z)\varphi''(z)], \\ \widetilde{A_3}(\varphi(z)) = \mu(z) v_1(z) \varphi'^2(z), \\ \quad \varphi_{\#_i}(z) = \frac{\widetilde{A_i}(\varphi(z))}{(1-|\varphi(z)|^2)^{\alpha+n-1+i}}, \quad i \in \{0,1,2,3\}. \end{cases} \tag{6}$$

In the same way, we can define $\widetilde{A}_i(\psi(z))$, $\psi_{\#_i}(z)$ by replacing $u_1, v_1$ and $\varphi$ with $u_2, v_2$ and $\psi$, respectively.

The following two propositions are routinely proved (see e.g., [36–38]).

**Proposition 3.1.** *Let $n \in \mathbb{N}$, $\alpha > 0$, $u_1, v_1 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and $\mu \in W(\mathbb{D})$. Then the following statements are equivalent.*

(1) $T^n_{u_1,v_1,\varphi} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ *is bounded.*

(2) $T^n_{u_1,v_1,\varphi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_\mu$ *is bounded.*

(3)

$$\sum_{i=0}^{3} \sup_{z\in\mathbb{D}} \left|\varphi_{\#_i}(z)\right| = \sum_{i=0}^{3} \sup_{z\in\mathbb{D}} \frac{\left|\widetilde{A}_i(\varphi(z))\right|}{(1-|\varphi(z)|^2)^{\alpha+n-1+i}} < \infty.$$

**Proposition 3.2.** *Let $n \in \mathbb{N}$, $\alpha > 0$, $u_1, v_1 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and $\mu \in W(\mathbb{D})$. Suppose that $T^n_{u_1,v_1,\varphi} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ is bounded, then the following statements are equivalent.*

(1) $T^n_{u_1,v_1,\varphi} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ *is compact.*

(2) $T^n_{u_1,v_1,\varphi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_\mu$ *is compact.*

(3)

$$\sum_{i=0}^{3} \limsup_{|\varphi(z)|\to 1} \left|\varphi_{\#_i}(z)\right| = \sum_{i=0}^{3} \limsup_{|\varphi(z)|\to 1} \frac{\left|\widetilde{A}_i(\varphi(z))\right|}{(1-|\varphi(z)|^2)^{\alpha+n-1+i}} = 0.$$

Define

$$\begin{aligned}
&\text{condition } a_0: \quad \sup_{z\in\mathbb{D}} \left|\varphi_{\#_0}(z)\right| \rho(z) < \infty, \quad \sup_{z\in\mathbb{D}} \left|\varphi_{\#_0}(z) - \psi_{\#_0}(z)\right| < \infty;\\
&\text{condition } a_1: \quad \sup_{z\in\mathbb{D}} \left|\varphi_{\#_1}(z)\right| \rho(z) < \infty, \quad \sup_{z\in\mathbb{D}} \left|\varphi_{\#_1}(z) - \psi_{\#_1}(z)\right| < \infty;\\
&\text{condition } a_2: \quad \sup_{z\in\mathbb{D}} \left|\varphi_{\#_2}(z)\right| \rho(z) < \infty, \quad \sup_{z\in\mathbb{D}} \left|\varphi_{\#_2}(z) - \psi_{\#_2}(z)\right| < \infty;\\
&\text{condition } a_3: \quad \sup_{z\in\mathbb{D}} \left|\varphi_{\#_3}(z)\right| \rho(z) < \infty, \quad \sup_{z\in\mathbb{D}} \left|\varphi_{\#_3}(z) - \psi_{\#_3}(z)\right| < \infty.
\end{aligned}$$

**Theorem 3.3.** *Let $n \in \mathbb{N}$, $\alpha > 0$, $u_1, u_2, v_1, v_2 \in H(\mathbb{D})$, $\varphi$, $\psi \in S(\mathbb{D})$ and $\mu \in W(\mathbb{D})$. Suppose that any three of conditions $a_0, a_1, a_2, a_3$ are valid. Then the following statements are equivalent.*

(i)$T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ is bounded;

(ii)$T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_\mu$ is bounded;

(iii) The remaining condition holds.

*Proof.* It is evident that (i) ⇒ (ii). Now, assume that the conditions $a_0, a_1, a_2, a_3$ hold, then for any $f \in \mathcal{B}^\alpha$, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}\mu(z)\left|\left((T^n_{u_1,v_1,\varphi}-T^n_{u_2,v_2,\psi})f\right)''(z)\right| \\
=&\sup_{z\in\mathbb{D}}\left|\sum_{i=0}^{3}\left[f^{(n+i)}(\varphi(z))\widetilde{A}_i(\varphi(z))-f^{(n+i)}(\psi(z))\widetilde{A}_i(\psi(z))\right]\right| \\
=&\sup_{z\in\mathbb{D}}\left|\sum_{i=0}^{3}\left[\varphi_{\#_i}(z)\left(1-|\varphi(z)|^2\right)^{\alpha+n-1+i}f^{(n+i)}(\varphi(z))-\psi_{\#_i}(z)\left(1-|\psi(z)|^2\right)^{\alpha+n-1+i}f^{(n+i)}(\psi(z))\right]\right| \\
=&\sup_{z\in\mathbb{D}}\left|\sum_{i=0}^{3}\left\{\varphi_{\#_i}(z)\left[\left(1-|\varphi(z)|^2\right)^{\alpha+n-1+i}f^{(n+i)}(\varphi(z))-\left(1-|\psi(z)|^2\right)^{\alpha+n-1+i}f^{(n+i)}(\psi(z))\right]\right.\right. \\
&\left.\left.+\left(1-|\psi(z)|^2\right)^{\alpha+n-1+i}f^{(n+i)}(\psi(z))\left(\varphi_{\#_i}(z)-\psi_{\#_i}(z)\right)\right\}\right| \\
\lesssim&\sum_{i=0}^{3}\sup_{z\in\mathbb{D}}\left(|\varphi_{\#_i}(z)|\rho(z)+|\varphi_{\#_i}(z)-\psi_{\#_i}(z)|\right)\cdot\|f\|_{\mathcal{B}^\alpha}<\infty,
\end{aligned}\tag{7}$$

where the inequality derives from Lemma 2.1 and Lemma 2.2. Further, repeating application of Lemma 2.1 enables us to obtain

$$\begin{aligned}
&\left|(T^n_{u_1,v_1,\varphi}-T^n_{u_2,v_2,\psi})f(0)\right| \\
\leqslant&\left|u_1(0)f^{(n)}(\varphi(0))+v_1(0)f^{(n+1)}(\varphi(0))\right|+\left|u_2(0)f^{(n)}(\psi(0))+v_2(0)f^{(n+1)}(\psi(0))\right| \\
=&\left|\left(1-|\varphi(0)|^2\right)^{\alpha+n-1}f^{(n)}(\varphi(0))\frac{u_1(0)}{(1-|\varphi(0)|^2)^{\alpha+n-1}}+\left(1-|\varphi(0)|^2\right)^{\alpha+n}f^{(n+1)}(\varphi(0))\frac{v_1(0)}{(1-|\varphi(0)|^2)^{\alpha+n}}\right| \\
&+\left|\left(1-|\psi(0)|^2\right)^{\alpha+n-1}f^{(n)}(\psi(0))\frac{u_2(0)}{(1-|\psi(0)|^2)^{\alpha+n-1}}+\left(1-|\psi(0)|^2\right)^{\alpha+n}f^{(n+1)}(\psi(0))\frac{v_2(0)}{(1-|\psi(0)|^2)^{\alpha+n}}\right| \\
\lesssim&\|f\|_{\mathcal{B}^\alpha}\cdot\left(\frac{|u_1(0)|}{(1-|\varphi(0)|^2)^{\alpha+n-1}}+\frac{|v_1(0)|}{(1-|\varphi(0)|^2)^{\alpha+n}}\right)+\|f\|_{\mathcal{B}^\alpha}\cdot\left(\frac{|u_2(0)|}{(1-|\psi(0)|^2)^{\alpha+n-1}}+\frac{|v_2(0)|}{(1-|\psi(0)|^2)^{\alpha+n}}\right) \\
\leqslant&\|f\|_{\mathcal{B}^\alpha}\cdot\left(\frac{|u_1(0)|+|v_1(0)|}{(1-|\varphi(0)|^2)^{\alpha+n}}+\frac{|u_2(0)|+|v_2(0)|}{(1-|\psi(0)|^2)^{\alpha+n}}\right) \\
\lesssim&\|f\|_{\mathcal{B}^\alpha}
\end{aligned}\tag{8}$$

and

$$\begin{aligned}
&\left|(T^n_{u_1,v_1,\varphi}-T^n_{u_2,v_2,\psi})f'(0)\right| \\
\lesssim&\|f\|_{\mathcal{B}^\alpha}\cdot\left(\frac{\left|u_1'(0)\right|+\left|u_1(0)\varphi'(0)+v_1'(0)\right|+\left|v_1(0)\varphi'(0)\right|}{(1-|\varphi(0)|^2)^{\alpha+n+1}}+\frac{\left|u_2'(0)\right|+\left|u_2(0)\psi'(0)+v_2'(0)\right|+\left|v_2(0)\psi'(0)\right|}{(1-|\psi(0)|^2)^{\alpha+n+1}}\right) \\
\lesssim&\|f\|_{\mathcal{B}^\alpha}.
\end{aligned}\tag{9}$$

Combining (7), (8), (9) and the fact that $f \in \mathcal{B}^\alpha$, we have

$$\begin{aligned}
&\left\|\left(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}\right) f\right\|_{\mathcal{Z}_\mu} \\
=&\left|(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi})f(0)\right| + \left|(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi})f'(0)\right| + \sup_{z\in\mathbb{D}} \mu(z) \left|\left((T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi})f\right)''(z)\right| \\
\lesssim&\|f\|_{\mathcal{B}^\alpha} + \|f\|_{\mathcal{B}^\alpha} + \sum_{i=0}^{3} \sup_{z\in\mathbb{D}} \left(\left|\varphi_{\#_i}(z)\right| \rho(z) + \left|\varphi_{\#_i}(z) - \psi_{\#_i}(z)\right|\right) \cdot \|f\|_{\mathcal{B}^\alpha} \\
<&\infty,
\end{aligned}$$

which implies $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ is bounded. Thus, (iii) ⇒ (i) is established.

It remains to show (ii) ⇒ (iii). Assume $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_\mu$ is bounded. Without loss of generality, we suppose that conditions $a_1, a_2, a_3$ hold, then prove that condition $a_0$ also holds. To shorten notations, we write $\mathbb{D}_1 = \{w \in \mathbb{D} : \varphi(w) = 0\}$ and $\mathbb{D}_2 = \{w \in \mathbb{D} : \psi(w) = 0\}$. If $w \notin \mathbb{D}_1$, then taking the functions $g_{n,0,w} \in \mathcal{B}^\alpha_0$ in Lemma 2.3 it follows that (1) holds, we have

$$\begin{aligned}
\infty >& \left\|\left(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}\right) g_{n,0,\varphi(w)}\right\|_{\mathcal{Z}_\mu} \\
\geqslant& \sup_{z\in\mathbb{D}} \mu(z) \left|\left[u_1(z) g^{(n)}_{n,0,\varphi(w)}(\varphi(z)) + v_1(z) g^{(n+1)}_{n,0,\varphi(w)}(\varphi(z)) - u_2(z) g^{(n)}_{n,0,\varphi(w)}(\psi(z)) - v_2(z) g^{(n+1)}_{n,0,\varphi(w)}(\psi(z))\right]''\right| \\
\geqslant& \left|\sum_{i=0}^{3} \left[\widetilde{A}_i(\varphi(w)) g^{(n+i)}_{n,0,\varphi(w)}(\varphi(w)) - \widetilde{A}_i(\psi(w)) g^{(n+i)}_{n,0,\varphi(w)}(\psi(w))\right]\right| \\
\geqslant& \left|\varphi_{\#_0}(w) - \psi_{\#_0}(w) \left(\frac{(1-|\varphi(w)|^2)(1-|\psi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^2}\right)^{\alpha+n-1}\right| \\
&- \sum_{i=1}^{3} \left|\left[\widetilde{A}_i(\varphi(w)) g^{(n+i)}_{n,0,\varphi(w)}(\varphi(w)) - \widetilde{A}_i(\psi(w)) g^{(n+i)}_{n,0,\varphi(w)}(\psi(w))\right]\right| \\
\geqslant& \left|\varphi_{\#_0}(w) - \psi_{\#_0}(w) \left(\frac{(1-|\varphi(w)|^2)(1-|\psi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^2}\right)^{\alpha+n-1}\right| \\
&- \sum_{i=1}^{3} \left|\varphi_{\#_i}(w) - \psi_{\#_i}(w)\right| \cdot \left(1-|\varphi(w)|^2\right)^{\alpha+n-1+i} g^{(n+i)}_{n,0,\varphi(w)}(\varphi(w)) \\
&- \sum_{i=1}^{3} \left|\psi_{\#_i}(w)\right| \cdot \left|\left(1-|\varphi(w)|^2\right)^{\alpha+n-1+i} g^{(n+i)}_{n,0,\varphi(w)}(\varphi(w)) - \left(1-|\psi(w)|^2\right)^{\alpha+n-1+i} g^{(n+i)}_{n,0,\varphi(w)}(\psi(w))\right| \\
\gtrsim& \left|\varphi_{\#_0}(w) - \psi_{\#_0}(w) \left(\frac{(1-|\varphi(w)|^2)(1-|\psi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^2}\right)^{\alpha+n-1}\right| \\
&- \sum_{i=1}^{3} \left(\left|\varphi_{\#_i}(w) - \psi_{\#_i}(w)\right| + \left|\psi_{\#_i}(w)\right| \rho(w)\right).
\end{aligned} \tag{10}$$

By choosing the functions $h_{n,i,\varphi(w)} \in \mathcal{B}^\alpha_0$ in Lemma 2.4 (1) it follows that (2) holds, in the same manner, we have

$$\begin{aligned}
\infty &> \left\| \left(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}\right) h_{n,0,\varphi(w)} \right\|_{\mathcal{Z}_\mu} \\
&\geqslant \left| \widetilde{A}_0(\varphi(w)) \cdot 0 - \widetilde{A}_0(\psi(w)) \frac{\left(1-|\varphi(w)|^2\right)^{\alpha+n-1}(\varphi(w)-\psi(w))}{(1-\overline{\varphi(w)}\psi(w))^{2(\alpha+n-1)+1}} \right| \\
&\quad - \sum_{i=1}^{3} \left| \left[ \widetilde{A}_i(\varphi(w)) h^{(n+i)}_{n,0,\varphi(w)}(\varphi(w)) - \widetilde{A}_i(\psi(w)) h^{(n+i)}_{n,0,\varphi(w)}(\psi(w)) \right] \right| \\
&\gtrsim \left| \psi_{\#_0}(w) \left( \frac{\left(1-|\varphi(w)|^2\right)\left(1-|\psi(w)|^2\right)}{(1-\overline{\varphi(w)}\psi(w))^2} \right)^{\alpha+n-1} \right| \rho(w) - \sum_{i=1}^{3} \left( \left| \varphi_{\#_i}(w) - \psi_{\#_i}(w) \right| + \left| \psi_{\#_i}(w) \right| \rho(w) \right).
\end{aligned} \tag{11}$$

Based on conditions $a_1, a_2, a_3$, (10) implies that

$$\begin{aligned}
\infty &> \left| \varphi_{\#_0}(w) - \psi_{\#_0}(w) \left( \frac{(1-|\varphi(w)|^2)(1-|\psi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^2} \right)^{\alpha+n-1} \right| \\
&\geqslant \left| \varphi_{\#_0}(w) - \psi_{\#_0}(w) \left( \frac{(1-|\varphi(w)|^2)(1-|\psi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^2} \right)^{\alpha+n-1} \right| \rho(w) \\
&\geqslant \left| \varphi_{\#_0}(w) \right| \rho(w) - \left| \psi_{\#_0}(w) \left( \frac{(1-|\varphi(w)|^2)(1-|\psi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^2} \right)^{\alpha+n-1} \right| \rho(w),
\end{aligned}$$

and (11) implies that

$$\left| \psi_{\#_0}(w) \left( \frac{(1-|\varphi(w)|^2)(1-|\psi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^2} \right)^{\alpha+n-1} \right| \rho(w) < \infty,$$

so we have

$$\sup_{w \in \mathbb{D} \setminus \mathbb{D}_1} \left| \varphi_{\#_0}(w) \right| \cdot \rho(w) < \infty.$$

Therefore, considering that conditions $a_1, a_2, a_3$ hold, we obtain

$$\sup_{w \in \mathbb{D} \setminus \mathbb{D}_1} \left| \varphi_{\#_i}(w) \right| \cdot \rho(w) < \infty, \quad i \in \{0,1,2,3\}. \tag{12}$$

Likewise, we can deduce

$$\sup_{w \in \mathbb{D} \setminus \mathbb{D}_2} \left| \psi_{\#_i}(w) \right| \cdot \rho(w) < \infty, \quad i \in \{0,1,2,3\}. \tag{13}$$

By (10) and conditions $a_1, a_2, a_3$, we also have

$$\begin{aligned}
\infty &> \left| \varphi_{\#_0}(w) - \psi_{\#_0}(w) \left( \frac{(1-|\varphi(w)|^2)(1-|\psi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^2} \right)^{\alpha+n-1} \right| \\
&= \left| \varphi_{\#_0}(w) - \psi_{\#_0}(w) + \psi_{\#_0}(w) - \psi_{\#_0}(w) \left( \frac{(1-|\varphi(w)|^2)(1-|\psi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^2} \right)^{\alpha+n-1} \right| \\
&\geqslant \left| \varphi_{\#_0}(w) - \psi_{\#_0}(w) \right| - \left| \psi_{\#_0}(w) \left[ \left(1-|\varphi(w)|^2\right)^{\alpha+n-1} g^{(n)}_{n,0,\varphi(w)}(\varphi(w)) - \left(1-|\psi(w)|^2\right)^{\alpha+n-1} g^{(n)}_{n,0,\varphi(w)}(\psi(w)) \right] \right| \\
&\gtrsim \left| \varphi_{\#_0}(w) - \psi_{\#_0}(w) \right| - \left| \psi_{\#_0}(w) \right| \cdot \rho(w)
\end{aligned} \tag{14}$$

for any $w \in \mathbb{D}\backslash\mathbb{D}_1$. Substituting (13) into (14) shows that

$$\sup_{w\in\mathbb{D}\backslash(\mathbb{D}_1\cup\mathbb{D}_2)} \left|\varphi_{\#_0}(w) - \psi_{\#_0}(w)\right| < \infty. \tag{15}$$

Then, combining (12) with (15), we can assert that

$$\sup_{w\in\mathbb{D}\backslash(\mathbb{D}_1\cup\mathbb{D}_2)} \left(\left|\varphi_{\#_0}(w)\right| \cdot \rho(w) + \left|\varphi_{\#_0}(w) - \psi_{\#_0}(w)\right|\right) < \infty. \tag{16}$$

If $w \in \mathbb{D}_1 \cap \mathbb{D}_2$, then $\rho(w) = 0$. Taking the functions $r_{n,i,\varphi(w)} \in \mathcal{B}_0^\alpha$ in Lemma 2.4 (2) it follows that (3) holds, we deduce that

$$\begin{aligned}
\infty &> \left\|\left(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}\right) r_{n,0,\varphi(w)}\right\|_{\mathcal{Z}_\mu} \\
&\geqslant \left|\widetilde{A}_0(\varphi(w)) - \widetilde{A}_0(\psi(w))\right| \\
&= \left|\varphi_{\#_0}(w) - \psi_{\#_0}(w)\right| \\
&= \left|\varphi_{\#_0}(w)\right| \rho(w) + \left|\varphi_{\#_0}(w) - \psi_{\#_0}(w)\right|.
\end{aligned} \tag{17}$$

If $w \in \mathbb{D}_2\backslash\mathbb{D}_1$, then $\rho(w) = |\varphi(w)|$. Choosing the functions $p_{n,i,\varphi(w)} \in \mathcal{B}_0^\alpha$ in Lemma 2.4 (3) it follows that (4) holds, we obtain

$$\begin{aligned}
\infty &> \left\|\left(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}\right) p_{n,0,\varphi(w)}\right\|_{\mathcal{Z}_\mu} \\
&\geqslant \left|\widetilde{A}_0(\varphi(w))\frac{\varphi(w)}{(1-|\varphi(w)|^2)^{\alpha+n-1}} - \widetilde{A}_0(\psi(w))\frac{\psi(w)}{\left(1-\overline{\varphi(w)}\psi(w)\right)^{\alpha+n-1}}\right| \\
&\quad - \sum_{i=1}^{3}\left|\left[\widetilde{A}_i(\varphi(w))p^{(n+i)}_{n,0,\varphi(w)}(\varphi(w)) - \widetilde{A}_i(\psi(w))p^{(n+i)}_{n,0,\varphi(w)}(\psi(w))\right]\right| \\
&\gtrsim \left|\varphi_{\#_0}(w)\right| \rho(w) - \sum_{i=1}^{3}\left(\left|\varphi_{\#_i}(w) - \psi_{\#_i}(w)\right| + \left|\psi_{\#_i}(w)\right| \rho(w)\right).
\end{aligned}$$

Further, applying the functions $q_{n,i,\varphi(w)} \in \mathcal{B}_0^\alpha$ in Lemma 2.4 (4) it follows that (5) holds, it follows that

$$\begin{aligned}
\infty &> \left\|\left(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}\right) q_{n,0,\varphi(w)}\right\|_{\mathcal{Z}_\mu} \\
&\geqslant \left|\widetilde{A}_0(\varphi(w)) \cdot 0 - \widetilde{A}_0(\psi(w)) \cdot (\varphi(w) - \psi(w))\right| \\
&\quad - \sum_{i=1}^{3}\left|\left[\widetilde{A}_i(\varphi(w))q^{(n+i)}_{n,0,\varphi(w)}(\varphi(w)) - \widetilde{A}_i(\psi(w))q^{(n+i)}_{n,0,\varphi(w)}(\psi(w))\right]\right| \\
&\gtrsim \left|\psi_{\#_0}(w)\right| \rho(w) - \sum_{i=1}^{3}\left(\left|\varphi_{\#_i}(w) - \psi_{\#_i}(w)\right| + \left|\psi_{\#_i}(w)\right| \rho(w)\right).
\end{aligned} \tag{18}$$

(18) together with (14) and conditions $a_1, a_2, a_3$ entail that

$$\sup_{w\in\mathbb{D}_2\backslash\mathbb{D}_1} \left|\varphi_{\#_0}(w) - \psi_{\#_0}(w)\right| < \infty.$$

Hence

$$\sup_{w\in\mathbb{D}_2\backslash\mathbb{D}_1} \left(\left|\varphi_{\#_0}(w)\right| \rho(w) + \left|\varphi_{\#_0}(w) - \psi_{\#_0}(w)\right|\right) < \infty. \tag{19}$$

In the same way, it can be proved that

$$\sup_{w\in\mathbb{D}_1\setminus\mathbb{D}_2}\left(\left|\varphi_{\#_0}(w)\right|\rho(w)+\left|\varphi_{\#_0}(w)-\psi_{\#_0}(w)\right|\right)<\infty. \tag{20}$$

In summary, by (16), (17), (19) and (20), we conclude that condition $a_0$ holds. Similarly, if we want to apply the other three conditions to prove that condition $a_i$ holds, we only need to select the corresponding $g_{n,i,\varphi(w)}, h_{n,i,\varphi(w)}, r_{n,i,\varphi(w)}, p_{n,i,\varphi(w)}, q_{n,i,\varphi(w)}$ and refer to the above derivation process to complete the proof. □

Next we sharpen this result and use Theorem 3.3 to show there exist $u_1, u_2, v_1, v_2 \in H(\mathbb{D})$ and $\varphi, \psi \in S(\mathbb{D})$ such that neither $T^n_{u_1,v_1,\varphi}$ nor $T^n_{u_2,v_2,\psi}$ is bounded but $T^n_{u_1,v_1,\varphi}-T^n_{u_2,v_2,\psi}$ is bounded through an example. Referring to Example 3.3 in [9], we provide the following example.

**Example 3.4.** *Let $n\in\mathbb{N}$, $\alpha=1$ and $\mu(z)=\left(1-|z|^2\right)^\beta$ with $n+\frac{5}{2}<\beta<n+3$. We set*

$$\varphi(z)=\frac{1+z}{2},\quad \psi(z)=\frac{1+z}{2}+t\cdot\frac{1-z}{2}.$$

*Since*

$$\sup_{z\in\mathbb{D}}\left|\frac{1+z}{2}\right|=1,$$

*and*

$$\sup_{z\in\mathbb{D}}\left|\frac{1+z}{2}+t\cdot\frac{1-z}{2}\right|=\sup_{z\in\mathbb{D}}\left|\frac{1+t}{2}+\frac{1-t}{2}\cdot z\right|=1,$$

*we have $\varphi,\psi\in S(\mathbb{D})$ whenever $t$ is sufficiently small. Specifically, we consider the following special case, and we are still exploring more general examples. Choosing*

$$u_1(z)=u_2(z)=0,\quad v_1(z)=(1-z)\log(1-z)+z,$$
$$v_2(z)=(1-z)\log(1-z)+z+\sqrt{1-z}.$$

*In this case, by some calculations, we deduce that*

$$\left|\varphi_{\#_0}(r)\right|=\frac{\left(1-r^2\right)^\beta\left|u_1''(r)\right|}{(1-|\varphi(r)|^2)^n}=0,$$
$$\left|\varphi_{\#_1}(r)\right|=\frac{\left(1-r^2\right)^\beta\left|u_1'(r)+v_1''(r)\right|}{(1-|\varphi(r)|^2)^{n+1}}=\frac{4^{n+1}(1+r)^\beta(1-r)^{\beta-n-2}}{(3+r)^{n+1}}\to 0,$$
$$\left|\varphi_{\#_2}(r)\right|=\frac{\left(1-r^2\right)^\beta\left|u_1(r)\cdot\frac{1}{4}+v_1'(r)\right|}{(1-|\varphi(r)|^2)^{n+2}}=\frac{4^{n+2}(1+r)^\beta(1-r)^{\beta-n-2}\cdot\log\frac{1}{1-r}}{(3+r)^{n+2}}\to 0,$$
$$\left|\varphi_{\#_3}(r)\right|=\frac{\left(1-r^2\right)^\beta\left|v_1(r)\cdot\frac{1}{4}\right|}{(1-|\varphi(r)|^2)^{n+3}}=\frac{4^{n+2}(1+r)^\beta\cdot[(1-r)\log(1-r)+r]}{(1-r)^{n+3-\beta}(3+r)^{n+3}}\to\infty,$$

*as $r\to 1$. Then Proposition 3.1 implies $T^n_{u_1,v_1,\varphi}$ is not bounded. For the same reason, whenever $t$ is sufficiently small, we have*

$$\left|\psi_{\#_i}(r)\right|\to 0,\quad i\in\{0,1,2\},\ \text{and}\ \left|\psi_{\#_3}(r)\right|\to\infty,$$

*as $r\to 1$. Then Proposition 3.1 implies $T^n_{u_2,v_2,\psi}$ is also not bounded. On the other hand,*

$$|\varphi_{\#_0}(z)|\rho(z)+\left|\varphi_{\#_0}(z)-\psi_{\#_0}(z)\right|=0,$$

*and*

$$|\varphi_{\#_i}(z)|\rho(z) + \left|\varphi_{\#_i}(z) - \psi_{\#_i}(z)\right| < \infty, \quad i \in \{0,1,2,3\},$$

*for all $z \in \mathbb{D}$. Thus Theorem 3.3 ensures the boundedness of $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$.*

Next, we proceed with the compactness of $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$. Let us denote the sets

$$\Gamma(\varphi) = \left\{\{z_N\} \subset \mathbb{D} : \left|\varphi(z_N)\right| \to 1\right\}, \ \ \Gamma(\psi) = \left\{\{z_N\} \subset \mathbb{D} : \left|\psi(z_N)\right| \to 1\right\},$$

$$D(\varphi) = \left\{\{z_N\} \subset \mathbb{D} : \left|\varphi(z_N)\right| \to 1, \sum_{i=0}^{3} \left|\varphi_{\#_i}(z_N)\right| \nrightarrow 0\right\},$$

$$D(\psi) = \left\{\{z_N\} \subset \mathbb{D} : \left|\psi(z_N)\right| \to 1, \sum_{i=0}^{3} \left|\psi_{\#_i}(z_N)\right| \nrightarrow 0\right\}.$$

Moreover for $\{z_N\} \in \Gamma(\varphi) \cap \Gamma(\psi)$, we define

$$\text{condition } b_0 : \quad \lim_{N\to\infty} \left|\varphi_{\#_0}(z_N)\right| \rho(z_N) = 0, \quad \lim_{N\to\infty} \left|\varphi_{\#_0}(z_N) - \psi_{\#_0}(z_N)\right| = 0;$$

$$\text{condition } b_1 : \quad \lim_{N\to\infty} \left|\varphi_{\#_1}(z_N)\right| \rho(z_N) = 0, \quad \lim_{N\to\infty} \left|\varphi_{\#_1}(z_N) - \psi_{\#_1}(z_N)\right| = 0;$$

$$\text{condition } b_2 : \quad \lim_{N\to\infty} \left|\varphi_{\#_2}(z_N)\right| \rho(z_N) = 0, \quad \lim_{N\to\infty} \left|\varphi_{\#_2}(z_N) - \psi_{\#_2}(z_N)\right| = 0;$$

$$\text{condition } b_3 : \quad \lim_{N\to\infty} \left|\varphi_{\#_3}(z_N)\right| \rho(z_N) = 0, \quad \lim_{N\to\infty} \left|\varphi_{\#_3}(z_N) - \psi_{\#_3}(z_N)\right| = 0.$$

**Theorem 3.5.** *Let $n \in \mathbb{N}$, $\alpha > 0$, $u_1, u_2, v_1, v_2 \in H(\mathbb{D})$, $\varphi$, $\psi \in S(\mathbb{D})$ and $\mu \in W(\mathbb{D})$. Suppose that neither $T^n_{u_1,v_1,\varphi}$ nor $T^n_{u_2,v_2,\psi}$ is compact but they are both bounded and any three of conditions $b_0, b_1, b_2, b_3$ are valid. Then the following statements are equivalent.*

(1) $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ *is compact.*

(2) $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_\mu$ *is compact.*

(3) (i)$D(\varphi) = D(\psi)$;

(ii)*For $\{z_N\} \in \Gamma(\varphi) \cap \Gamma(\psi)$, the remaining condition holds.*

*Proof.* It follows immediately that (1) ⇒ (2). Now we assume (3) and conditions $b_0, b_1, b_2, b_3$ hold but $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_\mu$ is not compact. Then by Lemma 2.5, we can find some $\varepsilon > 0$ and a bounded sequence $\{f_N\}$ in $\mathcal{B}^\alpha_0$, which converges to 0 uniformly on every compact subsets of $\mathbb{D}$ such that

$$\left\|\left(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}\right) f_N\right\|_{\mathcal{Z}_\mu} > \varepsilon,$$

for all $N$. From the boundedness of $T^n_{u_1,v_1,\varphi}$ and $T^n_{u_2,v_2,\psi}$, it follows that $\left|\left(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}\right) f_N(0)\right| \to 0$ and $\left|\left(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}\right) f'_N(0)\right| \to 0$ as $N \to \infty$. Consequently, for any $N$, there exists $z_N \in \mathbb{D}$ such that

$$\left|\sum_{i=0}^{3} \left[\varphi_{\#_i}(z_N)\left(1 - \left|\varphi(z_N)\right|^2\right)^{\alpha+n-1+i} f_N^{(n+i)}(\varphi(z_N)) - \psi_{\#_i}(z_N)\left(1 - \left|\psi(z_N)\right|^2\right)^{\alpha+n-1+i} f_N^{(n+i)}(\psi(z_N))\right]\right| > \varepsilon. \quad (21)$$

We have four cases to be considered.

① $\{z_N\} \notin \Gamma(\varphi)$ and $\{z_N\} \notin \Gamma(\psi)$

Since $\{z_N\} \notin \Gamma(\varphi)$, there exists a compact set $K \subset \mathbb{D}$. Then we choose a set of subsequences such that $\{\varphi(z_N)\} \subset K$. It can be inferred from the Cauchy formula that $f_N^{(n+i)}(\varphi(z_N)) \to 0$ as $N \to \infty$. On account of the boundedness of $T^n_{u_1,v_1,\varphi}$, we further deduce

$$\left|\varphi_{\#_i}(z_N)\left(1 - \left|\varphi(z_N)\right|^2\right)^{\alpha+n-1+i} f_N^{(n+i)}(\varphi(z_N))\right| \to 0, \quad N \to \infty,$$

for each $i \in \{0,1,2,3\}$. In the same manner we can see that

$$\left|\psi_{\#_i}(z_N)\left(1-\left|\psi(z_N)\right|^2\right)^{\alpha+n-1+i} f_N^{(n+i)}(\psi(z_N))\right| \to 0, \quad N \to \infty,$$

for each $i \in \{0,1,2,3\}$. These contradict (21).

② $\{z_N\} \in \Gamma(\varphi)$ but $\{z_N\} \notin \Gamma(\psi)$

In this case, we have $\{z_N\} \notin D(\varphi)$ because $D(\varphi) \subseteq \Gamma(\psi) \cap \Gamma(\varphi)$ by (i). Therefore $\sum_{i=0}^{3} |\varphi_{\#_i}(z_N)| \to 0$ as $N \to \infty$. From Lemma 2.1, it yields that

$$\left|\varphi_{\#_i}(z_N)\right| \cdot \left(1-\left|\varphi(z_N)\right|^2\right)^{\alpha+n-1+i} \left|f_N^{(n+i)}(\varphi(z_N))\right| \to 0, \quad N \to \infty,$$

for each $i \in \{0,1,2,3\}$. By the same way as ①, we can get

$$\left|\psi_{\#_i}(z_N)\right| \cdot \left(1-\left|\psi(z_N)\right|^2\right)^{\alpha+n-1+i} \left|f_N^{(n+i)}(\psi(z_N))\right| \to 0, \quad N \to \infty,$$

for $\left|\psi(z_N)\right| \nrightarrow 1$ and $i \in \{0,1,2,3\}$. From the above two equations, we obtain a contradiction with (21).

③ $\{z_N\} \in \Gamma(\psi)$ but $\{z_N\} \notin \Gamma(\varphi)$

This follows by the same method as in ②.

④ $\{z_N\} \in \Gamma(\varphi) \cap \Gamma(\psi)$

In this case, according to conditions $b_0, b_1, b_2, b_3$, we have

$$\begin{aligned}
&\left|\sum_{i=0}^{3}\left[\varphi_{\#_i}(z_N)\cdot\left(1-|\varphi(z_N)|^2\right)^{\alpha+n-1+i} f_N^{(n+i)}(\varphi(z_N)) - \psi_{\#_i}(z_N)\cdot\left(1-|\psi(z_N)|^2\right)^{\alpha+n-1+i} f_N^{(n+i)}(\psi(z_N))\right]\right| \\
\leqslant &\left|\sum_{i=0}^{3}\varphi_{\#_i}(z_N)\cdot\left[\left(1-|\varphi(z_N)|^2\right)^{\alpha+n-1+i} f_N^{(n+i)}(\varphi(z_N)) - \left(1-|\psi(z_N)|^2\right)^{\alpha+n-1+i} f_N^{(n+i)}(\psi(z_N))\right]\right| \\
&\quad + \left|\sum_{i=0}^{3}\left(1-|\psi(z_N)|^2\right)^{\alpha+n-1+i} f_N^{(n+i)}(\psi(z_N))\cdot\left(\varphi_{\#_i}(z_N)-\psi_{\#_i}(z_N)\right)\right| \\
\lesssim &\sum_{i=0}^{3}\left|\varphi_{\#_i}(z_N)\right|\cdot\rho(z_N) + \left|\sum_{i=0}^{3}\left(1-|\psi(z_N)|^2\right)^{\alpha+n-1+i} f_N^{(n+i)}(\psi(z_N))\cdot\left(\varphi_{\#_i}(z_N)-\psi_{\#_i}(z_N)\right)\right| \\
\lesssim &\sum_{i=0}^{3}\left[\left|\varphi_{\#_i}(z_N)\right|\rho(z_N) + \left|\varphi_{\#_i}(z_N)-\psi_{\#_i}(z_N)\right|\right] \to 0, \ \ N \to \infty,
\end{aligned}$$

where the second and third inequalities follow from Lemma 2.2 and Lemma 2.1, respectively. This is also a contradiction according to (21). Hence, we have $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ is compact and thus (3) ⇒ (1) is established.

We only need to prove (2) ⇒ (3). Assume that $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ is compact, but neither $T^n_{u_1,v_1,\varphi}$ nor $T^n_{u_2,v_2,\psi}$ is compact. Without loss of generality, we suppose that conditions $b_1, b_2, b_3$ hold, then prove that condition $b_0$ holds. Proposition 3.2 implies that there exists a sequence $\{z_N\} \in D(\varphi)$ with $\left|\varphi(z_N)\right| \to 1$ and $\sum_{i=0}^{3}\left|\varphi_{\#_i}(z_N)\right| \nrightarrow 0$ as $N \to \infty$. The functions $g_{n,i,\varphi(z_N)}$ and $h_{n,i,\varphi(z_N)}$ defined in Lemma 2.3 and Lemma 2.4 (1) are all bounded in $\mathcal{B}_0^\alpha$ and converge to 0 uniformly on every compact subset of $\mathbb{D}$ as $N \to \infty$. By Lemma 2.5, taking the functions $g_{n,0,\varphi(z_N)}$ and $h_{n,0,\varphi(z_N)}$, we have

$$\begin{aligned}0 \leftarrow & \left\|\left(T^n_{u_1,v_1,\varphi}-T^n_{u_2,v_2,\psi}\right) g_{n,0,\varphi(z_N)}\right\|_{\mathcal{Z}_\mu} \\ \geqslant & \left|\widetilde{A}_0(\varphi(z_N))\frac{1}{(1-|\varphi(z_N)|^2)^{\alpha+n-1}}-\widetilde{A}_0(\psi(z_N))\left(\frac{1-|\varphi(z_N)|^2}{(1-\overline{\varphi(z_N)}\psi(z_N))^2}\right)^{\alpha+n-1}\right| \\ & -\sum_{i=1}^{3}\left\|\left[\widetilde{A}_i(\varphi(z_N))g^{(n+i)}_{n,0,\varphi(z_N)}(\varphi(z_N))-\widetilde{A}_i(\psi(z_N))g^{(n+i)}_{n,0,\varphi(z_N)}(\psi(z_N))\right]\right\| \\ \gtrsim & \left|\varphi_{\#_0}(z_N)-\psi_{\#_0}(z_N)\left(\frac{\left(1-|\varphi(z_N)|^2\right)\left(1-|\psi(z_N)|^2\right)}{(1-\overline{\varphi(z_N)}\psi(z_N))^2}\right)^{\alpha+n-1}\right| \\ & -\sum_{i=1}^{3}\left(\left|\varphi_{\#_i}(z_N)-\psi_{\#_i}(z_N)\right|+\left|\psi_{\#_i}(z_N)\right|\rho(z_N)\right)\end{aligned} \tag{22}$$

and

$$\begin{aligned}0 \leftarrow & \left\|\left(T^n_{u_1,v_1,\varphi}-T^n_{u_2,v_2,\psi}\right) h_{n,0,\varphi(z_N)}\right\|_{\mathcal{Z}_\mu} \\ \geqslant & \left|\widetilde{A}_0(\varphi(z_N))\cdot 0-\widetilde{A}_0(\psi(z_N))\frac{\left(1-|\varphi(z_N)|^2\right)^{\alpha+n-1}(\varphi(z_N)-\psi(z_N))}{(1-\overline{\varphi(z_N)}\psi(z_N))^{2(\alpha+n-1)+1}}\right| \\ & -\sum_{i=1}^{3}\left\|\left[\widetilde{A}_i(\varphi(z_N))g^{(n+i)}_{n,0,\varphi(z_N)}(\varphi(z_N))-\widetilde{A}_i(\psi(z_N))g^{(n+i)}_{n,0,\varphi(z_N)}(\psi(z_N))\right]\right\| \\ \gtrsim & \left|\psi_{\#_0}(z_N)\left(\frac{\left(1-|\varphi(z_N)|^2\right)\left(1-|\psi(z_N)|^2\right)}{(1-\overline{\varphi(z_N)}\psi(z_N))^2}\right)^{\alpha+n-1}\right|\rho(z_N) \\ & -\sum_{i=1}^{3}\left(\left|\varphi_{\#_i}(z_N)-\psi_{\#_i}(z_N)\right|+\left|\psi_{\#_i}(z_N)\right|\rho(z_N)\right).\end{aligned} \tag{23}$$

Based on conditions $b_1, b_2, b_3$, combining (22) with (23), we obtain

$$\lim_{N\to\infty}\left|\varphi_{\#_0}(z_N)\right|\rho(z_N)=0.$$

Therefore, considering that conditions $b_1, b_2, b_3$, we have

$$\lim_{N\to\infty}\left|\varphi_{\#_i}(z_N)\right|\rho(z_N)=0,\quad i\in\{0,1,2,3\}, \tag{24}$$

which implies that

$$\lim_{N\to\infty}\left|\frac{\varphi(z_N)-\psi(z_N)}{1-\overline{\varphi(z_N)}\psi(z_N)}\right|=\lim_{N\to\infty}\rho(z_N)=0$$

according to $\{z_N\}\in D(\varphi)$. It follows that $|\psi(z_N)|\to 1$ as $N\to\infty$, i.e. $\{z_N\}\in\Gamma(\psi)$. Thus $D(\varphi)\subseteq\Gamma(\varphi)\cap\Gamma(\psi)$.

Further for any $\{z_N\} \in \Gamma(\varphi) \cap \Gamma(\psi)$, it yields from (22) and conditions $b_1, b_2, b_3$ that

$$\begin{aligned}0 \leftarrow & \left| \varphi_{\#_0}(z_N) - \psi_{\#_0}(z_N) \cdot \left( \frac{\left(1-|\varphi(z_N)|^2\right)\left(1-|\psi(z_N)|^2\right)}{(1-\overline{\varphi(z_N)}\psi(z_N))^2} \right)^{\alpha+n-1} \right| \\ & \geqslant \left|\varphi_{\#_0}(z_N) - \psi_{\#_0}(z_N)\right| - \left|\psi_{\#_0}(z_N) \cdot \left[ \left(1-|\varphi(z_N)|^2\right)^{\alpha+n-1} g^{(n)}_{n,0,\varphi(z_N)}(\varphi(z_N)) \right.\right. \\ & \quad \left.\left. - \left(1-|\psi(z_N)|^2\right)^{\alpha+n-1} g^{(n)}_{n,0,\varphi(z_N)}(\psi(z_N)) \right]\right| \\ & \gtrsim \left|\varphi_{\#_0}(z_N) - \psi_{\#_0}(z_N)\right| - \left|\psi_{\#_0}(z_N)\right| \cdot \rho(z_N)\end{aligned}$$

as $N \to \infty$. Then

$$\lim_{N\to\infty} \left|\varphi_{\#_0}(z_N) - \psi_{\#_0}(z_N)\right| = 0, \tag{25}$$

since $\rho(z_N) \to 0$. Hence for arbitrary $\{z_N\} \in \Gamma(\varphi)\cap\Gamma(\psi)$, combining (24) with (25), we conclude that condition $b_0$ holds. Furthermore (25) together with the fact that $D(\varphi) \subseteq \Gamma(\varphi) \cap \Gamma(\psi)$ confirms that $D(\varphi) \subseteq D(\psi)$. In the same way, $D(\psi) \subseteq D(\varphi)$. Therefore $D(\varphi) = D(\psi)$. We can assert that (2) ⇒ (3) holds. □

To further deepen our cognition of this result, we will provide an example to apply Theorem 3.5. The following example refer to Example 3.3 in [9].

**Example 3.6.** *Let $n \in \mathbb{N}$, $\alpha = 1$ and $\mu(z) = \left(1-|z|^2\right)^\beta$ with $\beta = (n+3)/2$. Set*

$$\varphi(z) = \frac{\sqrt{1+z} - \sqrt{1-z}}{\sqrt{1+z} + \sqrt{1-z}}, \quad \psi(z) = 1 - \sqrt{2} \cdot \sqrt{1-z},$$

*then $\varphi, \psi \in S(\mathbb{D})$. In particular, we consider the following special case. We are still exploring more general examples. Choosing*

$$u_1(z) = u_2(z) = 0, \quad v_1(z) = 1 - z^2, \quad v_2(z) = 2z - 2z^2.$$

*It is easily to check that*

$$\begin{aligned}|\varphi_{\#_0}(r)| &\approx \left(1-r^2\right)^{\beta-\frac{n}{2}} \cdot \left|u_1''(r)\right| = 0, \\ |\varphi_{\#_1}(r)| &\approx \left(1-r^2\right)^{\beta-\frac{n+1}{2}} \to 0, \\ |\varphi_{\#_2}(r)| &\approx \left(1-r^2\right)^{\beta-\frac{n+1}{2}} \to 0, \\ |\varphi_{\#_3}(r)| &\approx (1-r)^{\beta-\frac{n+3}{2}} = 1 \nrightarrow 0,\end{aligned}$$

*as $r \to 1$, i.e. $|\varphi(r)| \to 1$. And*

$$\sup_{z\in\mathbb{D}} \left|\varphi_{\#_i}(z)\right| < \infty, \quad i \in \{0,1,2,3\}.$$

*On one hand, Proposition 3.1 and Proposition 3.2 imply $T^n_{u_1,v_1,\varphi}$ is bounded but not compact. In the same way, it confirms that $T^n_{u_2,v_2,\psi}$ is also bounded but not compact. On the other hand, we deduce that $\Gamma(\varphi) = \Gamma(\psi) = D(\varphi) = D(\psi) = \{\{z_N\} \subset \mathbb{D} : z_N \to 1$ or $z_N \to -1, \quad N \to \infty\}$. It follows that*

$$\lim_{N\to\infty} \left[ \left|\varphi_{\#_i}(z_N)\right| \rho(z_N) + \left|\varphi_{\#_i}(z_N) - \psi_{\#_i}(z_N)\right| \right] = 0, \quad i \in \{0,1,2,3\},$$

*for $\{z_N\} \in \Gamma(\varphi) \cap \Gamma(\psi)$. Then Theorem 3.5 shows that the difference $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_\mu$ is compact.*

## 4. Boundedness and Compactness of $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_{\mu,0}$

In this section, we concentrate on the boundedness and compactness of $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_{\mu,0}$. Define

$$\begin{aligned}
&\text{condition } c_0: \quad \lim_{|z|\to1} \left|\varphi_{\#_0}(z)\right| \rho(z) = 0, \quad \lim_{|z|\to1} \left|\varphi_{\#_0}(z) - \psi_{\#_0}(z)\right| = 0;\\
&\text{condition } c_1: \quad \lim_{|z|\to1} \left|\varphi_{\#_1}(z)\right| \rho(z) = 0, \quad \lim_{|z|\to1} \left|\varphi_{\#_1}(z) - \psi_{\#_1}(z)\right| = 0;\\
&\text{condition } c_2: \quad \lim_{|z|\to1} \left|\varphi_{\#_2}(z)\right| \rho(z) = 0, \quad \lim_{|z|\to1} \left|\varphi_{\#_2}(z) - \psi_{\#_2}(z)\right| = 0;\\
&\text{condition } c_3: \quad \lim_{|z|\to1} \left|\varphi_{\#_3}(z)\right| \rho(z) = 0, \quad \lim_{|z|\to1} \left|\varphi_{\#_3}(z) - \psi_{\#_3}(z)\right| = 0.
\end{aligned}$$

**Theorem 4.1.** *Let $n \in \mathbb{N}$, $\alpha > 0$, $u_1, u_2, v_1, v_2 \in H(\mathbb{D})$, $\varphi, \psi \in S(\mathbb{D})$ and $\mu \in W(\mathbb{D})$. Suppose that $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ is bounded and any three of conditions $c_0, c_1, c_2, c_3$ are valid, then the following statements are equivalent.*

(1) $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha \to \mathcal{Z}_{\mu,0}$ *is bounded.*

(2) $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_{\mu,0}$ *is bounded.*

(3) *The remaining condition holds.*

(4) *For $i \in \{0, 1, 2, 3\}$,*

$$\lim_{|z|\to1} |\varphi(z) - \psi(z)| \cdot \max\left\{\left|\widetilde{A}_i(\varphi(z))\right|, \left|\widetilde{A}_i(\psi(z))\right|\right\} = 0, \tag{26}$$

*and*

$$\lim_{|z|\to1} \left|\widetilde{A}_i(\varphi(z)) - \widetilde{A}_i(\psi(z))\right| = 0.$$

(5) *For $i \in \{0, 1, 2, 3\}$,*

$$\lim_{|z|\to1} \left|\varphi(z)\widetilde{A}_i(\varphi(z)) - \psi(z)\widetilde{A}_i(\psi(z))\right| = 0, \tag{27}$$

*and*

$$\lim_{|z|\to1} \left|\widetilde{A}_i(\varphi(z)) - \widetilde{A}_i(\psi(z))\right| = 0.$$

*Proof.* (1) ⇒ (2). The proof is straightforward.

(2) ⇒ (3). Analysis similar to that in the proof of (2) ⇒ (3) in Theorem 3.3 shows that it is trivial.

(3) ⇒ (1). This follows by the same method as in the proof of (3) ⇒ (1) in Theorem 3.3.

(3) ⇒ (4). Setting $f(z) = z^n$, we have

$$\left|\widetilde{A}_0(\varphi(z)) - \widetilde{A}_0(\psi(z))\right| = \mu(z) \left|\left(\left(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}\right) f\right)''(z)\right| \to 0, \tag{28}$$

as $|z| \to 1$. Since

$$|\varphi(z) - \psi(z)| \cdot \left|\widetilde{A}_i(\varphi(z))\right| \leqslant \left|\varphi_{\#_i}(z)\right| \cdot \rho(z), \quad i \in \{0, 1, 2, 3\},$$

we have

$$\lim_{|z|\to1} |\varphi(z) - \psi(z)| \cdot \left|\widetilde{A}_i(\varphi(z))\right| = 0, \quad i \in \{0, 1, 2, 3\}.$$

Likewise, we obtain

$$\lim_{|z|\to1} |\varphi(z) - \psi(z)| \cdot \left|\widetilde{A}_i(\psi(z))\right| = 0, \quad i \in \{0, 1, 2, 3\}.$$

Therefore, it can be inferred from the above two equalities that (26) holds.

For $i \in \{1,2,3\}$, by the Taylor series of $(1-z)^\alpha$, we deduce that

$$\begin{aligned}
&\left|\widetilde{A_i}(\varphi(z)) - \widetilde{A_i}(\psi(z))\right| \\
&= \left|\frac{\widetilde{A_i}(\varphi(z))}{(1-|\varphi(z)|^2)^{\alpha+n-1+i}} \cdot \left(1-|\varphi(z)|^2\right)^{\alpha+n-1+i} - \frac{\widetilde{A_i}(\psi(z))}{(1-|\psi(z)|^2)^{\alpha+n-1+i}} \cdot \left(1-|\psi(z)|^2\right)^{\alpha+n-1+i}\right| \\
&= \left|\varphi_{\#_i}(z) \cdot \left(1 + \sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k+1)} \cdot \left(|\varphi(z)|^2\right)^k\right)\right. \\
&\quad \left. -\psi_{\#_i}(z) \cdot \left(1 + \sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k+1)} \cdot \left(|\psi(z)|^2\right)^k\right)\right| \\
&= \left|\varphi_{\#_i}(z) - \psi_{\#_i}(z) + \varphi_{\#_i}(z) \cdot \sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k+1)} \cdot \left(|\varphi(z)|^2\right)^k\right. \\
&\quad - \varphi_{\#_i}(z) \cdot \sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k+1)} \cdot \left(|\psi(z)|^2\right)^k \\
&\quad + \varphi_{\#_i}(z) \cdot \sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k+1)} \cdot \left(|\psi(z)|^2\right)^k \\
&\quad \left. -\psi_{\#_i}(z) \cdot \sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k+1)} \cdot \left(|\psi(z)|^2\right)^k\right| \\
&\leqslant \left|\varphi_{\#_i}(z) - \psi_{\#_i}(z)\right| \\
&\quad + \left|\varphi_{\#_i}(z)\right| \cdot \sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k+1)} \cdot \left[\left(|\varphi(z)|^2\right)^k - \left(|\psi(z)|^2\right)^k\right] \\
&\quad + \left|\varphi_{\#_i}(z) - \psi_{\#_i}(z)\right| \cdot \sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k+1)} \cdot \left(|\psi(z)|^2\right)^k. \qquad (29)
\end{aligned}$$

Firstly, we observe that

$$\sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k+1)} \cdot \left(\left|\psi(z)\right|^2\right)^k < \infty.$$

Since

$$\begin{aligned}
&\left|\sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k+1)} \left(|\psi(z)|^2\right)^k\right| \\
&\leqslant \sum_{k=1}^{\infty} \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k+1)},
\end{aligned}$$

and the calculations

$$\begin{aligned}
&\lim_{k\to\infty} k\left(1 - \frac{\frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k-1)\Gamma(k+2)}}{\frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k+1)}}\right) \\
&= \lim_{k\to\infty} k \cdot \left(1 - \frac{k+2-n-i-\alpha}{k+1}\right) = \alpha+n-1+i > 1,
\end{aligned}$$

it easily follows that

$$\sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k+1)} \cdot \left(\left|\psi(z)\right|^2\right)^k < \infty.$$

Consequently, we can estimate the first and third terms of (29) as $i \in \{1,2,3\}$

$$\begin{aligned}&\left|\varphi_{\#_i}(z) - \psi_{\#_i}(z)\right| \\ &\quad + \left|\varphi_{\#_i}(z) - \psi_{\#_i}(z)\right| \cdot \sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k+1)} \cdot \left(\left|\psi(z)\right|^2\right)^k \\ &\lesssim \left|\varphi_{\#_i}(z) - \psi_{\#_i}(z)\right|. \end{aligned} \tag{30}$$

For the second term of (29), we assume $|\varphi(z)| \geqslant |\psi(z)|$ and apply the fact

$$\sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k)} \cdot \left(|\varphi(z)|^2\right)^{k-1} < \infty,$$

to deduce that

$$\begin{aligned}&|\varphi_{\#_i}(z)| \cdot \sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k+1)} \cdot \left[\left(|\varphi(z)|^2\right)^k - \left(|\psi(z)|^2\right)^k\right] \\ &= \left|\varphi_{\#_i}(z)\right| \cdot (|\varphi(z)|^2 - |\psi(z)|^2) \cdot \sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k+1)} \\ &\quad \cdot \left(|\varphi(z)|^{2(k-1)} + |\varphi(z)|^{2(k-2)} \cdot |\psi(z)|^2 + \cdots + |\varphi(z)|^2 \cdot |\psi(z)|^{2(k-2)} + |\psi(z)|^{2(k-1)}\right) \\ &\leqslant \left|\varphi_{\#_i}(z)\right| \cdot (|\varphi(z)|^2 - |\psi(z)|^2) \cdot \sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k+1)} \cdot k \cdot |\varphi(z)|^{2(k-1)} \\ &= \left|\varphi_{\#_i}(z)\right| \cdot (|\varphi(z)|^2 - |\psi(z)|^2) \cdot \sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k)} \cdot \left(|\varphi(z)|^2\right)^{k-1} \\ &\leqslant 4\left|\varphi_{\#_i}(z)\right| \cdot \rho(z) \cdot \sum_{k=1}^{\infty}(-1)^k \cdot \frac{\Gamma(\alpha+n-1+i+1)}{\Gamma(\alpha+n-1+i-k)\Gamma(k)} \cdot \left(|\varphi(z)|^2\right)^{k-1} \\ &\lesssim \left|\varphi_{\#_i}(z)\right| \cdot \rho(z). \end{aligned} \tag{31}$$

Substituting (30) and (31) into (29), it follows that

$$\left|\widetilde{A}_i(\varphi(z)) - \widetilde{A}_i(\psi(z))\right| \lesssim \left|\varphi_{\#_i}(z) - \psi_{\#_i}(z)\right| + \left|\varphi_{\#_i}(z)\right| \cdot \rho(z), \quad i \in \{1,2,3\}, \tag{32}$$

for $|\varphi(z)| \geqslant |\psi(z)|$. In the same manner, we also entail

$$\left|\widetilde{A}_i(\varphi(z)) - \widetilde{A}_i(\psi(z))\right| \lesssim \left|\varphi_{\#_i}(z) - \psi_{\#_i}(z)\right| + \left|\psi_{\#_i}(z)\right| \cdot \rho(z), \quad i \in \{1,2,3\}, \tag{33}$$

for $|\varphi(z)| < |\psi(z)|$. Hence, combining (32) and (33) with the condition (3), and considering the conclusion of (28) we conclude that

$$\lim_{|z|\to 1}\left|\widetilde{A}_i(\varphi(z)) - \widetilde{A}_i(\psi(z))\right| = 0, \quad i \in \{0,1,2,3\}.$$

(4) ⇒ (5). For $i \in \{0,1,2,3\}$,

$$\begin{aligned}&\left|\varphi(z)\widetilde{A}_i(\varphi(z)) - \psi(z)\widetilde{A}_i(\psi(z))\right|\\ \lesssim&\left|(\varphi(z)-\psi(z))\cdot\left(\widetilde{A}_i(\varphi(z))+\widetilde{A}_i(\psi(z))\right)\right| + \left|(\varphi(z)+\psi(z))\cdot\left(\widetilde{A}_i(\varphi(z))-\widetilde{A}_i(\psi(z))\right)\right|\\ \lesssim&|\varphi(z)-\psi(z)|\cdot\left(\left|\widetilde{A}_i(\varphi(z))\right|+\left|\widetilde{A}_i(\psi(z))\right|\right)+\left|\widetilde{A}_i(\varphi(z))-\widetilde{A}_i(\psi(z))\right|.\end{aligned}$$

Therefore, based on the assumption, we can prove (27) holds.

(5) ⇒ (2). We set $f_l(z) = z^l, l \in \mathbb{N}_0$, then $f_l \in \mathcal{B}_0^\alpha$,

$$\begin{aligned}&\mu(z)\left|\left(\left(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}\right)f_l\right)''(z)\right|\\ =&\left|\sum_{i=0}^{3}\left[f_l^{(n+i)}(\varphi(z))\widetilde{A}_i(\varphi(z)) - f_l^{(n+i)}(\psi(z))\widetilde{A}_i(\psi(z))\right]\right|\\ =&\left|\sum_{i=0}^{3}\left[\frac{l!}{(n+i)!}(\varphi(z))^{l-n-i}\widetilde{A}_i(\varphi(z)) - \frac{l!}{(n+i)!}(\psi(z))^{l-n-i}\widetilde{A}_i(\psi(z))\right]\right|.\end{aligned}$$

When $l > n+4$, we have

$$\begin{aligned}&\left|\sum_{i=0}^{3}\left[\frac{l!}{(n+i)!}(\varphi(z))^{l-n-i}\widetilde{A}_i(\varphi(z)) - \frac{l!}{(n+i)!}(\psi(z))^{l-n-i}\widetilde{A}_i(\psi(z))\right]\right|\\ \leqslant&\sum_{i=0}^{3}\frac{l!}{(n+i)!}\left|(\varphi(z))^{l-n-i}\widetilde{A}_i(\varphi(z)) - (\psi(z))^{l-n-i}\widetilde{A}_i(\psi(z))\right|\\ \leqslant&\sum_{i=0}^{3}\frac{l!}{(n+i)!}\left[\left|\varphi^{l-n-i-1}(z)\right|\cdot\left|\varphi(z)\widetilde{A}_i(\varphi(z))-\psi(z)\widetilde{A}_i(\psi(z))\right|\right.\\ &+\left|\varphi^{l-n-i-1}(z)\psi(z)\right|\cdot\left|\widetilde{A}_i(\varphi(z))-\widetilde{A}_i(\psi(z))\right|\\ &\left.+|\psi(z)|\cdot\left|\varphi^{l-n-i-1}(z)\widetilde{A}_i(\varphi(z))-\psi^{l-n-i-1}(z)\widetilde{A}_i(\psi(z))\right|\right]\\ \lesssim&\sum_{i=0}^{3}\left[\left|\varphi(z)\widetilde{A}_i(\varphi(z))-\psi(z)\widetilde{A}_i(\psi(z))\right|\right.\\ &+\left|\widetilde{A}_i(\varphi(z))-\widetilde{A}_i(\psi(z))\right|\\ &\left.+\left|\varphi^{l-n-i-1}(z)\widetilde{A}_i(\varphi(z))-\psi^{l-n-i-1}(z)\widetilde{A}_i(\psi(z))\right|\right].\end{aligned}$$

Repeating the above iterative steps, we have

$$\begin{aligned}&\mu(z)\left|\left(\left(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}\right)f_l\right)''(z)\right|\\ \lesssim&\sum_{i=0}^{3}\left[\left|\varphi(z)\widetilde{A}_i(\varphi(z))-\psi(z)\widetilde{A}_i(\psi(z))\right|+\left|\widetilde{A}_i(\varphi(z))-\widetilde{A}_i(\psi(z))\right|\right].\end{aligned}\tag{34}$$

When $0 \leqslant l \leqslant n+4$, we only need to show that

$$\lim_{|z|\to 1}\left|(\varphi(z))^k\widetilde{A}_i(\varphi(z)) - (\psi(z))^k\widetilde{A}_i(\psi(z))\right| = 0,\tag{35}$$

where $i \in \{0,1,2,3\}$, $k \in \mathbb{N}_0$, such that $0 \leqslant k \leqslant 4-i$. It follows from the assumption that

$$\lim_{|z|\to 1}\left|\varphi(z)\widetilde{A_i}(\varphi(z)) - \psi(z)\widetilde{A_i}(\psi(z))\right| = 0,$$

$$\lim_{|z|\to 1}\left|\widetilde{A_i}(\varphi(z)) - \widetilde{A_i}(\psi(z))\right| = 0.$$

The rest of (35) can be obtained by applying the above iterative scaling method. Hence, when $0 \leqslant l \leqslant n+4$, (34) is also holds.

Thus, we obtain $\left(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}\right) f_l \in \mathcal{Z}_{\mu,0}$, so $\left(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}\right) p \in \mathcal{Z}_{\mu,0}$ for all polynomial $p$. Since the polynomial set is dense in $\mathcal{B}^\alpha_0$ and $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_\mu$ is bounded, we have $\left(T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}\right) f \in \mathcal{Z}_{\mu,0}$ for $f \in \mathcal{B}^\alpha_0$. Hence, the proof of boundedness of $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_{\mu,0}$ has been completed. □

Next, we characterize the compactness of $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_{\mu,0}$.

**Theorem 4.2.** *Let $n \in \mathbb{N}$, $\alpha > 0$, $u_1, u_2, v_1, v_2 \in H(\mathbb{D})$, $\varphi$, $\psi \in S(\mathbb{D})$ and $\mu \in W(\mathbb{D})$. Suppose that any three conditions $c_0, c_1, c_2, c_3$ are valid. Then $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_{\mu,0}$ is compact if and only if the remaining condition holds.*

*Proof. Sufficiency*. According to the equivalence relation between condition (2) and (3) in Theorem 4.1, we have $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_{\mu,0}$ is bounded.

Let $\mathcal{L} = \left\{f \in \mathcal{B}^\alpha_0 : \|f\|_{\mathcal{B}^\alpha} \leqslant 1\right\}$ be a closed subset in $\mathcal{B}^\alpha_0$. Following the calculations in (7), we have

$$\begin{aligned}
&\lim_{|z|\to 1}\sup_{f\in\mathcal{L}} \mu(z)\left|\left((T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi})f\right)''(z)\right| \\
&\lesssim \lim_{|z|\to 1}\sum_{i=0}^{3}\sup_{f\in\mathcal{L}}\left(\left|\varphi_{\#_i}(z)\right|\rho(z) + \left|\varphi_{\#_i}(z) - \psi_{\#_i}(z)\right|\right) = 0.
\end{aligned}$$

Lemma 2.6 implies the difference $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_{\mu,0}$ is compact.

*Necessity*. Assume $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_{\mu,0}$ is compact and any three of conditions $c_0, c_1, c_2, c_3$ are valid, then it is evident that $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_{\mu,0}$ is bounded. Similarly, according to the equivalence relation between condition (2) and (3) in Theorem 4.1, it is proved that the remaining condition holds.

In sum, we have proved the compactness equivalence condition of $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_{\mu,0}$. □

**Remark 4.3.** *The conclusions of Theorem 4.1 and Theorem 4.2 can be summarized by saying that the boundedness and compactness of $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_{\mu,0}$ are equivalent.*

## 5. Some corollaries

In this section, we degenerate the conclusions of $T^n_{u_1,v_1,\varphi} - T^n_{u_2,v_2,\psi}$. Let $v_1 = v_2 = 0$ in (6), the following corollaries are valid.

**Corollary 5.1.** *Let $n \in \mathbb{N}$, $\alpha > 0$, $u_1, u_2 \in H(\mathbb{D})$, $\varphi$, $\psi \in S(\mathbb{D})$ and $\mu \in W(\mathbb{D})$. Suppose that any two of conditions $a_0, a_1, a_2$ are valid. Then the following statements are equivalent.*

(1) $D^n_{\varphi,u_1} - D^n_{\psi,u_2} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ *is bounded.*

(2) $D^n_{\varphi,u_1} - D^n_{\psi,u_2} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_\mu$ *is bounded.*

(3) *The remaining condition holds.*

**Corollary 5.2.** *Let $n \in \mathbb{N}$, $\alpha > 0$, $u_1, u_2 \in H(\mathbb{D})$, $\varphi$, $\psi \in S(\mathbb{D})$ and $\mu \in W(\mathbb{D})$. Suppose that neither $D^n_{\varphi,u_1}$ nor $D^n_{\psi,u_2}$ is compact but they are both bounded and any two of conditions $b_0, b_1, b_2$ are valid. Then the following statements are equivalent.*

(1) $D^n_{\varphi,u_1} - D^n_{\psi,u_2} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ *is compact.*

(2) $D^n_{\varphi,u_1} - D^n_{\psi,u_2} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_\mu$ *is compact.*

(3) (i)$D(\varphi) = D(\psi)$;

(ii)*For* $\{z_N\} \in \Gamma(\varphi) \cap \Gamma(\psi)$, *the remaining condition hold.*

**Corollary 5.3.** *Let $n \in \mathbb{N}$, $\alpha > 0$, $u_1, u_2 \in H(\mathbb{D})$, $\varphi$, $\psi \in S(\mathbb{D})$ and $\mu \in W(\mathbb{D})$. Suppose that $D^n_{\varphi,u_1}$ nor $D^n_{\psi,u_2} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ is bounded and any two of conditions $c_0, c_1, c_2$ are valid, then the following statements are equivalent.*

(1) $D^n_{\varphi,u_1} - D^n_{\psi,u_2} : \mathcal{B}^\alpha \to \mathcal{Z}_{\mu,0}$ *is bounded.*

(2) $D^n_{\varphi,u_1} - D^n_{\psi,u_2} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_{\mu,0}$ *is bounded.*

(3) $D^n_{\varphi,u_1} - D^n_{\psi,u_2} : \mathcal{B}^\alpha_0 \to \mathcal{Z}_{\mu,0}$ *is compact.*

(4) *The remaining condition holds.*

(5) *For* $i \in \{0, 1, 2\}$,

$$\lim_{|z|\to 1} |\varphi(z) - \psi(z)| \cdot \max\left\{\left|\widetilde{A}_i(\varphi(z))\right|, \left|\widetilde{A}_i(\psi(z))\right|\right\} = 0,$$
$$\lim_{|z|\to 1} \left|\widetilde{A}_i(\varphi(z)) - \widetilde{A}_i(\psi(z))\right| = 0.$$

(6) *For* $i \in \{0, 1, 2\}$,

$$\lim_{|z|\to 1} \left|\varphi(z)\widetilde{A}_i(\varphi(z)) - \psi(z)\widetilde{A}_i(\psi(z))\right| = 0$$
$$\lim_{|z|\to 1} \left|\widetilde{A}_i(\varphi(z)) - \widetilde{A}_i(\psi(z))\right| = 0.$$

Further fixing $u_1 = u_2 = 1$, the above corollaries can degenerate the difference of $C_\varphi D^n - C_\psi D^n$. On the other hand, choosing $u_1 = u, u_2 = 0$ in the above corollaries, it confirms that the corresponding results in [1].

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